# Energy-to-Peak Output Tracking Control of Actuator Saturated Periodic Piecewise Time-Varying Systems With Nonlinear Perturbations

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***Abstract*—This article is focused on the design of an output tracking control scheme for a class of continuous-time periodic piecewise time-varying systems (PPTVSs) with actuator saturation and nonlinear perturbations. The energy-to-peak tracking performance is studied based on an equivalent condition on the definiteness property of matrix polynomials. Considering the actuator saturation and nonlinear perturbation, matrix polynomial-based sufficient conditions are derived through the Lyapunov method using periodic matrix functions. From a perspective of subinterval segmentation aimed at PPTVSs, the proposed conditions can achieve less conservatism for tracking the output of a periodic time-varying reference system, while the controller gains can be computed using convex optimization. Moreover, a heuristic algorithm is constructed to simultaneously guarantee the closed-loop state convergence and the output tracking performance. The reduction in conservatism and the effectiveness of algorithm are demonstrated by illustrative case studies.**

***Index Terms*—Actuator saturation, nonlinear perturbations, output tracking control, periodic systems, time-varying systems.**

## I. Introduction

PERIODIC time-varying systems are the simplest nonautonomous systems but widely present in both nature and industry. Under the assumption of a fixed fundamental period, the stability, the control and filtering issues of periodic time-varying systems have been extensively investigated in regarding to their widespread application, such as in mechanical systems, power systems, networked systems, and ecological systems [1]–[6]. Among the relevant studies, the controller synthesis and optimization for continuous-time periodic systems have been regarded as more difficult than those for discrete-time periodic systems [7]. Although the stability and stabilization of continuous-time periodic linear systems with exact model information may be tackled by numerical methods like the Floquet–Lyapunov theory [7], [8], the controller design under practical limitations is likely to encounter NP-hardness due to the underlying nonconvexity in system models.

To improve the convenience in analysis and synthesis, periodic piecewise system (PPS) models have been found efficient in facilitating the study on continuous-time periodic systems, which are without closed-form solutions and inapplicable to lifted models. A PPS can be achieved by dividing the known fundamental period into a number of subintervals, and the dynamics over each subinterval are characterized by a corresponding subsystem. The subsystems, which may be either time-invariant or time varying, can offer more technical freedom and amenability for convex optimization tools. In recent years, PPSs and their variants have drawn growing research interests, which have developed from the cases using time-invariant subsystems to those using time-varying subsystems. Based on the previous results on periodic system approximation [9] and switched systems [10]–[13], the stability analyses of periodic piecewise linear time-invariant (LTI) systems in time domain and frequency domain are investigated in [14] and [15], respectively. In [16], a constructive time-delay method to averaging of linear systems with almost periodic coefficients that are piecewise-continuous in time is proposed for analyzing exponential stability and input-to-state stability via direct Lyapunov functionals. Research efforts have also been extended to solving the control and filtering problems of PPSs constituted by LTI subsystems with or without uncertainties [17]–[19], positive LTI subsystems [20] and linear time-delay subsystems [21]–[23], as well as periodic switched impulsive linear systems [24]. Note that in most of the previous studies, the controller and filter gains are supposed to be periodic piecewise constant.

Since the polynomial-based study in [25], periodic time-varying control approaches have been considered to tackle the stabilization of closed-loop time-varying PPSs, which motivated the study on periodic piecewise time-varying systems (PPTVSs). Compared with the PPSs using LTI subsystems, the PPTVSs that comprise of several time-varying subsystems

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can provide more accurate approximation of periodic dynamics. The analysis of stability and $L_2$ performance for the class of PPTVSs with known linear time-varying (LTV) subsystems have been presented in [26], where a helpful property of matrix polynomials is proposed. In [27], a polynomial-based nonfragile control scheme is established for PPTVSs under controller perturbations with nonidentical time-varying coefficients. The use of matrix polynomials not only brings convenience to analysis, but also creates new possibility to solve more practical-oriented problems.

In our previous work, the state tracking control problem of PPTVSs has been studied with a periodic piecewise linear reference model [28]. However, the actuator saturation and nonlinear perturbations widely existing in control systems can degrade the stability and/or tracking performance [29]–[31], especially for output tracking issues that are usually more desirable in practice [32], [33]. Differing from the existing studies involving actuator saturation and nonlinear perturbations, in PPTVSs the coexistence of saturation and perturbations is more challenging to output tracking control, since the nonconvexity in the system, controller and perturbations can together result in coupled time-varying variables even after a proper model augmentation.

In this article, an energy-to-peak output tracking control scheme, which has not been reported for PPTVSs before, is established for a type of actuator saturated PPTVSs involving nonlinear perturbations. Compared to the existing studies on PPTVSs, the PPTVS model considered in this work involves both known and unknown nonlinear dynamics. The known dynamics over each period is represented by a series of LTV subsystems for convenience in both modelling and algorithm implementation, while the unknown dynamics in the subsystems are represented by Lipschitz nonlinear perturbations. Meanwhile, the reference model that generates the output signal for tracking can be periodic time varying. The actuator saturation is tackled by a bounding region condition proposed for PPTVSs. The output tracking performance is analyzed via the $L_2$-$L_\infty$ synthesis of an augmented system. To lower the conservatism in tracking performance, an important property of matrix polynomials is complemented based on the one given in [26] and applied to computing the periodic controller gains, which are constructed based on selectable subinterval segmentations. A heuristic iterative algorithm is therefore proposed to simultaneously ensure the closed-loop stability and the output tracking performance. The contributions of this work includes the following.

1) The proposed approach integrates the subinterval segmentation with the negative definiteness property of matrix polynomials, leading to lower conservatism in energy-to-peak output tracking performance than that based on the existing method in previous studies [26], [28].
2) The obtained periodic controller not only ensures the control inputs compliant to the actuator saturation, but also demonstrates its effectiveness in tracking the output of a periodic time-varying reference model.
3) The computation of key matrices in controller design is amenable to convex optimization, while the proposed algorithm provides an alternative to get the controller gains and the performance index simultaneously.

The article is organized as follows. Section II provides the problem formulation and theoretical preliminaries. Section III analyzes the closed-loop stability under energy-to-peak output tracking performance, with the relevant criteria and a control algorithm proposed. Section IV validates the proposed results and gives some discussions based on illustrative simulations. Section V concludes the article.

*Notation:* $\mathbb{R}^n$ and $\mathbb{Z}^+$ stand for the $n$-dimensional Euclidean space and the set of positive integers, respectively; $\|\cdot\|$ denotes the Euclidean norm of a vector; $I$ and 0 represent the identity matrix and zero matrix with appropriate dimensions; $P > 0$ ($P \geq 0$) denotes that $P$ is a real symmetric and positive definite (semi-definite) matrix. $P^{\mathrm{T}}$ and $P^{-1}$ denote the transpose and the inverse of matrix $P$, respectively. For convenience, let $\mathbf{sym}(P) = P^{\mathrm{T}} + P$. $\mathbf{diag}(\cdot)$ denotes a diagonal matrix constructed by the given block matrices. In block symmetric matrices, "$*$" is used as an ellipsis for the terms introduced by symmetry.

## II. Problem Formulation and Preliminaries

Consider the following actuator saturated PPTVS with a fundamental period $T_p$ and nonlinear perturbations:

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}(t)x(t) + f(t, x(t)) + \mathcal{B}(t)\mathrm{SAT}(u(t)) + \mathcal{E}(t)w(t)\\ z(t) &= \mathcal{C}(t)x(t) + \mathcal{D}(t)\mathrm{SAT}(u(t))\end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector and supposed to be continuous of $t$ for all $t \geq 0$; $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$ and $w(t) \in \mathbb{R}^{n_w}$ are the control input, system output and energy-bounded external disturbance, respectively; $f(t, x(t)) \in \mathbb{R}^{n_x}$ represents the time-varying and state-dependent nonlinear perturbations; $(\mathcal{A}(t), \mathcal{B}(t), \mathcal{C}(t), \mathcal{D}(t), \mathcal{E}(t))$ are $T_p$-periodic matrix functions characterized by $S$ parts over each period, that is, for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $l = 0, 1, \ldots$, $T_i = t_i - t_{i-1}$, $i \in \mathcal{S} \triangleq \{1, 2, \ldots, S\}$, $\Sigma_{i=1}^{S} T_i = T_p$, $t_0 = 0$ and $t_S = T_p$, with the relevant dynamics given as

$$\begin{cases}\mathcal{A}(t) = \mathcal{A}_i(t) = A_i + \frac{t-lT_p-t_{i-1}}{T_i}(A_{i+1} - A_i)\\ \mathcal{B}(t) = \mathcal{B}_i(t) = B_i + \frac{t-lT_p-t_{i-1}}{T_i}(B_{i+1} - B_i)\\ \mathcal{C}(t) = \mathcal{C}_i(t) = C_i + \frac{t-lT_p-t_{i-1}}{T_i}(C_{i+1} - C_i)\\ \mathcal{D}(t) = \mathcal{D}_i(t) = D_i + \frac{t-lT_p-t_{i-1}}{T_i}(D_{i+1} - D_i)\\ \mathcal{E}(t) = \mathcal{E}_i(t) = E_i + \frac{t-lT_p-t_{i-1}}{T_i}(E_{i+1} - E_i)\end{cases} \quad (2)$$

where $(A_i, B_i, C_i, D_i, E_i)$, $i \in \mathcal{S}$, are known constant matrices with appropriate dimensions. For all $t \geq 0$, nonlinear function $f(t, x(t))$ is supposed to satisfy $f(t, 0) = 0$ and consists of $S$ parts over each period, that is, $f(t, x(t)) = f_i(t, x(t))$, $i \in \mathcal{S}$. For $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $f_i(t, x(t))$ satisfies

$$\|f_i(t, x(t))\| \leq \alpha_i \|F_i x(t)\|, \quad i \in \mathcal{S} \quad (3)$$

where $F_i \in \mathbb{R}^{n_x \times n_x}$, $i \in \mathcal{S}$, are constant matrices and scalars $\alpha_i > 0$, $i \in \mathcal{S}$. The control input in (1) is subject to actuator saturation $\mathrm{SAT}(\cdot) : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$, that is

$$\mathrm{SAT}(u(t)) = \left[\mathrm{sat}(u_1), \mathrm{sat}(u_2), \ldots, \mathrm{sat}\left(u_{n_u}\right)\right]^{\mathrm{T}} \quad (4)$$

where $\text{sat}(u_j) = \text{sgn}(u_j) \cdot \min(1, |u_j|)$ for the $j$th actuator, $j = 1, 2, \ldots, n_u$.

*Remark 1:* The PPTVS model composed of LTV subsystems was first proposed in [26] to provide time-varying approximation and simplification for continuous-time periodic systems. In practice, however, it is usually difficult to obtain the exact LTV dynamics of each subsystem due to model uncertainties. In addition, when the system is affected by nonlinear perturbations, it can be challenging to use spline fitting to capture the subsystem dynamics. Thus, the nonlinear function $f(t, x(t))$ in PPTVS (1) not only helps compensate potential nonlinearities like model uncertainties and inaccuracy, but also characterizes the perturbations induced by the modelling process. Note that $f(t, x(t))$ may not be periodic, but it has a bound on its linear growth rate for $t \geq 0$.

In this article, the output tracking controller design is based on a stable periodic reference model sharing the same dimension of output with PPTVS (1)

$$\begin{aligned} \dot{x}_r(t) &= \mathcal{A}_r(t)x_r(t) + r(t) \\ z_r(t) &= \mathcal{C}_r(t)x_r(t) \end{aligned} \tag{5}$$

where $x_r(t) \in \mathbb{R}^{n_r}$, $z_r(t) \in \mathbb{R}^{n_z}$, and $r(t) \in \mathbb{R}^{n_r}$ are the reference state continuous for all $t \geq 0$, reference output and energy-bounded reference input, respectively; $\mathcal{A}_r(t) = \mathcal{A}_r(t + T_p)$, $\mathcal{E}_r(t) = \mathcal{E}_r(t + T_p)$, $\mathcal{C}_r(t) = \mathcal{C}_r(t + T_p)$ are prescribed periodic matrix functions.

*Remark 2:* Due to the wide application of periodic systems in practice, this work considers a periodic time-varying reference model, aiming to design a periodic controller which can stabilize the system and give desirable closed-loop behavior as well as performance. The reference model should be stable since the performance is considered from the energy-to-peak perspective.

Let $\xi(t) = [x^{\mathrm{T}}(t), x_r^{\mathrm{T}}(t)]^{\mathrm{T}}$, consider a periodic time-varying control law for output tracking

$$\begin{aligned} u(t) &= \tilde{\mathcal{K}}_i(t)\xi(t) = \begin{bmatrix} \mathcal{K}_{x,i}(t) & \mathcal{K}_{r,i}(t) \end{bmatrix}\xi(t) \\ & t \in [lT_p + t_{i-1}, lT_p + t_i) \end{aligned} \tag{6}$$

where $\mathcal{K}_{x,i}(t)$ and $\mathcal{K}_{r,i}(t)$ are continuous in the $i$th subsystem, and $\mathcal{K}_{x,i}(t) = \mathcal{K}_{x,i}(t+T_p)$, $\mathcal{K}_{r,i}(t) = \mathcal{K}_{r,i}(t+T_p)$, $i \in \mathcal{S}$. Define the output tracking error as $e_r(t) = z(t) - z_r(t)$. Combining PPTVS (1) with reference model (5) and control law (6), an augmented closed-loop system is obtained

$$\begin{aligned} \dot{\xi}(t) &= \tilde{\mathcal{A}}_i(t)\xi(t) + \tilde{f}_i(t, \xi(t)) + \tilde{\mathcal{B}}_i(t)\text{SAT}(\tilde{\mathcal{K}}_i(t)\xi(t)) \\ &\quad + \tilde{\mathcal{E}}_i(t)\varpi(t) \\ e_r(t) &= \tilde{\mathcal{C}}_i(t)\xi(t) + \tilde{\mathcal{D}}_i(t)\text{SAT}(\tilde{\mathcal{K}}_i(t)\xi(t)) \\ & t \in [lT_p + t_{i-1}, lT_p + t_i) \end{aligned} \tag{7}$$

where

$$\begin{aligned} \tilde{\mathcal{A}}_i(t) &= \begin{bmatrix} \mathcal{A}_i(t) & 0 \\ 0 & \mathcal{A}_r(t) \end{bmatrix}, \quad \tilde{\mathcal{B}}_i(t) = \begin{bmatrix} \mathcal{B}_i(t) \\ 0 \end{bmatrix} \\ \tilde{\mathcal{C}}_i(t) &= \begin{bmatrix} \mathcal{C}_i(t) & -\mathcal{C}_r(t) \end{bmatrix}, \quad \tilde{\mathcal{D}}_i(t) = \mathcal{D}_i(t) \\ \tilde{\mathcal{E}}_i(t) &= \begin{bmatrix} \mathcal{E}_i(t) & 0 \\ 0 & I \end{bmatrix} \\ \varpi(t) &= \begin{bmatrix} w(t) \\ r(t) \end{bmatrix}, \quad \tilde{f}_i(t, \xi(t)) = \begin{bmatrix} f_i(t, x(t)) \\ 0 \end{bmatrix}. \end{aligned}$$

The augmented system in (7) is a PPTVS with $S$ time-varying subsystems affected by saturated control input and nonlinear perturbations. From (3), one has

$$\left\|\tilde{f}_i(t, \xi(t))\right\| \leq \alpha_i \left\|\tilde{F}_i\xi(t)\right\|, \quad i \in \mathcal{S} \tag{8}$$

where $\tilde{F}_i = \begin{bmatrix} F_i & 0 \end{bmatrix}$, $i \in \mathcal{S}$. Aimed at the analysis of energy-to-peak performance, the following assumption is considered throughout this article.

*Assumption 1:* The energy-bounded disturbance $w(t)$ and reference input $r(t)$ satisfy $\|w(t)\| \leq w_{\max}$ and $\|r(t)\| \leq r_{\max}$, respectively, where $w_{\max}$ and $r_{\max}$ are known nonnegative constants.

In previous study [26], a helpful lemma on the property of a class of matrix polynomials is given. However, it only considered the sufficient condition for the definiteness property of matrix polynomials. In this article, the property is extended below to facilitate the following analysis.

*Lemma 1:* Let $g : [0, 1]^k \to \mathbb{R}^{n\times n}$ be a matrix polynomial function defined as

$$g(\eta_1, \eta_2, \ldots, \eta_k) = \Omega_0 + \eta_1\Omega_1 + \eta_1\eta_2\Omega_2 + \cdots + \left(\prod_{j=1}^{k} \eta_j\right)\Omega_k \tag{9}$$

where scalars $\eta_j \in [0, 1]$, $j = 1, 2, \ldots, k$, $k \in \mathbb{Z}^+$, and $\Omega_0, \Omega_1, \ldots, \Omega_k$ are $n \times n$ real symmetric matrices. Real symmetric matrix polynomial $g(\eta_1, \eta_2, \ldots, \eta_k) < 0$ (resp., $> 0$), if and only if

$$\sum_{j=0}^{d} \Omega_j < 0 \text{ (resp., } > 0), \quad d = 0, 1, \ldots, k. \tag{10}$$

*Proof:* Without loss of generality, one can take the proof of negative definiteness for example. The sufficiency that considers the negative definiteness of matrix polynomial $g(\eta_1, \eta_2, \ldots, \eta_k)$ for integer $k \geq 2$ has been proved, see the proof of [26, Lemma 2]. For $k = 1$, with $\eta_1 \in [0, 1]$, symmetric matrix inequalities $\Omega_0 < 0$ and $\Omega_0+\Omega_1 < 0$ following (10), it is easy to obtain

$$g(\eta_1) = \Omega_0 + \eta_1\Omega_1 = (1 - \eta_1)\Omega_0 + \eta_1(\Omega_0 + \Omega_1) < 0 \tag{11}$$

indicating that the sufficiency holds for $k \in \mathbb{Z}^+$.

The necessity that considers the negative definiteness of real symmetric matrix polynomial $g(\eta_1, \eta_2, \ldots, \eta_k)$ can be proved based on the property of convex combination: With $g(\eta_1, \eta_2, \ldots, \eta_k) < 0$ and scalars $\eta_j \in [0, 1]$, $j = 1, 2, \ldots, k$, $k \in \mathbb{Z}^+$, one has $g(\eta_1, \eta_2, \ldots, \eta_k) = \Omega_0 + \eta_1(\Omega_1 + \eta_2\Omega_2 + \cdots + (\prod_{j=2}^{k} \eta_j)\Omega_k) < 0$, which implies $\Omega_0 < 0$ and $\Omega_0 + \Omega_1 + \eta_2\Omega_2 + \cdots + (\prod_{j=2}^{k} \eta_j)\Omega_k = (\Omega_0 + \Omega_1) + \eta_2(\Omega_2 + \cdots + (\prod_{j=3}^{k} \eta_j)\Omega_k) < 0$, further indicating that $\Omega_0 + \Omega_1 < 0$ and $\Omega_0 + \Omega_1 + \Omega_2 + \eta_3(\Omega_3 \cdots + (\prod_{j=4}^{k} \eta_j)\Omega_k) < 0$. Through recursive implementation one can obtain matrix inequalities as follows:

$$\begin{aligned} \Omega_0 &< 0 \\ \Omega_0 + \Omega_1 &< 0 \\ \Omega_0 + \Omega_1 + \Omega_2 &< 0 \\ &\vdots \\ \Omega_0 + \Omega_1 + \Omega_2 + \cdots + \Omega_k &< 0 \end{aligned}$$

which can be summarized by (10) for all $k \geq 1$. The necessity is proved.

On the other hand, the proof for the positive definiteness of $g(\eta_1, \eta_2, \ldots, \eta_k)$ can be easily obtained by replacing "$<$" with "$>$" in the aforementioned procedures, which are omitted here. Thus, the lemma provides an equivalent condition for the property in either the negative definiteness or the positive definiteness of matrix polynomial $g(\eta_1, \eta_2, \ldots, \eta_k)$ for $\eta_j \in [0, 1]$, $j = 1, 2, \ldots, k$, $k \in \mathbb{Z}^+$. ∎

*Remark 3:* Given $k \geq 1$, $k \in \mathbb{Z}^+$, there are $2^k$ combinations of $(\eta_1, \eta_2, \ldots, \eta_{k-1}, \eta_k)$ taking the endpoint values 0 and 1, that is, $(\eta_1, \eta_2, \ldots, \eta_{k-1}, \eta_k) = \{(0, 0, \ldots, 0, 0), (0, 0, \ldots, 0, 1), (0, 0, \ldots, 1, 0), (0, 0, \ldots, 1, 1), \ldots, (1, 1, \ldots, 1, 1)\}$. The $(k + 1)$ matrix inequalities in (10) can also be seen as the result of substituting these combinations into (9).

In this article, one mainly focuses on limiting the upper bound of output tracking error. Hence, the energy-to-peak (also known as $L_2$-$L_\infty$ or generalized $H_2$) performance index is considered. The objectives are expressed from two aspects.

1) *Exponential Stability:* Augmented system (7) with $\varpi(t) = 0$ is exponentially stable.
2) *Tracking Performance:* For all nonzero $w, r \in L_2[0, \infty)$, the effect of $\varpi(t)$ on the output tracking error $e_r(t)$ is attenuated below a desired level $\overline{\gamma} > 0$. More specifically, under zero initial conditions, it is required that

$$\sup_{\forall t \geq 0} e_r^{\mathrm{T}}(t) e_r(t) < \overline{\gamma}^2 \int_0^\infty \varpi^{\mathrm{T}}(t)\varpi(t)\mathrm{d}t. \tag{12}$$

## III. Main Results

In this section, the closed-loop stability and tracking performance are analyzed under the effects of saturation and nonlinear perturbations. First, a bounding region condition is proposed to guarantee a norm-bounded control input. A sufficient condition of stability with energy-to-peak tracking performance index are hence derived as the basis of controller design. The criterion of output tracking control and a heuristic iterative convex optimization algorithm are established.

### A. Stability and Performance Analysis

For convenience, the $j$th row of $\tilde{\mathcal{K}}_i(t)$ for the $i$th subsystem is denoted as $\tilde{\mathcal{K}}_{ij}(t)$, which is corresponding to the $j$th actuator. Under Assumption 1, it holds that

$$\begin{cases} w^{\mathrm{T}}(t)w(t) \leq w_{\max}^2 \\ r^{\mathrm{T}}(t)r(t) \leq r_{\max}^2 \\ \varpi_{\max} \triangleq \sqrt{w_{\max}^2 + r_{\max}^2} \geq 0. \end{cases} \tag{13}$$

For the augmented system in (7), define its reachable set as

$$\mathscr{R}_\xi \triangleq \big\{\xi(t) \in \mathbb{R}^{n_x+n_r} \mid \xi(0) = 0 \\ \xi(t), \varpi(t) \text{ satisfy (7) and (13)}, \ t \geq 0\big\}. \tag{14}$$

The following bounding region of reachable set $\mathscr{R}_\xi$ is used to deal with the periodic piecewise time-varying dynamics:

$$\mathscr{E}(\mathcal{P}(t)) \triangleq \left\{\xi \in \mathbb{R}^{n_x+n_r} \mid \xi^{\mathrm{T}}\mathcal{P}(t)\xi \leq 1, \ \mathcal{P}(t) > 0\right\} \tag{15}$$

where $\mathcal{P}(t) > 0$ is a continuous $T_p$-periodic matrix function with

$$\mathcal{P}(t) = \mathcal{P}_i(t) > 0, \ t \in [lT_p + t_{i-1}, lT_p + t_i), \ i \in \mathcal{S} \tag{16}$$

and $\lim_{t \to lT_p + t_i^-} \mathcal{P}_i(t) = \mathcal{P}_{i+1}(lT_p + t_i)$, $l = 0, 1, \ldots$, $i = 1, 2, \ldots, S$, $\mathcal{P}_{S+1}(t) = \mathcal{P}_1(t)$. The upper right Dini derivative of $\mathcal{P}(t)$ is described as

$$\mathcal{D}^+\mathcal{P}(t) = \mathcal{D}^+\mathcal{P}_i(t) = \limsup_{h \to 0^+} \frac{\mathcal{P}_i(t+h) - \mathcal{P}_i(t)}{h} \\ t \in [lT_p + t_{i-1}, lT_p + t_i). \tag{17}$$

A region-bounding condition is proposed to deal with the actuator saturation.

*Theorem 1 (Region-Bounding Condition):* Consider augmented system (7) with the periodic control law in (6) and disturbance $\varpi$ satisfying (13). If there exist scalars $\beta_i > 0$, $\upsilon_i > 0$, $i = 1, 2, \ldots, S$, and real symmetric $T_p$-periodic, continuous and Dini-differentiable matrix function $\mathcal{P}(t)$ defined on $t \in [0, \infty)$ such that, for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $\mathcal{P}(t) = \mathcal{P}_i(t) > 0$, $i = 1, 2, \ldots, S$, $j = 1, 2, \ldots, n_u$, the following conditions hold:

$$\begin{bmatrix} \mathcal{P}_i(t) & * \\ \tilde{\mathcal{K}}_{ij}(t) & 1 \end{bmatrix} \geq 0 \tag{18}$$

$$\begin{bmatrix} \Gamma_i(t) & * & * \\ \mathcal{P}_i(t) & -\upsilon_i I & * \\ \tilde{\mathcal{E}}_i^{\mathrm{T}}(t)\mathcal{P}_i(t) & 0 & -\frac{\beta_i}{\varpi_{\max}^2} I \end{bmatrix} < 0 \tag{19}$$

where

$$\Gamma_i(t) = \mathbf{sym}\Big(\mathcal{P}_i(t)\tilde{\mathcal{A}}_i(t) + \mathcal{P}_i(t)\tilde{\mathcal{B}}_i(t)\tilde{\mathcal{K}}_i(t)\Big) \\ + \upsilon_i\alpha_i^2\tilde{F}_i^{\mathrm{T}}\tilde{F}_i + \mathcal{D}^+\mathcal{P}_i(t) + \beta_i\mathcal{P}_i(t) \tag{20}$$

then a region bounding the reachable set $\mathscr{R}_\xi$ of system (7) is given by $\mathscr{E}(\mathcal{P}(t))$ in form of (15). The control input $u(t)$ satisfies $\|u(t)\| \leq 1$ and is bounded within $\mathscr{E}(\mathcal{P}(t))$.

The detailed proof of Theorem 1 is given in the Appendix.

*Remark 4:* Theorem 1 serves as the constraint of control input $u(t)$ by bounding the system state in a desired reachable set. In the following controller synthesis based on conditions (18) and (19), one does not need to care about the size of the reachable set, but only needs to ensure that it can be appropriately bounded.

Based on Theorem 1, the following condition is presented to guarantee the closed-loop stability and energy-to-peak output tracking performance.

*Theorem 2 (Stability With Tracking Performance):* Consider augmented system (7) with output tracking control law (6) and nonzero $w, r \in L_2[0, \infty)$ under Assumption 1. Given a scalar $\lambda^* > 0$, if there exist scalars $\tilde{\upsilon}_i > 0$, $\lambda_i$, $i = 1, 2, \ldots, S$, $\lambda_{\min} \triangleq \min_{i \in \mathcal{S}}(\lambda_i)$, $\lambda_{\max} \triangleq \max_{i \in \mathcal{S}}(\lambda_i)$, $\gamma > 0$, and real symmetric $T_p$-periodic, continuous and Dini-differentiable matrix function $\mathcal{P}(t)$ defined on $t \in [0, \infty)$ such that, for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $i = 1, 2, \ldots, S$, $\mathcal{P}(t) = \mathcal{P}_i(t) > 0$, inequalities (18), (19), and the following conditions hold:

$$\begin{bmatrix} \mathbf{sym}\Big(\mathcal{P}_i(t)\tilde{\mathcal{A}}_i(t)+\mathcal{P}_i(t)\tilde{\mathcal{B}}_i(t)\tilde{\mathcal{K}}_i(t)\Big) & * & * \\ +\tilde{\upsilon}_i\alpha_i^2\tilde{F}_i^{\mathrm{T}}\tilde{F}_i+\mathcal{D}^+\mathcal{P}_i(t)+\lambda_i\mathcal{P}_i(t) & & \\ \mathcal{P}_i(t) & -\tilde{\upsilon}_i I & * \\ \tilde{\mathcal{E}}_i^{\mathrm{T}}(t)\mathcal{P}_i(t) & 0 & -I \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} -\mathcal{P}_i(t) & * \\ \tilde{\mathcal{C}}_i(t)+\tilde{\mathcal{D}}_i(t)\tilde{\mathcal{K}}_i(t) & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

$$2\lambda^* T_p - \sum_{i=1}^{S}\lambda_i T_i \le 0 \tag{23}$$

then the system is exponentially stable, and satisfies the energy-to-peak output tracking performance (12) with $\overline{\gamma} = \gamma e^{T_p \max(2\lambda^*-\lambda_{\min},0)}$.

*Proof:* Consider the Lyapunov function $V(t)$ in (A.1) for $\xi(t) \neq 0$, $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $i = 1, 2, \ldots, S$. When inequalities (18) and (19) hold, by Theorem 1 it is guaranteed that $u^{\mathrm{T}}(t)u(t) \le 1$. With scalars $\tilde{\upsilon}_i > 0$ and $\alpha_i > 0$, following the similar procedures in the proof of Theorem 1, one obtains:

$$\begin{aligned} &\mathcal{D}^+V_i(t)+\lambda_i V_i(t)+\tilde{\upsilon}_i\Big(\alpha_i^2\xi^{\mathrm{T}}(t)\tilde{F}_i^{\mathrm{T}}\tilde{F}_i\xi(t)-\tilde{f}_i^{\mathrm{T}}(t)\tilde{f}_i(t)\Big)\\ &\quad-\varpi^{\mathrm{T}}(t)\varpi(t)\\ &=\begin{bmatrix}\xi^{\mathrm{T}}(t) & \tilde{f}_i^{\mathrm{T}}(t) & \varpi^{\mathrm{T}}(t)\end{bmatrix}\Phi_i(t)\begin{bmatrix}\xi(t)\\ \tilde{f}_i(t)\\ \varpi(t)\end{bmatrix} \end{aligned} \tag{24}$$

where

$$\Phi_i(t)=\begin{bmatrix} \mathbf{sym}\Big(\mathcal{P}_i(t)\tilde{\mathcal{A}}_i(t)+\mathcal{P}_i(t)\tilde{\mathcal{B}}_i(t)\tilde{\mathcal{K}}_i(t)\Big) & * & * \\ +\tilde{\upsilon}_i\alpha_i^2\tilde{F}_i^{\mathrm{T}}\tilde{F}_i+\mathcal{D}^+\mathcal{P}_i(t)+\lambda_i\mathcal{P}_i(t) & & \\ \mathcal{P}_i(t) & -\tilde{\upsilon}_i I & * \\ \tilde{\mathcal{E}}_i^{\mathrm{T}}(t)\mathcal{P}_i(t) & 0 & -I. \end{bmatrix}.$$

From (8), one has $\tilde{\upsilon}_i(\alpha_i^2\xi^{\mathrm{T}}(t)\tilde{F}_i^{\mathrm{T}}\tilde{F}_i\xi(t)-\tilde{f}_i^{\mathrm{T}}(t)\tilde{f}_i(t)) \ge 0$. Combining condition (21) one has $\Phi_i(t) < 0$, which implies that for $t \in [lT_p + t_{i-1}, lT_p + t_i)$

$$\begin{aligned} \mathcal{D}^+V_i(t) &< -\lambda_i V_i(t)+\varpi^{\mathrm{T}}(t)\varpi(t)\\ &\quad-\tilde{\upsilon}_i\Big(\alpha_i^2\xi^{\mathrm{T}}(t)\tilde{F}_i^{\mathrm{T}}\tilde{F}_i\xi(t)-\tilde{f}_i^{\mathrm{T}}(t)\tilde{f}_i(t)\Big)\\ &< -\lambda_i V_i(t)+\varpi^{\mathrm{T}}(t)\varpi(t). \end{aligned} \tag{25}$$

Integrating (25), it follows that:

$$\begin{aligned} V(t) &\le e^{-\lambda_i(t-lT_p-t_{i-1})}V\big(lT_p+t_{i-1}\big)\\ &\quad+\int_{lT_p+t_{i-1}}^{t} e^{-\lambda_i(t-\tau)}\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau\\ &\quad t \in [lT_p+t_{i-1}, lT_p+t_i). \end{aligned} \tag{26}$$

When $\varpi(t) = 0$, by (21) and (23) it is easy to obtain that $\mathcal{D}^+V_i(t) < -\lambda_i V_i(t)$ for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, leading to $V(lT_p) \le e^{-2\lambda^* lT_p}V(0)$. According to the previous study on the stability of continuous-time PPTVSs [25], it can be proved that the augmented system in (7) with $\varpi(t) = 0$ is $\lambda^*$-exponentially stable, where the augmented state $\xi(t)$ satisfies

$$\|\xi(t)\| \le \kappa e^{-\lambda^* t}\|\xi(0)\| \quad \forall t \ge 0 \tag{27}$$

where $\kappa = e^{\lambda^* T_p}\sqrt{\overline{\boldsymbol{\lambda}}(\mathcal{P}(0))/\underline{\boldsymbol{\lambda}}(\mathcal{P}(0))}\prod_{i=1}^{S}\max(1, e^{\psi_i T_i}) \ge 1$, $\overline{\boldsymbol{\lambda}}(\cdot)$ and $\underline{\boldsymbol{\lambda}}(\cdot)$, respectively, denote the maximum and minimum eigenvalues; $\psi_i$ is a constant satisfying $\psi_i \ge (1/2)\overline{\boldsymbol{\lambda}}(\mathbf{sym}(\tilde{\mathcal{A}}_i(t)))$ for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $i \in \mathcal{S}$.

Moreover, consider nonzero $w, r \in L_2[0,\infty)$ leading to $\varpi \in L_2[0,\infty)$ and $\varpi(t) \neq 0$. Under zero initial conditions, for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, from (26) one has

$$\begin{aligned} V(t) &= \xi^{\mathrm{T}}(t)\mathcal{P}(t)\xi(t)\\ &\le \sum_{j=1}^{l}\sum_{k=1}^{S}\int_{(j-1)T_p+t_{k-1}}^{(j-1)T_p+t_k}\exp\Bigg(-\lambda_k\big((j-1)T_p+t_k-\tau\big)\\ &\qquad-\sum_{q=k+1}^{S}\lambda_q T_q-(l-j)\sum_{q=1}^{S}\lambda_q T_q\\ &\qquad-\sum_{q=1}^{i-1}\lambda_q T_q-\lambda_i\big(t-lT_p-t_{i-1}\big)\Bigg)\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau\\ &\quad+\sum_{k=1}^{i-1}\int_{lT_p+t_{k-1}}^{lT_p+t_k}\exp\Bigg(-\lambda_k\big(lT_p+t_k-\tau\big)\\ &\qquad-\sum_{q=k+1}^{i-1}\lambda_q T_q-\lambda_i\big(t-lT_p-t_{i-1}\big)\Bigg)\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau\\ &\quad+\int_{lT_p+t_{i-1}}^{t}\exp(-\lambda_i(t-\tau))\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau. \end{aligned} \tag{28}$$

Following a derivation process similar to those presented in [26] to tackle the exponential functions in (28) yields:

$$\begin{aligned} &\xi^{\mathrm{T}}(t)\mathcal{P}(t)\xi(t)\\ &\quad\le \int_0^t e^{-2\lambda^*(t-\tau)+2T_p\max(2\lambda^*-\lambda_{\min},0)}\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau\\ &\quad\le e^{2T_p\max(2\lambda^*-\lambda_{\min},0)}\int_0^t\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau. \end{aligned} \tag{29}$$

On the other hand, applying Schur complement equivalence to (22), for $t \in [lT_p + t_{i-1}, lT_p + t_i)$ one has

$$\begin{aligned} &\xi^{\mathrm{T}}(t)\Big(\tilde{\mathcal{C}}_i(t)+\tilde{\mathcal{D}}_i(t)\tilde{\mathcal{K}}_i(t)\Big)^{\mathrm{T}}\Big(\tilde{\mathcal{C}}_i(t)+\tilde{\mathcal{D}}_i(t)\tilde{\mathcal{K}}_i(t)\Big)\xi(t)\\ &\quad<\gamma^2\xi^{\mathrm{T}}(t)\mathcal{P}_i(t)\xi(t)\\ &\quad\le\gamma^2 e^{2T_p\max(2\lambda^*-\lambda_{\min},0)}\int_0^t\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau. \end{aligned}$$

Taking the supremum over $t > 0$, it follows that:

$$\sup_{\forall t} e_r^{\mathrm{T}}(t)e_r(t) < \overline{\gamma}^2\int_0^{\infty}\varpi^{\mathrm{T}}(\tau)\varpi(\tau)d\tau$$

where $\overline{\gamma} = \gamma e^{T_p\max(2\lambda^*-\lambda_{\min})} > 0$ indicates the energy-to-peak performance. Therefore, augmented system (7) with nonzero $w, r \in L_2[0,\infty)$ is exponentially stable and satisfies the output tracking performance described in (12). ■

*Remark 5:* In Theorem 2, conditions (18), (19), and (21)–(23) construct a general closed-loop stability framework with respect to the actuator saturation, nonlinear perturbations as well as the considered output tracking performance. Inequality (23) guarantees that the whole PPTVS is exponentially stable. Under appropriate optimization frameworks, one may not need to impose the values of $\lambda_i, i \in \mathcal{S}$, which creates more flexibility in the control scheme.

If one considers a class of PPTVSs given by (1) and (2) but without control input and nonlinear perturbations, a corollary of basic $L_2$-$L_\infty$ performance for stable PPTVSs can be derived as follows.

*Corollary 1 (Basic $L_2$-$L_\infty$ Performance for PPTVSs):* Consider PPTVS (1) with $f(t, x(t)) = 0$, $u(t) = 0$ and nonzero $w \in L_2[0, \infty)$. Given a scalar $\lambda^* > 0$, if there exist scalars $\lambda_i, i = 1, 2, \ldots, S$, $\lambda_{\min} \triangleq \min_{i\in\mathcal{S}}(\lambda_i)$, $\lambda_{\max} \triangleq \max_{i\in\mathcal{S}}(\lambda_i)$, $\gamma > 0$, and real symmetric $T_p$-periodic, continuous and Dini-differentiable matrix function $\mathcal{Z}(t)$ defined on $t \in [0, \infty)$ such that, for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $i = 1, 2, \ldots, S$, $\mathcal{Z}(t) = \mathcal{Z}_i(t) > 0$, inequality (23) and the following conditions hold:

$$\begin{bmatrix} \mathbf{sym}(\mathcal{Z}_i(t)\mathcal{A}_i(t)) \\ +\mathcal{D}^+\mathcal{Z}_i(t) + \lambda_i\mathcal{Z}_i(t) & * \\ \mathcal{E}_i^{\mathrm{T}}(t)\mathcal{Z}_i(t) & -I \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} -\mathcal{Z}_i(t) & * \\ \mathcal{C}_i(t) & -\gamma^2 I \end{bmatrix} < 0 \tag{31}$$

then the system is exponentially stable with $L_2$-$L_\infty$ performance described by $\overline{\gamma} = \gamma e^{T_p \max(2\lambda^* - \lambda_{\min}, 0)}$.

Note that the conditions in Theorems 1 and 2 contain time-varying terms that cannot be directly used in computing the controller gains. To solve the problem by convex optimization techniques, the controller design and optimization will be further discussed in the next section.

## B. Controller Design and Optimization

Since the known parts of each subsystem in PPTVS (1) are LTV as given by (2), from a perspective of subinterval segmentation for $t \in [lT_p+t_{i-1}, lT_p+t_i)$, $i \in \mathcal{S}$, the LTV matrix functions in augmented system (7) can be approximated by

$$\begin{cases} \tilde{\mathcal{A}}_i(t) = \tilde{A}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{A}_{i,m-1} \\ \tilde{\mathcal{B}}_i(t) = \tilde{B}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{B}_{i,m-1} \\ \tilde{\mathcal{C}}_i(t) = \tilde{C}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{C}_{i,m-1} \\ \tilde{\mathcal{D}}_i(t) = \tilde{D}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{D}_{i,m-1} \\ \tilde{\mathcal{E}}_i(t) = \tilde{E}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{E}_{i,m-1} \end{cases} \tag{32}$$

where

$$\begin{cases} \Delta\tilde{A}_{i,m-1} \triangleq \tilde{A}_{i,m} - \tilde{A}_{i,m-1} \\ \Delta\tilde{B}_{i,m-1} \triangleq \tilde{B}_{i,m} - \tilde{B}_{i,m-1} \\ \Delta\tilde{C}_{i,m-1} \triangleq \tilde{C}_{i,m} - \tilde{C}_{i,m-1} \\ \Delta\tilde{D}_{i,m-1} \triangleq \tilde{D}_{i,m} - \tilde{D}_{i,m-1} \\ \Delta\tilde{E}_{i,m-1} \triangleq \tilde{E}_{i,m} - \tilde{E}_{i,m-1} \end{cases} \tag{33}$$

and $\sigma_{i,m}(t) = M_i(t - (m-1)[T_i/M_i])/T_i \in [0, 1)$, $m = 1, 2, \ldots, M_i$, with prescribed $M_i \in \mathbb{Z}^+$, $i = 1, 2, \ldots, S$. Constant matrices $(\tilde{A}_{i,m}, \tilde{B}_{i,m}, \tilde{C}_{i,m}, \tilde{D}_{i,m}, \tilde{E}_{i,m})$ are obtained as follows:

$$\begin{cases} \tilde{A}_{i,m} = \begin{bmatrix} A_i + \frac{m}{M_i}(A_{i+1} - A_i) & 0 \\ 0 & A_r\left(lT_p + t_{i-1} + \frac{mT_i}{M_i}\right) \end{bmatrix} \\ \tilde{B}_{i,m} = \begin{bmatrix} B_i + \frac{m}{M_i}(B_{i+1} - B_i) \\ 0 \end{bmatrix} \\ \tilde{C}_{i,m} = \begin{bmatrix} C_i + \frac{m}{M_i}(C_{i+1} - C_i) & -C_r\left(lT_p + t_{i-1} + \frac{mT_i}{M_i}\right) \end{bmatrix} \\ \tilde{D}_{i,m} = D_i + \frac{m}{M_i}(D_{i+1} - D_i) \\ \tilde{E}_{i,m} = \begin{bmatrix} E_i + \frac{m}{M_i}(E_{i+1} - E_i) & 0 \\ 0 & I \end{bmatrix}. \end{cases} \tag{34}$$

Based on the results in Theorems 1 and 2, a condition for designing the output tracking controller is presented.

*Theorem 3 (Controller Design):* Consider augmented system (7) with fundamental period $T_p > 0$, output tracking control law (6) and nonzero $w, r \in L_2[0, \infty)$ under Assumption 1. Given $M_i \in \mathbb{Z}^+, i = 1, 2, \ldots, S$, and a scalar $\lambda^* > 0$, the system is exponentially stable and satisfies the energy-to-peak output tracking performance (12) with $\overline{\gamma} = \gamma e^{T_p \max(2\lambda^* - \lambda_{\min}, 0)}$ if there exist scalars $\upsilon_i > 0$, $\beta_i > 0$, $\tilde{\upsilon}_i > 0$, $\lambda_i$, $i = 1, 2, \ldots, S$, $\lambda_{\min} \triangleq \min_{i\in\mathcal{S}}(\lambda_i)$, $\lambda_{\max} \triangleq \max_{i\in\mathcal{S}}(\lambda_i)$, $\gamma > 0$, matrices $Q_{i,m} > 0$, and matrices $U_{i,m}$ with $U_{i,m,j}$ denoting its $j$th row, $i = 1, 2, \ldots, S$, $m = 1, 2, \ldots, M_i$, $j = 1, 2, \ldots, n_u$, such that inequality (23) and the following conditions hold:

$$\begin{bmatrix} Q_{i,m} & * \\ U_{i,m,j} & 1 \end{bmatrix} \geq 0 \tag{35}$$

$$\begin{bmatrix} \Theta_{0,i,m} & * \\ \mho_{0,i,m} & \Upsilon_i \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} \Theta_{0,i,m} + \Theta_{1,i,m} & * \\ \mho_{1,i,m} & \Upsilon_i \end{bmatrix} < 0 \tag{37}$$

$$\begin{bmatrix} \Theta_{0,i,m} + \Theta_{1,i,m} + \Theta_{2,i,m} & * \\ \mho_{1,i,m} & \Upsilon_i \end{bmatrix} < 0 \tag{38}$$

$$\begin{bmatrix} \Lambda_{0,i,m} & * \\ \overline{\mho}_{0,i,m} & \overline{\Upsilon}_i \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} \Lambda_{0,i,m} + \Lambda_{1,i,m} & * \\ \overline{\mho}_{1,i,m} & \overline{\Upsilon}_i \end{bmatrix} < 0 \tag{40}$$

$$\begin{bmatrix} \Lambda_{0,i,m} + \Lambda_{1,i,m} + \Lambda_{2,i,m} & * \\ \overline{\mho}_{1,i,m} & \overline{\Upsilon}_i \end{bmatrix} < 0 \tag{41}$$

$$\begin{bmatrix} -Q_{i,m-1} & * \\ \Xi_{0,i,m} & -\gamma^2 I \end{bmatrix} < 0 \tag{42}$$

$$\begin{bmatrix} -Q_{i,m} & * \\ \Xi_{0,i,m} + \Xi_{1,i,m} & -\gamma^2 I \end{bmatrix} < 0 \tag{43}$$

$$\begin{bmatrix} -Q_{i,m} & * \\ \Xi_{0,i,m} + \Xi_{1,i,m} + \Xi_{2,i,m} & -\gamma^2 I \end{bmatrix} < 0 \tag{44}$$

$$Q_{i,M_i} = Q_{i+1,0}, \quad Q_{S,M_i} = Q_{1,0} \tag{45}$$

where the approximated matrices in (32) and (34) are considered, and

$$\Delta Q_{i,m-1} = Q_{i,m} - Q_{i,m-1}, \quad \Delta U_{i,m-1} = U_{i,m} - U_{i,m-1}$$
$$\Theta_{0,i,m} = \mathbf{sym}\left(\tilde{A}_{i,m-1}Q_{i,m-1} + \tilde{B}_{i,m-1}U_{i,m-1}\right) - \frac{M_i}{T_i}\Delta Q_{i,m-1} + \beta_i Q_{i,m-1}$$

$$\begin{aligned}
\Theta_{1,i,m} &= \mathbf{sym}\Big(\tilde{A}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{A}_{i,m-1}Q_{i,m-1} \\
&\qquad + \tilde{B}_{i,m-1}\Delta U_{i,m-1} + \Delta\tilde{B}_{i,m-1}U_{i,m-1}\Big) \\
&\quad + \beta_i\Delta Q_{i,m-1} \\
\Theta_{2,i,m} &= \mathbf{sym}\Big(\Delta\tilde{A}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{B}_{i,m-1}\Delta U_{i,m-1}\Big) \\
\mho_{0,i,m} &= \begin{bmatrix} I & \tilde{E}_{i,m-1} & \upsilon_i Q_{i,m-1}\tilde{F}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
\mho_{1,i,m} &= \begin{bmatrix} I & \tilde{E}_{i,m} & \upsilon_i Q_{i,m}\tilde{F}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
\Upsilon_i &= \mathbf{diag}\left(-\upsilon_i I, -\frac{\beta_i}{\varpi_{\max}^2}I, -\frac{\upsilon_i}{\alpha_i^2}I\right) \\
\Lambda_{0,i,m} &= \mathbf{sym}\Big(\tilde{A}_{i,m-1}Q_{i,m-1} + \tilde{B}_{i,m-1}U_{i,m-1}\Big) \\
&\quad - \frac{M_i}{T_i}\Delta Q_{i,m-1} + \lambda_i Q_{i,m-1} \\
\Lambda_{1,i,m} &= \mathbf{sym}\Big(\tilde{A}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{A}_{i,m-1}Q_{i,m-1} \\
&\qquad + \tilde{B}_{i,m-1}\Delta U_{i,m-1} + \Delta\tilde{B}_{i,m-1}U_{i,m-1}\Big) \\
&\quad + \lambda_i\Delta Q_{i,m-1} \\
\Lambda_{2,i,m} &= \mathbf{sym}\Big(\Delta\tilde{A}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{B}_{i,m-1}\Delta U_{i,m-1}\Big) \\
\overline{\mho}_{0,i,m} &= \begin{bmatrix} I & \tilde{E}_{i,m-1} & \tilde{\upsilon}_i Q_{i,m-1}\tilde{F}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
\overline{\mho}_{1,i,m} &= \begin{bmatrix} I & \tilde{E}_{i,m} & \tilde{\upsilon}_i Q_{i,m}\tilde{F}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
\overline{\Upsilon}_i &= \mathbf{diag}\left(-\tilde{\upsilon}_i I, -I, -\frac{\tilde{\upsilon}_i}{\alpha_i^2}I\right) \\
\Xi_{0,i,m} &= \tilde{C}_{i,m-1}Q_{i,m-1} + \tilde{D}_{i,m-1}U_{i,m-1} \\
\Xi_{1,i,m} &= \Delta\tilde{C}_{i,m-1}Q_{i,m-1} + \Delta\tilde{D}_{i,m-1}U_{i,m-1} \\
&\quad + \tilde{C}_{i,m-1}\Delta Q_{i,m-1} + \tilde{D}_{i,m-1}\Delta U_{i,m-1} \\
\Xi_{2,i,m} &= \Delta\tilde{C}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{D}_{i,m-1}\Delta U_{i,m-1}.
\end{aligned}$$

The $T_p$-periodic output tracking controller gains can be computed by

$$\tilde{\mathcal{K}}(t) = \tilde{\mathcal{K}}_i(t) = \mathcal{U}_i(t)\mathcal{Q}_i^{-1}(t),\ \ t \in [lT_p + t_{i-1}, lT_p + t_i) \tag{46}$$

where for $t \in [lT_p+t_{i-1}+(m-1)[T_i/M_i], lT_p+t_{i-1}+m[T_i/M_i])$ over the $i$th subinterval, time-varying matrix functions $\mathcal{Q}(t)$ and $\mathcal{U}(t)$ are obtained by

$$\mathcal{Q}(t) = \mathcal{Q}_i(t) = Q_{i,m-1} + \sigma_{i,m}(t)\Delta Q_{i,m-1} \tag{47}$$

$$\mathcal{U}(t) = \mathcal{U}_i(t) = U_{i,m-1} + \sigma_{i,m}(t)\Delta U_{i,m-1} \tag{48}$$

with $\sigma_{i,m}(t) = M_i(t - (m-1)[T_i/M_i])/T_i \in [0,1)$, $i = 1,2,\ldots,S$, $m = 1,2,\ldots,M_i$.

*Proof:* First, from (45), (47), and (48), it can be seen that $\mathcal{Q}(t)$ is a periodic matrix function continuous at all the switching instants for $t \geq 0$. For $t \in [lT_p+t_{i-1}+(m-1)[T_i/M_i], lT_p+t_{i-1}+m[T_i/M_i])$, $M_i \in \mathbb{Z}^+$, $i = 1,2,\ldots,S$, it has the following upper right Dini derivative:

$$\mathcal{D}^+\mathcal{Q}(t) = \frac{M_i}{T_i}\big(Q_{i,m} - Q_{i,m-1}\big) = \frac{M_i}{T_i}\Delta Q_{i,m-1}. \tag{49}$$

From (35), one has

$$\begin{bmatrix} \mathcal{Q}_i(t) & * \\ \mathcal{U}_{ij}(t) & 1 \end{bmatrix} \geq 0,\quad j = 1,2,\ldots,n_u \tag{50}$$

where $\mathcal{U}_{ij}(t)$ denotes the $j$th row of matrix function $\mathcal{U}_i(t)$, $i \in \mathcal{S}$.

Based on (32), (34), and the case of $k = 2$ in Lemma 1, consider $\eta_1 = \eta_2 = \sigma_{i,m}(t) \in [0,1) \subset [0,1]$ for $g(\eta_1,\eta_2)$. Since

$$\begin{aligned}
\mho_{1,i,m} &= \begin{bmatrix} I & \tilde{E}_{i,m} & \upsilon_i Q_{i,m}\tilde{F}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} I & \tilde{E}_{i,m-1} + \Delta\tilde{E}_{i,m-1} & \upsilon_i\big(Q_{i,m-1} + \Delta Q_{i,m-1}\big)\tilde{F}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}
\end{aligned}$$

from Lemma 1 and conditions (36)–(38), for $t \in [lT_p + t_{i-1} + (m-1)[T_i/M_i], lT_p + t_{i-1} + m[T_i/M_i])$ one has

$$\begin{bmatrix}
\begin{matrix}\mathbf{sym}\Big(\tilde{\mathcal{A}}_i(t)\mathcal{Q}_i(t) \\ \quad + \tilde{\mathcal{B}}_i(t)\mathcal{U}_i(t)\Big) \\ -\mathcal{D}^+\mathcal{Q}_i(t) + \beta_i\mathcal{Q}_i(t)\end{matrix} & * & * & * \\
I & -\upsilon_i I & * & * \\
\tilde{\mathcal{E}}_i^{\mathrm{T}}(t) & 0 & -\frac{\beta_i}{\varpi_{\max}^2}I & * \\
\upsilon_i\tilde{F}_i\mathcal{Q}_i(t) & 0 & 0 & -\frac{\upsilon_i}{\alpha_i^2}I
\end{bmatrix} < 0 \tag{51}$$

where

$$\begin{aligned}
&\mathbf{sym}\Big(\tilde{\mathcal{A}}_i(t)\mathcal{Q}_i(t) + \tilde{\mathcal{B}}_i(t)\mathcal{U}_i(t)\Big) - \mathcal{D}^+\mathcal{Q}_i(t) + \beta_i\mathcal{Q}_i(t) \\
&= \mathbf{sym}\Big(\big(\tilde{A}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{A}_{i,m-1}\big) \\
&\qquad \times \big(Q_{i,m-1} + \sigma_{i,m}(t)\Delta Q_{i,m-1}\big) \\
&\qquad + \big(\tilde{B}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{B}_{i,m-1}\big) \\
&\qquad \times \big(U_{i,m-1} + \sigma_{i,m}(t)\Delta U_{i,m-1}\big)\Big) \\
&\quad - \frac{M_i}{T_i}\Delta Q_{i,m-1} + \beta_i\big(Q_{i,m-1} + \sigma_{i,m}(t)\Delta Q_{i,m-1}\big) \\
&= \mathbf{sym}\Big(\tilde{A}_{i,m-1}Q_{i,m-1} + \tilde{B}_{i,m-1}U_{i,m-1}\Big) \\
&\quad - \frac{M_i}{T_i}\Delta Q_{i,m-1} + \beta_i Q_{i,m-1} \\
&\quad + \sigma_{i,m}(t)\Big(\mathbf{sym}\Big(\tilde{A}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{A}_{i,m-1}Q_{i,m-1} \\
&\qquad\qquad + \tilde{B}_{i,m-1}\Delta U_{i,m-1} + \Delta\tilde{B}_{i,m-1}U_{i,m-1}\Big) \\
&\qquad\qquad + \beta_i\Delta Q_{i,m-1}\Big) \\
&\quad + \sigma_{i,m}^2(t)\mathbf{sym}\Big(\Delta\tilde{A}_{i,m-1}\Delta Q_{i,m-1} + \Delta\tilde{B}_{i,m-1}\Delta U_{i,m-1}\Big) \\
&= \Theta_{0,i,m} + \sigma_{i,m}(t)\Theta_{1,i,m} + \sigma_{i,m}^2(t)\Theta_{2,i,m}
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} I \\ \tilde{\mathcal{E}}_i^{\mathrm{T}}(t) \\ \upsilon_i\tilde{F}_i\mathcal{Q}_i(t) \end{bmatrix} &= \begin{bmatrix} I \\ \big(\tilde{E}_{i,m-1} + \sigma_{i,m}(t)\Delta\tilde{E}_{i,m-1}\big)^{\mathrm{T}} \\ \upsilon_i\tilde{F}_i\big(Q_{i,m-1} + \sigma_{i,m}(t)\Delta Q_{i,m-1}\big) \end{bmatrix} \\
&= \begin{bmatrix} I \\ \tilde{E}_{i,m-1}^{\mathrm{T}} \\ \upsilon_i\tilde{F}_i Q_{i,m-1} \end{bmatrix} + \sigma_{i,m}(t)\begin{bmatrix} 0 \\ \Delta\tilde{E}_{i,m-1}^{\mathrm{T}} \\ \upsilon_i\tilde{F}_i\Delta Q_{i,m-1}. \end{bmatrix}.
\end{aligned}$$

Similarly, combing conditions (39)–(38) and (42)–(44) with Lemma 1, respectively, the following results can be derived:

$$\begin{bmatrix}
\begin{matrix}\mathbf{sym}\Big(\tilde{\mathcal{A}}_i(t)\mathcal{Q}_i(t) \\ +\tilde{\mathcal{B}}_i(t)\mathcal{U}_i(t)\Big) \\ -\mathcal{D}^+\mathcal{Q}_i(t) + \lambda_i\mathcal{Q}_i(t)\end{matrix} & * & * & * \\
I & -\tilde{\upsilon}_i I & * & * \\
\tilde{\mathcal{E}}_i^{\mathrm{T}}(t) & 0 & -I & * \\
\tilde{\upsilon}_i\tilde{F}_i\mathcal{Q}_i(t) & 0 & 0 & -\frac{\tilde{\upsilon}_i}{\alpha_i^2}I
\end{bmatrix} < 0 \tag{52}$$

$$\begin{bmatrix} -\mathcal{Q}_i(t) & * \\ \tilde{\mathcal{C}}_i(t)\mathcal{Q}_i(t)+\tilde{\mathcal{D}}_i(t)\mathcal{U}_i(t) & -\gamma^2 I \end{bmatrix} < 0 \tag{53}$$

where

$$\begin{aligned}
&\mathbf{sym}\Big(\tilde{\mathcal{A}}_i(t)\mathcal{Q}_i(t)+\tilde{\mathcal{B}}_i(t)\mathcal{U}_i(t)\Big)-\mathcal{D}^+\mathcal{Q}_i(t)+\lambda_i\mathcal{Q}_i(t)\\
&\quad=\Lambda_{0,i,m}+\sigma_{i,m}(t)\Lambda_{1,i,m}+\sigma_{i,m}^2(t)\Lambda_{2,i,m}\\
&\tilde{\mathcal{C}}_i(t)\mathcal{Q}_i(t)+\tilde{\mathcal{D}}_i(t)\mathcal{U}_i(t)\\
&\quad=\Big(\tilde{C}_{i,m-1}+\sigma_{i,m}(t)\Delta\tilde{C}_{i,m-1}\Big)\big(Q_{i,m-1}+\sigma_{i,m}(t)\Delta Q_{i,m-1}\big)\\
&\qquad+\Big(\tilde{D}_{i,m-1}+\sigma_{i,m}(t)\Delta\tilde{D}_{i,m-1}\Big)\big(U_{i,m-1}+\sigma_{i,m}(t)\Delta U_{i,m-1}\big)\\
&\quad=\Xi_{0,i,m}+\sigma_{i,m}(t)\Xi_{1,i,m}+\sigma_{i,m}^2(t)\Xi_{2,i,m}.
\end{aligned}$$

Multiply both sides of (51) and (52) with $\mathbf{diag}(\mathcal{Q}_i^{-1}(t), I, I, I)$, and multiply both sides of inequalities (50) and (53) with $\mathbf{diag}(\mathcal{Q}_i^{-1}(t), I)$. Using Schur complement equivalence and the fact $\mathcal{D}^+\mathcal{Q}_i^{-1}(t) = -\mathcal{Q}_i^{-1}(t)\mathcal{D}^+\mathcal{Q}_i(t)\mathcal{Q}_i^{-1}(t)$, one can derive conditions in the form of (18), (19), (21), and (22) as given in Theorems 1 and 2. Therefore, when conditions (23) and (35)–(45) hold, augmented system (7) with periodic time-varying controller (6) is exponentially stable, and satisfies the energy-to-peak output tracking performance described by $\overline{\gamma} = \gamma e^{T_p \max(2\lambda^*-\lambda_{\min},0)}$. ■

*Remark 6:* The conditions in Theorem 3 are constructed based on the segmentations of subintervals defined by $M_i$, $i \in \mathcal{S}$. The periodic matrix functions in linear interpolative forms provide an approximation of the time-varying dynamics in the augmented system. Using Lemma 1, the conditions in Theorem 2 can hence be achieved by those in Theorem 3 that are amenable to convex optimization.

In Theorem 3, periodic matrix function $\mathcal{Q}(t)$ is continuous of $t$ at all the switching instants for $t \geq 0$, while $\mathcal{U}(t)$ is only continuous over each subinterval and not necessarily continuous at the switching instants. If one imposes the continuity of matrix function $\mathcal{U}(t)$ at all the switching instants, the periodic time-varying controller gains $\tilde{\mathcal{K}}(t)$ will become continuous for $t \geq 0$, improving the smoothness of dynamics in practical applications. The condition for computing continuous output tracking controller gains is given in the following corollary.

*Corollary 2:* Consider augmented system (7) with fundamental period $T_p > 0$, output tracking control law (6) and nonzero $w, r \in L_2[0,\infty)$ under Assumption 1. Given $M_i \in \mathbb{Z}^+, i = 1, 2, \ldots, S$, and a scalar $\lambda^* > 0$, the system is exponentially stable and satisfies the energy-to-peak output tracking performance (12) with $\overline{\gamma} = \gamma e^{T_p \max(2\lambda^*-\lambda_{\min},0)}$ if there exist scalars $\upsilon_i > 0$, $\beta_i > 0$, $\tilde{\upsilon}_i > 0$, $\lambda_i$, $i = 1, 2, \ldots, S$, $\lambda_{\min} \triangleq \min_{i\in\mathcal{S}}(\lambda_i)$, $\lambda_{\max} \triangleq \max_{i\in\mathcal{S}}(\lambda_i)$, $\gamma > 0$, matrices $Q_{i,m} > 0$, and matrices $U_{i,m}$ with $U_{i,m,j}$ denoting its $j$th row, $i = 1, 2, \ldots, S$, $m = 1, 2, \ldots, M_i$, $j = 1, 2, \ldots, n_u$, such that conditions (23), (35)–(45), and the following equations hold:

$$U_{i,M_i} = U_{i+1,0},\ \ U_{S,M_i} = U_{1,0}. \tag{54}$$

The $T_p$-periodic controller gains can be computed by (46)–(48) and are continuous of $t$ for $t \geq 0$.

*Remark 7:* Theorem 3 and Corollary 2 share the same number of matrix variables $Q_{i,m}$ as $\sum_{i=1}^S M_i$, while the corresponding numbers of matrix variables $U_{i,m}$ are $S + \sum_{i=1}^S M_i$ and $\sum_{i=1}^S M_i$, respectively. The decision variable numbers are as follows.

1) *Theorem 3:* $1+4S+[1/2](n_x+n_r)(n_x+n_r+1)\sum_{i=1}^S M_i + n_u(n_x+n_r)(S+\sum_{i=1}^S M_i)$.
2) *Corollary 2:* $1 + 4S + (n_x + n_r)(n_u + [(n_x+n_r+1)/2])\sum_{i=1}^S M_i$.

For fixed system parameters, the computational complexity mainly depends on the values of $M_i, i \in \mathcal{S}$. Larger $M_i$ can be helpful to improve the feasibility and tracking performance, while inevitably increases the computational burden. The tradeoff between $M_i$ and the desirable performance may be referred to the previous work [34] on PPSs. Although the method in [34] uses a similar periodic time-varying matrix function, without Lemma 1 it is not applicable to the problem in this article due to the time-varying terms in (51)–(53).

The feasibility problems, in Theorem 3 and Corollary 2 involving parameters $\upsilon_i$, $\tilde{\upsilon}_i$, $\beta_i$, $\lambda_i$ and $\lambda^*$, contain bilinear matrix inequality constraints. Since $\lambda^* > 0$, condition (23) can be simplified by letting $\lambda^* = ([\sum_{i=1}^S \lambda_i T_i]/2T_p) > 0$, which implies

$$\sum_{i=1}^{S} \lambda_i T_i > 0. \tag{55}$$

To tackle the bilinear terms, two types of tracking performance can be considered through the following objectives of optimization, respectively.

1) *Energy-to-Peak Performance:* Fix the positive scalars $\upsilon_i$, $\tilde{\upsilon}_i$, $\beta_i$, and scalars $\lambda_i$ satisfying (55), $i = 1, 2, \ldots, S$, $m = 1, 2, \ldots, M_i$, solve the following optimization problem to obtain matrices $Q_{i,m} > 0$, $U_{i,m}$ and scalar $\gamma > 0$:

$$\min_{Q_{i,m}, U_{i,m}, \gamma} \gamma^2 \quad \text{subject to (35)–(45).} \tag{56}$$

2) *Mixed Performance:* Solve the following optimization problem to obtain matrices $Q_{i,m} > 0$, $U_{i,m}$ and scalars $\gamma > 0$, $\upsilon_i > 0$, $\tilde{\upsilon}_i > 0$, $\lambda_i$, $\beta_i > 0$:

$$\begin{aligned}&\min_{Q_{i,m}, U_{i,m}, \gamma, \upsilon_i, \tilde{\upsilon}_i, \lambda_i, \beta_i} \ \gamma^2 - \sum_{i=1}^{S} \lambda_i T_i\\ &\quad\text{subject to} \quad \text{(35)–(45) and (55).}\end{aligned} \tag{57}$$

*Remark 8:* The first objective of optimization in (56) using given parameters is common in the relevant studies involving bilinear matrix inequality constraints [26], [28]. With some fixed initial parameters, one can convert the optimization problem to an LMI feasibility problem that can be directly solved at the price of some conservatism. The parameters may either be given based on trail-and-error, or searched via some programs like genetic algorithm (GA) [35] by setting some initial objectives to guarantee the feasibility.

The optimization problem in this article is essentially a multiobjective one. To optimize the energy-to-peak performance and the state convergence at the same time, a heuristic iterative algorithm for output tracking control (Algorithm OTC) based on the mixed performance is proposed.

*Remark 9:* In Algorithm OTC, the objectives of optimization in steps 2 and 4 can lead to equivalent

TABLE I
PARAMETER MATRICES AT THE ENDPOINTS OF SUBINTERVALS

| $i$ | $A_i$ | $B_i$ | $C_i$ | $D_i$ | $E_i$ |
|---|---|---|---|---|---|
| 1 | $\begin{bmatrix} -2 & 1 & 0 \\ 0.5 & 1 & 1 \\ 0.8 & 2 & -3 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0.5 & 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \end{bmatrix}$ |
| 2 | $\begin{bmatrix} -4 & 1 & 0 \\ 0 & 0.6 & 1 \\ 0.5 & 1 & -1.5 \end{bmatrix}$ | $\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0.6 & 0.5 \end{bmatrix}$ | $\begin{bmatrix} 1 & 3 \end{bmatrix}$ | $\begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \end{bmatrix}$ |
| 3 | $\begin{bmatrix} -3 & 1 & 0 \\ 0.5 & 1 & 1 \\ 0.5 & 1 & -2 \end{bmatrix}$ | $\begin{bmatrix} 4 & 0 \\ 0 & 1 \\ 1.5 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0.5 & 0.8 & 1.5 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0.5 \\ 0.3 \\ 0.1 \end{bmatrix}$ |

**Algorithm 1** OTC

**Step 1:** For $i = 1, 2, \dots, S$, given $M_i \in \mathbb{Z}^+$, initial parameters $\upsilon_i^{(0)} > 0$, $\tilde{\upsilon}_i^{(0)} > 0$, $\beta_i^{(0)} > 0$, $\lambda_i^{(0)} = 0$. Set $\mathcal{J}_1^{(0)} = \mathcal{J}_2^{(0)} = 0$, a sufficiently small tolerance $\varrho > 0$ and the current number of iteration $\mu = 1$.
**Step 2:** With fixed parameters $\upsilon_i^{(\mu-1)}$, $\tilde{\upsilon}_i^{(\mu-1)}$, $\beta_i^{(\mu-1)}$, $\lambda_i^{(\mu-1)}$, get scalar $\gamma > 0$, matrices $Q_{i,m} > 0$ and $U_{i,m}$, $m = 1, 2, \dots, M_i$, $i = 1, 2, \dots, S$, through solving the following optimization problem:

$$\min_{Q_{i,m}, U_{i,m}, \gamma} \gamma^2 \quad \text{subject to}$$
$$\begin{cases} \text{(35)–(45), if } \tilde{\mathcal{K}}(t) \text{ discontinuous at switching instants} \\ \text{(35)–(45), (54), if } \tilde{\mathcal{K}}(t) \text{ continuous at switching instants} \end{cases}$$

Let $Q_{i,m}^{(\mu)} = Q_{i,m}$, $U_{i,m}^{(\mu)} = U_{i,m}$, $\mathcal{J}_1^{(\mu)} = \gamma^2 - \sum_{i=1}^{S} \lambda_i^{(\mu-1)} T_i$.
**Step 3:** If $|\mathcal{J}_1^{(\mu)} - \mathcal{J}_2^{(\mu-1)}| < \varrho$, let $\lambda^* = \frac{1}{2T_p} \sum_{i=1}^{S} \lambda_i^{(\mu-1)} T_i$, STOP. Otherwise, go to **Step 4**.
**Step 4:** With fixed matrices $Q_{i,m}^{(\mu)}$, $U_{i,m}^{(\mu)}$, get scalars $\gamma > 0$, $\upsilon_i > 0$, $\tilde{\upsilon}_i > 0$, $\beta_i > 0$ and $\lambda_i$, $i = 1, 2, \dots, S$, through solving the following optimization problem:

$$\min_{\upsilon_i, \tilde{\upsilon}_i, \lambda_i, \beta_i} \gamma^2 - \sum_{i=1}^{S} \lambda_i T_i \quad \text{subject to (36)–(44) and (55)}$$

Let $\beta_i^{(\mu)} = \beta_i$, $\upsilon_i^{(\mu)} = \upsilon_i$, $\lambda_i^{(\mu)} = \lambda_i$, $\tilde{\upsilon}_i^{(\mu)} = \tilde{\upsilon}_i$. Denote $\mathcal{J}_2^{(\mu)} = \gamma^2 - \sum_{i=1}^{S} \lambda_i^{(\mu)} T_i$.
**Step 5:** If $|\mathcal{J}_2^{(\mu)} - \mathcal{J}_1^{(\mu)}| < \varrho$, let $\lambda^* = \frac{1}{2T_p} \sum_{i=1}^{S} \lambda_i^{(\mu)} T_i$, STOP. Otherwise, set $\mu = \mu + 1$, then go to **Step 2**.
**Step 6:** Output the final solutions of scalars $\gamma$, $\lambda^*$, $\upsilon_i$, $\tilde{\upsilon}_i$, $\beta_i$, $\lambda_i$ and matrices $Q_{i,m}$, $U_{i,m}$, $m = 1, 2, \dots, M_i$, $i = 1, 2, \dots, S$. Compute the controller gains based on (46)–(48), and the energy-to-peak performance index $\overline{\gamma} = \gamma e^{T_p \max(2\lambda^* - \lambda_{\min}, 0)}$.

effects as the one in (57). Since the values of $\lambda_i, i \in \mathcal{S}$, are fixed in step 2, one only needs to minimize $\gamma^2$ to continue the optimization process. By steps 3 and 5, the mixed performance indices are gradually reduced, which guarantees the convergence of the algorithm. With initial parameters $\lambda_i^{(0)} = 0, i \in \mathcal{S}$, the initialization of positive scalars $\upsilon_i^{(0)}$, $\tilde{\upsilon}_i^{(0)}$ and $\beta_i^{(0)}$ just needs to ensure the feasibility of the first iteration. Although the obtained results are locally optimal, Algorithm OTC enables a simultaneous optimization of both the tracking performance and the convergence rate of closed-loop state. Based on the optimized $\lambda_i$, $i \in \mathcal{S}$, the closed-loop state satisfies an exponential decay rate $\lambda^* = (1/2T_p) \sum_{i=1}^{S} \lambda_i T_i$.

## IV. ILLUSTRATIVE EXAMPLE

To validate the proposed output tracking control scheme, one considers an actuator saturated PPTVS with three subsystems and corresponding nonlinear perturbations in form of (1). Consider a fundamental period $T_p = 3.5$ with $(T_1, T_2, T_3) = (1, 1.5, 1)$ in appropriate time unit, and real constant matrices $(A_i, B_i, C_i, D_i, E_i)$, $i = 1, 2, 3$, satisfying (2). The parameter matrices at the periodic subinterval endpoints (in other words, the switching instants) are presented in Table I. The settings of nonlinear perturbations for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $i = 1, 2, 3$, are given by

$$f_i(t, x(t)) = \alpha_i \begin{bmatrix} \frac{t}{t+1} \sin(x_1(t)) \\ \sin(x_2(t)) \\ \sin(x_1(t) + x_3(t)) \end{bmatrix}$$
$$F_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{58}$$

with $(\alpha_1, \alpha_2, \alpha_3) = (0.3, 0.4, 0.2)$, which guarantee the requirement in (3). For $x(0) = [1, 1, 1]^{\mathrm{T}}$, the open-loop state trajectory shown in Fig. 1 indicates that the PPTVS is unstable. Given a stable $T_p$-periodic reference system satisfying (5), where

$$A_r(t) = \begin{bmatrix} -2 + \sin(2\pi t/3.5) & 0 \\ 1 - \cos(2\pi t/3.5) & -3 \end{bmatrix}$$
$$C_r(t) = \begin{bmatrix} 1.5 \sin(2\pi t/3.5) & 1 \end{bmatrix}. \tag{59}$$

To control the output $z(t)$ aimed at tracking the reference output $z_r(t)$, given the reference input and disturbance under Assumption 1

$$r(t) = \begin{bmatrix} e^{-0.1t} \cos(2\pi t/3.5) \\ e^{-0.1t} \sin(2\pi t/3.5) \end{bmatrix}$$
$$w(t) = 0.75 e^{-0.5t} \cos(t), \quad t \geq 0 \tag{60}$$

which indicates $\varpi_{\max} = 1.25$. The following two cases are considered to illustrate the effectiveness of the proposed approach using the MATLAB solver SeDuMi.

*Case 1 (Comparison of Conservatism):* First, consider the case with fixed parameters $\lambda_i$ to analyze the conservatism of

TABLE II
COMPARISON OF TRACKING PERFORMANCE INDEX $\overline{\gamma}$ OBTAINED BY DIFFERENT VALUES OF $M$ AND PARAMETER SETS

| $M$ | | 1 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|---|
| $\overline{\gamma}$ (parameter set 1) | Discontinuous controller gains | 1.8016 | 1.3206 | 1.0956 | 1.0211 | 0.9719 | 0.9383 |
| | Continuous controller gains | 1.9451 | 1.4137 | 1.1652 | 1.0590 | 0.9945 | 0.9543 |
| $\overline{\gamma}$ (parameter set 2) | Discontinuous controller gains | 2.6667 | 1.8214 | 1.4762 | 1.3698 | 1.2974 | 1.2512 |
| | Continuous controller gains | 2.8921 | 1.9688 | 1.5840 | 1.4269 | 1.3349 | 1.2780 |

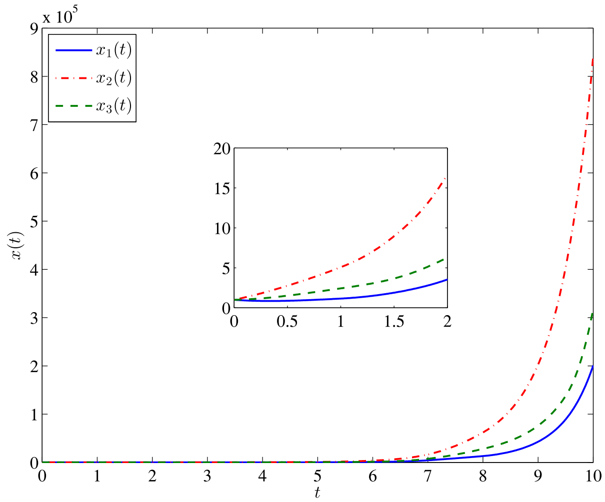


Fig. 1. Open-loop system state trajectory.

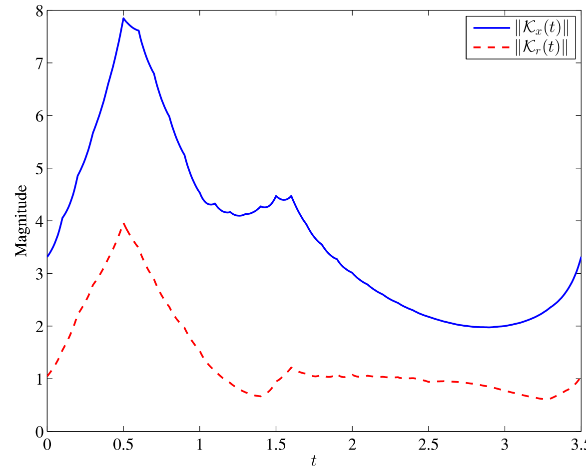


Fig. 2. Variations of $\|\mathcal{K}_x(t)\|$ and $\|\mathcal{K}_r(t)\|$ over one period.

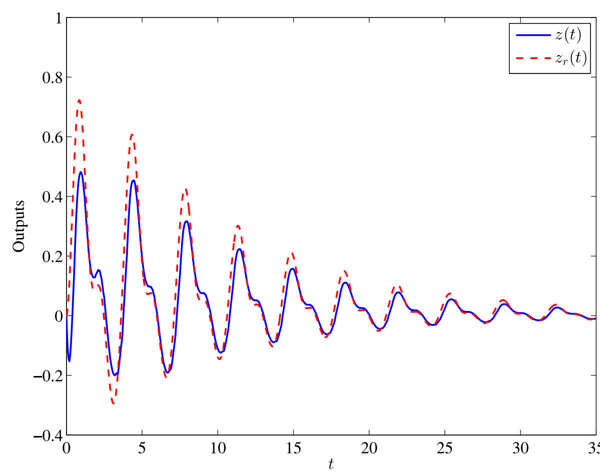


Fig. 3. Output tracking performance.

the proposed tracking control approach based on the actuator saturated PPTVS represented by (1), (2), (58)–(60), and Table I. For convenience of comparison, let $M_i = M \geq 1$, $i = 1, 2, 3$, and consider the following two sets of parameters.

1) *Parameter Set 1:* $\beta_i = 1$, $\upsilon_i = 5$, $\tilde{\upsilon}_i = 5$, $\lambda_i = 1$, $i = 1, 2, 3$.
2) *Parameter Set 2:* $\beta_1 = 1.5$, $\beta_2 = 1.2$, $\beta_3 = 1.6$, $\upsilon_1 = 4$, $\upsilon_2 = 5$, $\upsilon_3 = 6$, $\tilde{\upsilon}_1 = 5$, $\tilde{\upsilon}_2 = 6$, $\tilde{\upsilon}_3 = 4.5$, $\lambda_1 = 1$, $\lambda_2 = 0.9$, $\lambda_3 = 1.1$.

For different values of $M$ and parameter sets, the periodic controller gains $\tilde{\mathcal{K}}(t)$ can be straightforwardly obtained by solving the optimization problem in (56). With controller gains $\tilde{\mathcal{K}}(t)$ that are discontinuous or continuous at the switching instants under different values of $M$, the results of energy-to-peak output tracking performance $\overline{\gamma}$ are shown in Table II.

From Table II, it can be observed that for both parameter sets, the results of $\overline{\gamma}$ obtained with discontinuous $\tilde{\mathcal{K}}(t)$ are smaller than those obtained with continuous $\tilde{\mathcal{K}}(t)$, since the discontinuous controller gains can provide more flexibility in the solutions of matrix variables. In addition, larger values of $M$ can achieve less conservative results in energy-to-peak performance $\overline{\gamma}$, implying smaller upper bounds of output tracking errors. Note that when $M = 1$, the corresponding conditions in Algorithm OTC can be regarded as the extensions based on the existing method in previous studies on PPTVSs [26], [28]. For both cases with discontinuous and continuous controller gains, the values of $\overline{\gamma}$ obtained by $M > 1$ are smaller than those obtained by $M = 1$, which indicates a lower conservatism achieved by the proposed approach than the existing method.

*Case 2 (Iterative Optimization Scheme):* For the considered PPTVS, one uses Algorithm OTC initialized by $M_1 = 10$, $M_2 = 15$, $M_3 = 10$, $\lambda_i^{(0)} = 0$, $\beta_i^{(0)} = 2$, $\upsilon_i^{(0)} = \tilde{\upsilon}_i^{(0)} = 5$, $i = 1, 2, 3$. $\tilde{\mathcal{K}}(t)$ is set to be continuous at all the switching instants to obtain smoothly time-varying controller gains. After 4 iterations, the final parameters are obtained as follows:

$$(\beta_1, \beta_2, \beta_3) = (1.5892, 1.4900, 1.6979)$$
$$(\upsilon_1, \upsilon_2, \upsilon_3) = (7.9039, 5.8647, 6.9556)$$
$$(\tilde{\upsilon}_1, \tilde{\upsilon}_2, \tilde{\upsilon}_3) = (8.1332, 5.7542, 13.1367)$$
$$(\lambda_1, \lambda_2, \lambda_3) = (1.5611, 1.5649, 1.6507)$$

and the value of energy-to-peak performance index is obtained as $\overline{\gamma} = 0.8984$. According to (6), the time-varying controller

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



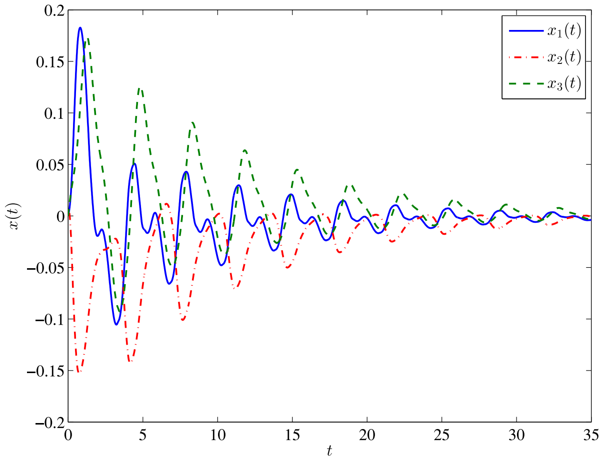


Fig. 4. Closed-loop system state trajectory.

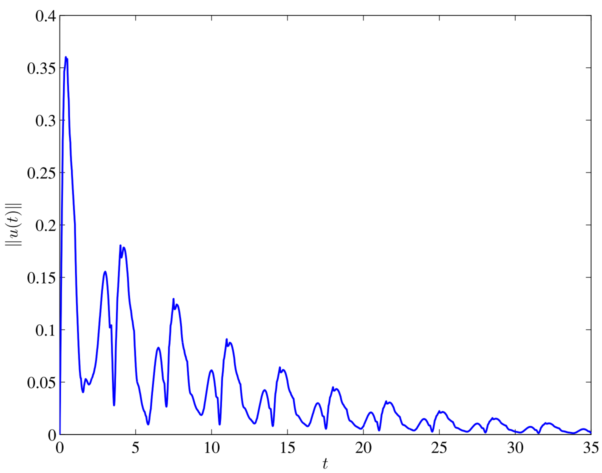


Fig. 5. Norm of the obtained control input $u(t)$.

gains for $t \geq 0$ can be denoted as

$$\tilde{\mathcal{K}}(t) = [\mathcal{K}_x(t) \quad \mathcal{K}_r(t)]. \tag{61}$$

For convenience of illustration, the variations of $\|\mathcal{K}_x(t)\|$ and $\|\mathcal{K}_r(t)\|$ over one period are given in Fig. 2. The output tracking performance, closed-loop system state trajectory over 10 periods are shown in Figs. 3 and 4, respectively. It can be seen that the controlled output $z(t)$ under the designed controller can track the variation of reference output $z_r(t)$, meanwhile the closed-loop system is stable. Moreover, the norm of the obtained control input $u(t)$ is shown in Fig. 5, which indicates $u^{\mathrm{T}}(t)u(t) \leq 1$ and thus, avoids the actuator saturation.

## V. Conclusion

In this article, the energy-to-peak output tracking controller synthesis of a type of actuator saturated PPTVSs with nonlinear perturbations has been presented. Combing the augmented system approach and the negative definiteness property of matrix polynomial inequalities, sufficient conditions for closed-loop stability and controller design have been proposed based on the Lyapunov functions under subinterval segmentation approach. An iterative algorithm framework has been established to provide the solutions of both state convergence rate and energy-to-peak tracking performance. The results of simulation and comparison under different cases of subinterval segmentation have demonstrated the reduction of conservatism in tracking performance, and the effectiveness of the proposed Algorithm OTC has been illustrated. In future work, tracking control issues under strong nonlinear conditions and more general actuator saturations will be considered. Some polynomial-based techniques [36] and helpful properties of convex hulls [37] could also be integrated to deal with the difficulties in periodic control brought by more complicated saturations.

## Appendix

*Proof of Theorem 1:* With (13)–(17), consider a quadratic Lyapunov function candidate

$$V(t) = \xi^{\mathrm{T}}(t)\mathcal{P}(t)\xi(t) \tag{A.1}$$

with periodic matrix function $\mathcal{P}(t)$ satisfying (16) and (17). For $\xi(t) \neq 0$, $V(t)$ can be rewritten as $V(t) = V_i(t) = \xi^{\mathrm{T}}(t)\mathcal{P}_i(t)\xi(t) > 0$, $t \in [lT_p + t_{i-1}, lT_p + t_i)$. When condition (19) holds, for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, denote $\tilde{f}_i(t) \triangleq \tilde{f}_i(t, \xi(t))$ for conciseness, then one has

$$\begin{aligned}
&\mathcal{D}^+V_i(t) + \beta_i V_i(t) + \upsilon_i\Big(\alpha_i^2\xi^{\mathrm{T}}(t)\tilde{F}_i^{\mathrm{T}}\tilde{F}_i\xi(t) - \tilde{f}_i^{\mathrm{T}}(t)\tilde{f}_i(t)\Big)\\
&\quad - \frac{\beta_i}{\varpi_{\max}^2}\varpi^{\mathrm{T}}(t)\varpi(t)\\
&= \xi^{\mathrm{T}}(t)\Big(\mathbf{sym}\Big(\mathcal{P}_i(t)\tilde{\mathcal{A}}_i(t) + \mathcal{P}_i(t)\tilde{\mathcal{B}}_i(t)\tilde{\mathcal{K}}_i(t)\Big)\\
&\qquad + \upsilon_i\alpha_i^2\tilde{F}_i^{\mathrm{T}}\tilde{F}_i + \mathcal{D}^+\mathcal{P}_i(t) + \beta_i\mathcal{P}_i(t)\Big)\xi(t)\\
&\quad + \mathbf{sym}\Big(\tilde{f}_i^{\mathrm{T}}(t)\mathcal{P}_i(t)\xi(t) + \varpi^{\mathrm{T}}(t)\tilde{\mathcal{E}}_i^{\mathrm{T}}(t)\mathcal{P}_i(t)\xi(t)\Big)\\
&\quad - \upsilon_i\tilde{f}_i^{\mathrm{T}}(t)\tilde{f}_i(t) - \frac{\beta_i}{\varpi_{\max}^2}\varpi^{\mathrm{T}}(t)\varpi(t)\\
&= \begin{bmatrix}\xi^{\mathrm{T}}(t) & \tilde{f}_i^{\mathrm{T}}(t) & \varpi^{\mathrm{T}}(t)\end{bmatrix}\\
&\quad \times \begin{bmatrix}\Gamma_i(t) & \mathcal{P}_i(t) & \mathcal{P}_i(t)\tilde{\mathcal{E}}_i(t)\\ \mathcal{P}_i(t) & -\upsilon_i I & 0\\ \tilde{\mathcal{E}}_i^{\mathrm{T}}(t)\mathcal{P}_i(t) & 0 & -\frac{\beta_i}{\varpi_{\max}^2}I\end{bmatrix}\begin{bmatrix}\xi(t)\\ \tilde{f}_i(t)\\ \varpi(t)\end{bmatrix}\\
&< 0
\end{aligned} \tag{A.2}$$

where $\Gamma_i(t)$ is defined by (20). By (8) with $\upsilon_i > 0$, $i \in \mathcal{S}$, it always follows that:

$$\upsilon_i\Big(\alpha_i^2\xi^{\mathrm{T}}(t)\tilde{F}_i^{\mathrm{T}}\tilde{F}_i\xi(t) - \tilde{f}_i^{\mathrm{T}}(t)\tilde{f}_i(t)\Big) \geq 0 \tag{A.3}$$

which indicates that

$$\mathcal{D}^+V_i(t) + \beta_i V_i(t) - \frac{\beta_i}{\varpi_{\max}^2}\varpi^{\mathrm{T}}(t)\varpi(t) < 0. \tag{A.4}$$

Considering the bounded disturbance $\varpi(t)$ satisfying (13) and (A.4), one obtains

$$\mathcal{D}^+V_i(t) + \beta_i V_i(t) < \frac{\beta_i}{\varpi_{\max}^2}\varpi^{\mathrm{T}}(t)\varpi(t) \leq \beta_i \tag{A.5}$$

for $t \in [lT_p + t_{i-1}, lT_p + t_i)$. By integrating (A.5) over $[lT_p + t_{i-1}, t]$, one has

$$\begin{aligned}
V(t) &\leq V\big(lT_p + t_{i-1}\big)e^{-\beta_i(t-(lT_p+t_{i-1}))}\\
&\quad + \int_{lT_p+t_{i-1}}^{t}\beta_i e^{-\beta_i(t-\tau)}d\tau
\end{aligned}$$

$$
\begin{aligned}
&= V\big(lT_p + t_{i-1}\big)e^{-\beta_i\left(t-(lT_p+t_{i-1})\right)}\\
&\quad + \beta_i\left(\frac{1}{\beta_i} - \frac{1}{\beta_i}e^{-\beta_i\left(t-(lT_p+t_{i-1})\right)}\right)\\
&= 1 + \big(V\big(lT_p + t_{i-1}\big) - 1\big)e^{-\beta_i\left(t-(lT_p+t_{i-1})\right)}. \qquad \text{(A.6)}
\end{aligned}
$$

For the augmented system (7) under zero initial conditions, $V(t_0) = V(0) = 0 \leq 1$. Based on (A.6) and $\beta_i > 0$, for $t \in [t_0, t_1)$, one has $V(t) \leq 1 + (V(0) - 1)e^{-\beta_1 t} \leq 1$. Since the continuous state $\xi(t)$ and matrix function $\mathcal{P}(t)$ guarantee the continuity of $V(t)$, one has $V(T_p) \leq 1$, hence

$$V(t) \leq 1 + \big(V\big(T_p\big) - 1\big)e^{-\beta_1 t} \leq 1, \quad t \in \big[T_p, T_p + t_1\big). \qquad \text{(A.7)}$$

By induction, it follows that $V(lT_p + t_{i-1}) \leq 1$. For $t \in [lT_p + t_{i-1}, lT_p + t_i)$, inequality

$$V(t) \leq 1 + \big(V\big(lT_p + t_{i-1}\big) - 1\big)e^{-\beta_i\left(t-(lT_p+t_{i-1})\right)} \leq 1 \qquad \text{(A.8)}$$

holds for all $i \in \mathcal{S}$. Hence, the augmented state $\xi(t)$ will not escape from the bounding region $\mathscr{E}(\mathcal{P}(t))$. According to Schur complement equivalence, condition (18) implies that for $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $j = 1, 2, \ldots, n_u$, one has

$$
\begin{aligned}
u_j^{\mathrm{T}}(t)u_j(t) &= \xi^{\mathrm{T}}(t)\tilde{\mathcal{K}}_{ij}^{\mathrm{T}}(t)\tilde{\mathcal{K}}_{ij}(t)\xi(t)\\
&\leq \xi^{\mathrm{T}}(t)\mathcal{P}_i(t)\xi(t) \leq 1 \qquad \text{(A.9)}
\end{aligned}
$$

which is applicable for all the actuators. Therefore, it can be concluded that for $t \geq 0$, the control input satisfies $\|u(t)\| \leq 1$ such that $\mathrm{SAT}(u(t)) = u(t)$. In other words, an estimate of reachable set $\mathscr{R}_\xi$ for augmented system (7) can be given via $\mathscr{E}(\mathcal{P}(t))$, which prevents $u(t)$ from being saturated for all $t \geq 0$. The proof is complete.

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