A Bernstein Polynomial Approach to Estimating Reachable Set of Periodic Piecewise Polynomial Systems

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***Abstract*—In this paper, a Bernstein polynomial approach is first applied to the estimation of reachable set for a class of periodic piecewise polynomial systems (PPPSs), whose subsystems are time-varying and can be expanded to Bernstein polynomial forms. A lemma on the negativity/positivity for a class of Bernstein polynomial matrix functions is presented, which can provide a feasible set larger than that by the existing method. Based on the integration of the presented lemma and the theory of matrix polynomials, two tractable sufficient conditions are developed. For comparison of conservatism, the reachable set estimation is achieved through optimizing the ellipsoidal bounding region. Four sets of constraints with different conservatism are derived and compared. The effectiveness and superiority of the Bernstein polynomial approach in reachable set estimation are demonstrated via an illustrative example. The results show that the proposed approach enables lower conservatism in reachable set estimation, providing an intuitive route to tackle time-varying parameter products with high powers.**

***Index Terms*—Bernstein polynomial, periodic piecewise polynomial systems, reachable set estimation, time-varying systems.**

I. Introduction

Systems with periodic time-varying dynamics in continuous time domain have aroused extensive research interests, not only because of their wide application in various fields like mechanical vibration, power regulation, ecological balance and economic adjustment [1], but also due to the attractive aim of seeking for more efficient tools compared to conventional methods like Floquet theory, see [2]–[4] and the references therein. In recent years, periodic piecewise systems have received increasing attention, which approximate the system dynamics over each period using piecewise constant subsystems [5], [6], piecewise uncertain linear subsystems [7], [8], or piecewise linear time-varying subsystems [9], [10]. Such approximations not only provide easier access to analyzing complex periodic systems without necessarily closed-form expressions, but also enable the problems involving periodic dynamics more compliant to convex optimization tools such as linear matrix inequalities (LMIs).

This work was supported by the National Natural Science Foundation of China (NSFC) under Grant 61973259, the Innovation and Technology Commission (ITC) of Hong Kong under Grant UIM/353, and the Research Grants Council (RGC) of Hong Kong under Grants (17200918, 17206818, 17202317). (*Corresponding author: James Lam.*)

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In practice, periodic time-varying dynamics are usually nonlinear, which are more desirable to be characterized in polynomial forms. To this end, it is favorable to approximate a periodic system by a number of time-varying polynomial subsystems, that is, a periodic piecewise polynomial system (PPPS). Given the fundamental period divided into several subintervals, each subsystem is described as a matrix polynomial function with a prescribed degree. However, high-order polynomials will lead to time-varying coefficients and parameter-dependent LMI constraints, which generally result in non-convex problems and NP-hardness in optimization [11], [12]. The stability and H_∞ control issues of PPPSs are studied in [13] and [14], where the interval time-varying coefficients are tackled via a useful property on the definiteness of matrix polynomials proposed in [9].

It is worth noting that the polynomial functions in PPPSs can be formulated on the Bernstein polynomial basis, which is a versatile tool for creating parametric Bézier curves [15] and generating polynomial functions by the interpolation of vertex points (also known as "control points") [16], [17]. Bernstein polynomials provide tighter bounds for the range of polynomially parameter-dependent conditions, which enables more relaxation in dealing with structured continuous optimization problems. The arithmetic study on the bounds of univariate interval Bernstein polynomials was started by Rokne [18]. Garloff, Jansson and Smith studied the bounds, constructions and computations of multivariate Bernstein polynomials [19], [20], which were extended to checking robust stability [21]. In [17], Kojima proposed the solution of parameter-dependent LMIs using the Bernstein polynomial basis. In [22], Gao *et al.* used Bernstein polynomials to generate piecewise safe trajectories for quadrotors.

Given an initial state to a dynamic system, the set of all terminal states to which the system can be transferred is referred to as the reachable set, possibly under control constraints [23]. The estimation of reachable set is usually achieved by finding a superset which includes all the states starting from the origin by peak-bounded inputs. Driven by the needs in safety monitoring and verification [24], the reachable set estimation of periodic systems has drawn increasing attention, but mostly in the discrete-time domain [25], [26]. In the authors' previous work [27], a reachable set estimation approach is developed for continuous-time periodic piecewise linear time-varying systems. However, the approach cannot deal with PPPSs involving high-order polynomial subsystems. In this paper, a Bernstein polynomial approach is first applied to the estimation of reachable set for a class of PPPSs. Aimed at tackling PPPSs with lower conservatism than the lemma in [9], the Bernstein approach is developed from a lemma on the negativity/positivity of Bernstein polynomial matrix functions. It is proved that the lemma can provide more relaxed constraints than those based on the existing lemma [9], [14] with identical time-varying coefficients. Thus, two sufficient conditions are proposed for reachable set estimation. Integrating the proposed lemmas with the existing results on matrix polynomials [9], [14], four sets of constraints are established to optimize the bounding region of reachable set characterized by ellipsoid, which can provide an intuitive means for comparing the conservatism in the results. The contributions of this work are as follows:

1) The considered PPPS formulation enables more generality and flexibility compared to those in the previous studies on PPPSs.
2) The proposed Bernstein polynomial approach effectively free the LMI constraints from the terms dependent on time-varying parameters, leading to tractable conditions.
3) The proposed approach provides more relaxed conditions than the existing PPPS methods, achieving lower conservatism in optimizing the bounding region of reachable set.

The paper is organized as follows. In Section II, the reachable set estimation problem for PPPSs is formulated, and some useful preliminaries are presented. In Section III, a Bernstein polynomial approach is proposed, and the related tractable conditions are established. The effectiveness of the proposed approach is validated and discussed in Section IV based on an illustrative example. The conclusions and future work are given in Section V.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of natural numbers (including zero) and the set of positive integers, respectively. For $n \in \mathbb{N}$, $n!$ denotes the factorial of n. I_n denotes an $n \times n$ identity matrix (if the subscript is omitted, the dimension is consistent with the context), and 0 denotes a zero matrix of appropriate dimension. P^T and P^{-1} are the transpose and inverse of matrix P, respectively. $\det(P)$ denotes the determinant of a square matrix P. For real symmetric matrices P and Q, the notation $P \geq Q$ (resp., $P > Q$) means that the matrix $P - Q$ is positive semi-definite (resp., positive definite). Throughout the paper, $\mathbf{sym}(P) = P^T + P$, and $\otimes$ denotes the Kronecker product. $\mathscr{P}_N^{m\times n}[\tau_1, \tau_2]$ denotes the set of $m \times n$ matrix polynomials of degree no greater than N over the interval $[\tau_1, \tau_2]$. $\mathcal{D}^+ g(t)$ denotes the upper right Dini derivative of continuous and Dini-differentiable function $g(t)$.

II. Problem Formulation and Preliminaries

Consider a T_p-periodic time-varying system with peak-bounded disturbances:

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \Omega(t)\omega(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $\omega(t) \in \mathbb{R}^{n_\omega}$ are the state vector and the disturbance vector, respectively; $\mathcal{A}(t) = \mathcal{A}(t + T_p)$, $\Omega(t) = \Omega(t + T_p)$ are continuously periodic time-varying matrix functions for $t \geq 0$. With a known scalar bound $\bar{\omega} > 0$, disturbance vector $\omega(t)$ satisfies

$$\omega^T(t)\omega(t) \leq \bar{\omega}, \quad \forall t \geq 0. \tag{2}$$

Partitioning each time interval of fundamental period T_p into S subintervals denoted as $[lT_p + t_{i-1}, lT_p + t_i)$, $l = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, S$, where $t_0 = 0$, $t_S = T_p$. The dwell time of the i-th subinterval is defined as $T_i \triangleq t_i - t_{i-1}$, $i \in \mathcal{S} \triangleq \{1, 2, \ldots, S\}$, and $\sum_{i=1}^S T_i = T_p$. Periodic system (1) is approximated by the following PPPS:

$$\dot{x}(t) = \mathcal{A}_i(t)x(t) + \Omega_i(t)\omega(t), t \in [lT_p + t_{i-1}, lT_p + t_i), \tag{3}$$

where the matrix functions over the i-th subinterval satisfy:

$$\begin{aligned}\mathcal{A}_i(t) &= A_{i,0} + \sigma_i(t)A_{i,1} + \cdots + \sigma_i^{N_i}(t)A_{i,N_i} \\ &= \sum_{j=0}^{N_i} \sigma_i^j(t)A_{i,j} \in \mathscr{P}_{N_i}^{n_x\times n_x}[lT_p + t_{i-1}, lT_p + t_i), \end{aligned} \tag{4}$$

$$\Omega_i(t) = \sum_{j=0}^{N_i} \sigma_i^j(t)\Omega_{i,j} \in \mathscr{P}_{N_i}^{n_x\times n_\omega}[lT_p + t_{i-1}, lT_p + t_i), \tag{5}$$

with $\sigma_i(t) \triangleq (t - lT_p - t_{i-1})/T_i$ and constant matrices $A_{i,j} \in \mathbb{R}^{n_x\times n_x}$, $\Omega_{i,j} \in \mathbb{R}^{n_x\times n_\omega}$, $i \in \mathcal{S}$, $j \in \mathcal{N}_i \triangleq \{0, 1, \ldots, N_i\}$, $N_i \in \mathbb{N}$. Over the i-th subinterval of the period from lT_p to $(l+1)T_p$, $\mathcal{A}(t) = \mathcal{A}_i(t)$, $\Omega(t) = \Omega_i(t)$, meanwhile $\mathcal{A}_i(t)$ and $\Omega_i(t)$ respectively described by (4) and (5) are right continuous. Moreover, take periodic matrix function $\mathcal{A}(t)$ for example, if

$$A_{1,0} = \sum_{j=0}^{N_S} A_{S,j}, \tag{6}$$

$$A_{i,0} = \sum_{j=0}^{N_{i-1}} A_{i-1,j}, \ i = 2, 3, \ldots, S, \tag{7}$$

then $\mathcal{A}(t)$ will be continuous at all switching instants for $t \in [0, \infty)$ with $\lim_{t\to lT_p+t_i^-} \mathcal{A}_i(t) = \mathcal{A}_{i+1}(lT_p + t_i)$, $i \in \mathcal{S}$. One may choose the continuity of $\mathcal{A}(t)$ and/or $\Omega(t)$ at switching instants based on the requirements of analysis and synthesis in practice. The state of PPPS (3) is continuous for all $t \geq 0$.

Remark 1: Compared with the previous studies on PPPSs [13], [14], the PPPS model given by (3)–(5) enables more generality and flexibility in system description due to the selectable polynomial degrees N_i, $i \in \mathcal{S}$, over different subintervals. When $N_i = 0$, $\forall i \in \mathcal{S}$, PPPS (3) will reduce to a periodic piecewise constant system as discussed in [5]. When $N_i = 1$, $\forall i \in \mathcal{S}$, PPPS (3) will reduce to a periodic piecewise linear time-varying system as discussed in [9], [10], [27], [28]. When $S = 1$, it indicates that each periodic matrix function in periodic system (1) will be approximated by one matrix polynomial over each period.

Concerning the reachability of state $x(t)$, the reachable set of PPPS (3) is defined as

$$\mathcal{R}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x(0) = 0, x(t) \text{ and } \omega(t) \text{ satisfy (3) and (2)}, \forall t \geq 0\}. \tag{8}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.3037041, IEEE Transactions on Automatic Control

To estimate the reachable set of PPPS (3), an intuitive method is sought for a bounding region as small as possible for $\mathcal{R}_x$, which can be described by $\mathscr{R} \triangleq \bigcup_{0\le t\le T_p} \mathcal{E}(\mathcal{P}(t))$, where

$$\mathcal{E}(\mathcal{P}(t)) \triangleq \{x \in \mathbb{R}^{n_x} \mid x^T\mathcal{P}(t)x \le 1, \mathcal{P}(t) > 0\}, \quad (9)$$

with a continuous time-varying matrix function $\mathcal{P}(t)$.

Remark 2: $\mathcal{P}(t)$ provides generality for both periodic piecewise linear and nonlinear time-varying systems. When $\mathcal{P}(t) = P > 0$, $\forall t \ge 0$, $\mathcal{E}(\mathcal{P}(t))$ will reduce to a bounding ellipsoid $E(P)$ determined by a constant matrix $P \in \mathbb{R}^{n_x\times n_x}$:

$$E(P) \triangleq \{x \in \mathbb{R}^{n_x} | x^T Px \le 1, P > 0\}, \quad (10)$$

which is a widely-applied characterization in the reachable set estimation issues of dynamic systems [24], [29], [30].

Lemma 1: (*Stability of periodic piecewise time-varying systems under peak-bounded disturbances* [27]) Consider PPPS (3) with peak-bounded disturbance $\omega(t)$ satisfying (2). Given a quadratic Lyapunov function candidate $V(t) = V_i(t)$, $t \in [lT_p + t_{i-1}, lT_p + t_i)$, $i \in \mathcal{S}$, if there exist scalars $\alpha_i > 0$, $i \in \mathcal{S}$, such that

$$\mathcal{D}^+V_i(t) + \alpha_i V_i(t) - \frac{\alpha_i}{\bar{\omega}}\omega^T(t)\omega(t) \le 0, \quad (11)$$

then the system is asymptotically stable and under zero initial conditions, $V(t) \le 1$.

In [9], [14], a lemma concerning the negativity/positivity property for a class of matrix polynomials is found helpful in analyzing periodic piecewise time-varying systems. For PPPS (3), the lemma is specialized and given below.

Lemma 2: (*Negativity/positivity property for a class of matrix polynomials* [9], [14]) Consider a bounded n-th degree symmetric matrix polynomial function $f : [0, 1] \to \mathbb{R}^{d\times d}$ defined as

$$f(\beta) = \Xi_0 + \beta\Xi_1 + \beta^2\Xi_2 + \cdots + \beta^n\Xi_n, \quad (12)$$

where $\beta \in [0, 1]$ is a scalar, and $\Xi_k \in \mathbb{R}^{d\times d}$ are real symmetric matrices, $k = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $d \in \mathbb{N}^+$. Symmetric matrix polynomial function $f(\beta) < 0$ (resp., > 0) if the following inequalities hold:

$$\sum_{q=0}^{k} \Xi_q < 0 \text{ (resp.,} > 0), \quad k = 0, 1, \ldots, n. \quad (13)$$

Remark 3: A more general version of Lemma 2 for $n \ge 2$ and its proof can be found in [9]. For $n = 0$, $f(\beta) = \Xi_0 < 0$ is obvious. For $n = 1$ with $\beta \in [0, 1]$, when $\Xi_0 < 0$ (resp., > 0) and $\Xi_0 + \Xi_1 < 0$ (resp., > 0) hold, it is clear that

$$f(\beta) = \Xi_0 + \beta\Xi_1 = (1-\beta)\Xi_0 + \beta(\Xi_0 + \Xi_1) < 0 \text{ (resp.,} > 0).$$

Thus, Lemma 2 is applicable for all $n \in \mathbb{N}$.

III. Main Results

A. Bernstein Polynomial Approach

Note that (9) involves time-varying function $\mathcal{P}(t)$ that may be difficult to tackle over each period. To solve the problem in an efficient way, one may use the Bernstein polynomial basis that helps provide more freedom in solutions.

Consider a scalar $\beta \in [0, 1]$ and constant matrices $\Gamma_k \in \mathbb{R}^{d\times d}$, $k = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $d \in \mathbb{N}^+$. Any n-th degree matrix polynomial in the form of

$$p(\beta) = \sum_{k=0}^{n} \beta^k\Gamma_k, \quad (14)$$

can be expanded to a Bernstein polynomial:

$$p(\beta) = \sum_{k=0}^{n} B_k(\beta)\Lambda_k. \quad (15)$$

The Bernstein polynomial basis can be characterized by

$$B_k(\beta) = \binom{n}{k}\beta^k(1-\beta)^{n-k} \quad (16)$$

with binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and Bernstein coefficient matrices Λ_k, $k = 0, 1, \ldots, n$, can be obtained by matrices $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$:

$$\Lambda_k = \sum_{q=0}^{k} \frac{\binom{k}{q}}{\binom{n}{q}}\Gamma_q, \; k = 0, 1, \ldots, n. \quad (17)$$

Using real symmetric matrices Ξ_k to replace Γ_k, $k = 0, 1, \ldots, n$, one can specialize $p(\beta)$ to the $f(\beta)$ in (12). Based on the previous studies on Bernstein polynomials and their applications [17], [18], [20], one can derive the following lemma for the negativity/positivity of matrix polynomial function $f(\beta)$.

Lemma 3: (*Negativity/positivity property on the Bernstein polynomial basis*) Consider an n-th degree symmetric matrix polynomial function $f : [0, 1] \to \mathbb{R}^{d\times d}$ defined in (12) with scalar $\beta \in [0, 1]$ and real symmetric matrices $\Xi_k \in \mathbb{R}^{d\times d}$, $k = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $d \in \mathbb{N}^+$. Symmetric matrix polynomial function $f(\beta) < 0$ (resp., > 0) if the following inequalities hold:

$$\sum_{q=0}^{k} \frac{\binom{k}{q}}{\binom{n}{q}}\Xi_q < 0 \text{ (resp.,} > 0), \; k = 0, 1, \ldots, n. \quad (18)$$

Proof: For $\beta = 0$ or $\beta = 1$, it is obvious that $f(0) = \Xi_0 < 0$ (resp., > 0) or $f(1) = \sum_{k=0}^{n} \Xi_k < 0$ (resp., > 0) based on (18). For $\beta \in (0, 1)$, from (16) with $\beta > 0$ and $1 - \beta > 0$, one has $B_k(\beta) > 0$, $k = 0, 1, \ldots, n$, and thus $f(\beta) < 0$ (resp., > 0) when the inequalities in (18) hold. Hence, for $\beta \in [0, 1]$, matrix polynomial $f(\beta) < 0$ (resp., > 0) if the matrix inequalities in (18) hold. The result also goes for the cases when β is time-varying but bounded in $[0, 1]$, which completes the proof. □

Take $f(\beta) < 0$ for example, one notices that the negativity of polynomial matrix function $f(\beta)$ can be tackled either by Lemma 2 or by Lemma 3. Using real symmetric matrices $\Xi_0, \Xi_1, \ldots, \Xi_n$ as decision variables, define the feasible sets based on Lemma 2 and Lemma 3 as follows:

$$\mathscr{S}_O \triangleq \left\{(\Xi_0, \Xi_1, \ldots, \Xi_n) \,\middle|\, \sum_{q=0}^{k} \Xi_q < 0, \; k = 0, 1, \ldots, n\right\}, \quad (19)$$

$$\mathscr{S}_B \triangleq \left\{(\Xi_0, \Xi_1, \ldots, \Xi_n) \,\middle|\, \sum_{q=0}^{k} \frac{\binom{k}{q}}{\binom{n}{q}}\Xi_q < 0, \; k = 0, 1, \ldots, n\right\}. \quad (20)$$

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For $n = 0, 1$, one has $\mathscr{S}_O = \mathscr{S}_B$, namely, the constraints obtained by the two lemmas are the same. For $n \geq 2$, a comparison of the constraints obtained by the two lemmas is given in **Table I** with $n = 2, 3, 4, 5$. Compared to Lemma 2, it can be seen that for $n \geq 2$, Lemma 3 provides smaller coefficients associated with $\Xi_q,\ q = 1, 2, \ldots, k$, in the $(k+1)$-th constraint, $k = 1, 2, \ldots, n-1$. To show the advantage of Lemma 3 over the existing Lemma 2, a theorem on the feasible sets of the two lemmas is proposed and proved as follows.

Theorem 1: Consider feasible sets $\mathscr{S}_O$ and $\mathscr{S}_B$ defined in (19) and (20), respectively. For any $n \geq 2$, $\mathscr{S}_O \subset \mathscr{S}_B$.

Proof: The theorem can be proved from two aspects:

(i) $\forall(\Xi_0, \Xi_1, \ldots, \Xi_n) \in \mathscr{S}_O \Rightarrow (\Xi_0, \Xi_1, \ldots, \Xi_n) \in \mathscr{S}_B$: First, for $n \geq 2$, consider the following set of constraints established by $\Xi_k \in \mathbb{R}^{d\times d}$, $k = 0, 1, \ldots, n$:

$$(Y \otimes I_d)\zeta \leq 0, \tag{21}$$

where $\zeta = \begin{bmatrix}\Xi_0^T & \Xi_1^T & \cdots & \Xi_n^T\end{bmatrix}^T \in \mathbb{R}^{(n+1)d\times d}$, and Y is a lower triangular matrix whose nonzero elements are 1:

$$Y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{22}$$

When (21) holds, the constraints of $\mathscr{S}_O$ in (19) hold, and ζ contains all the solutions of $(\Xi_0, \Xi_1, \ldots, \Xi_n) \in \mathscr{S}_O$. Consider slack matrix variables $G_k \in \mathbb{R}^{d\times d},\ k = 0, 1, \ldots, n,\ G_k > 0$, such that

$$(Y \otimes I_d)\zeta + \vartheta = 0, \tag{23}$$

where $\vartheta = \begin{bmatrix}G_0^T & G_1^T & \cdots & G_n^T\end{bmatrix}^T \in \mathbb{R}^{(n+1)d\times d}$. Since Y is invertible, from (23) one has

$$\zeta = -(Y \otimes I_d)^{-1}\vartheta = -(Y^{-1} \otimes I_d)\vartheta, \tag{24}$$

where

$$Y^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{25}$$

For $n \geq 2$, one can use a lower triangular matrix Z to help parameterize the feasible set $\mathscr{S}_B$ in (20):

$$Z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & z_{2,2} & 0 & \cdots & 0 & 0 \\ 1 & z_{3,2} & z_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_{n,2} & z_{n,3} & \cdots & z_{n,n} & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \tag{26}$$

where the parameter elements are computed by

$$z_{k+1,q+1} \triangleq \frac{\binom{k}{q}}{\binom{n}{q}} \in (0, 1),\ k = 1, 2, \ldots, n-1,\ q \leq k, \tag{27}$$

and here one has

$$z_{k+1,q+1} = \frac{\binom{k}{q}}{\binom{n}{q}} = \frac{k(k-1)\cdots(k-q+2)(k-q+1)}{n(n-1)\cdots(n-q+2)(n-q+1)}$$

$$< z_{k+1,q} = \frac{k(k-1)\cdots(k-q+2)}{n(n-1)\cdots(n-q+2)}, \tag{28}$$

indicating that the positive elements in the $(k+1)$-th row of matrix Z are strictly decreasing from left to right for $k = 1, 2, \ldots, n-1$. It follows that $\det(Z) = \prod_{k=2}^{n} z_{k,k} \in (0, 1)$ and matrix Z is invertible. From (24), the feasible set $\mathscr{S}_B$ is parameterized as

$$(Z\otimes I_d)\zeta = -(Z\otimes I_d)(Y^{-1}\otimes I_d)\vartheta = (-ZY^{-1}\otimes I_d)\vartheta. \tag{29}$$

With (25), (26) and (28), one can find that all the nonzero elements of $-ZY^{-1}$ are negative. From (29), for any ϑ and $n \geq 2$, the negative definiteness can always be guaranteed for

$$(\delta_{k+1}Z \otimes I_d)\zeta < 0,\ k = 0, 1, \ldots, n, \tag{30}$$

where $\delta_{k+1} \in \mathbb{R}^{1\times(n+1)}$ is obtained by

$$\delta_{k+1} = [\ \underbrace{0 \cdots 0}_{k}\ 1\ \underbrace{0 \cdots 0}_{n-k}\ ],\ k = 0, 1, \ldots, n. \tag{31}$$

Inequality (30) implies that all the constraints of $\mathscr{S}_B$ hold. Thus, for $n \geq 2$, any solution of $(\Xi_0, \Xi_1, \ldots, \Xi_n) \in \mathscr{S}_O$ can be contained in feasible set $\mathscr{S}_B$.

(ii) $\exists(\Xi_0, \Xi_1, \ldots, \Xi_n) \in \mathscr{S}_B \Rightarrow (\Xi_0, \Xi_1, \ldots, \Xi_n) \notin \mathscr{S}_O$: Conversely, consider $(Z \otimes I_d)\zeta \leq 0$, and the constraints of $\mathscr{S}_B$ in (20) thus hold. The feasible set $\mathscr{S}_O$ in (19) can be similarly parameterized as

$$(Y\otimes I_d)\zeta = -(Y\otimes I_d)(Z^{-1}\otimes I_d)\vartheta = (-YZ^{-1}\otimes I_d)\vartheta. \tag{32}$$

Let $C_{i,j}^Z$ denote the cofactor of the (i,j) entry of matrix Z, $i, j = 1, 2, \ldots, n+1$. One has $Z^{-1} \in \mathbb{R}^{(n+1)\times(n+1)}$ satisfying

$$Z^{-1} = \frac{1}{\det(Z)} \begin{bmatrix} C_{1,1}^Z & C_{2,1}^Z & \cdots & C_{n+1,1}^Z \\ C_{1,2}^Z & C_{2,2}^Z & \cdots & C_{n+1,2}^Z \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n+1}^Z & C_{2,n+1}^Z & \cdots & C_{n+1,n+1}^Z \end{bmatrix}. \tag{33}$$

Take the second row of $-YZ^{-1}$ for example. From (22), (27) and (33), the row vector can be obtained by

$$-\frac{1}{\det(Z)}\left[\ C_{1,1}^Z + C_{1,2}^Z\ \ C_{2,1}^Z + C_{2,2}^Z\ \ 0\ \cdots\ 0\ \right]$$

$$= \left[\ -1 + z_{2,2}^{-1}\ \ -z_{2,2}^{-1}\ \ 0\ \cdots\ 0\ \right], \tag{34}$$

where $-1 + z_{2,2}^{-1} > 0$, $-z_{2,2}^{-1} < 0$ since $z_{2,2}^{-1} > 1$. Based on (31) with $k = 1$, one has $(\delta_2 Y \otimes I_d)\zeta > 0$ for some ϑ with $(1 - z_{2,2})G_1 > G_2 > 0$. Thus, the constraints of $\mathscr{S}_O$ do not necessarily hold for any $n \geq 2$. Combining (i) and (ii), for any $n \geq 2$ one has $\mathscr{S}_O \subset \mathscr{S}_B$. The proof is complete. □

Feasible sets $\mathscr{S}_O$ and $\mathscr{S}_B$ correspond to the solution spaces of (13) in Lemma 2 and (18) in Lemma 3, respectively. Similarly, one can find that Theorem 1 also holds when $f(\beta) > 0$ is considered. It implies that the feasible set based on Lemma 3 is strictly larger than that based on Lemma 2. Therefore, compared to Lemma 2, the proposed Lemma 3 can provide more freedom in the solutions of matrices $\Xi_0, \Xi_1, \ldots, \Xi_n$ due to its more relaxed constraint set.

Remark 4: Lemma 3 is focused on the univariate case of $\beta \in [0, 1]$. In this paper, the use of Lemma 3 is aimed at reducing the conservatism in reachable set estimation particularly for PPPS (3). For more details about the multivariate case in LMI-based optimization, the readers are referred to [17].

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TABLE I: Comparison of Inequality Constraints Obtained by Lemma 2 and Lemma 3 ($n=2,3,4,5$)

Theory	$n=2$	$n=3$	$n=4$	$n=5$
Lemma 2 [9]	$\Xi_0<0$ $\Xi_0+\Xi_1<0$ $\Xi_0+\Xi_1+\Xi_2<0$	$\Xi_0<0$ $\Xi_0+\Xi_1<0$ $\Xi_0+\Xi_1+\Xi_2<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3<0$	$\Xi_0<0$ $\Xi_0+\Xi_1<0$ $\Xi_0+\Xi_1+\Xi_2<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3+\Xi_4<0$	$\Xi_0<0$ $\Xi_0+\Xi_1<0$ $\Xi_0+\Xi_1+\Xi_2<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3+\Xi_4<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3+\Xi_4+\Xi_5<0$
Lemma 3	$\Xi_0<0$ $\Xi_0+\frac{1}{2}\Xi_1<0$ $\Xi_0+\Xi_1+\Xi_2<0$	$\Xi_0<0$ $\Xi_0+\frac{1}{3}\Xi_1<0$ $\Xi_0+\frac{2}{3}\Xi_1+\frac{1}{3}\Xi_2<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3<0$	$\Xi_0<0$ $\Xi_0+\frac{1}{4}\Xi_1<0$ $\Xi_0+\frac{1}{2}\Xi_1+\frac{1}{6}\Xi_2<0$ $\Xi_0+\frac{3}{4}\Xi_1+\frac{1}{2}\Xi_2+\frac{1}{4}\Xi_3<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3+\Xi_4<0$	$\Xi_0<0$ $\Xi_0+\frac{1}{5}\Xi_1<0$ $\Xi_0+\frac{2}{5}\Xi_1+\frac{1}{10}\Xi_2<0$ $\Xi_0+\frac{3}{5}\Xi_1+\frac{3}{10}\Xi_2+\frac{1}{10}\Xi_3<0$ $\Xi_0+\frac{4}{5}\Xi_1+\frac{3}{5}\Xi_2+\frac{2}{5}\Xi_3+\frac{1}{5}\Xi_4<0$ $\Xi_0+\Xi_1+\Xi_2+\Xi_3+\Xi_4+\Xi_5<0$

B. Reachable Set Estimation

To analyze the stability and reachability of PPPS (3), one constructs a continuous Lyapunov function with matrix polynomial functions:

$$V(t)=x^T(t)\mathcal{P}(t)x(t), \tag{35}$$

where $\mathcal{P}(t)=\mathcal{P}(t+T_p)>0$, and for $t\in[lT_p+t_{i-1},lT_p+t_i)$, $i\in\mathcal{S}$,

$$V(t)=V_i(t)=x^T(t)\mathcal{P}_i(t)x(t), \tag{36}$$

$$\mathcal{P}(t)=\mathcal{P}_i(t)=\sum_{m=0}^{M_i}\sigma_i^m(t)P_{i,m} \in \mathscr{P}_{M_i}^{n_x\times n_x}[lT_p+t_{i-1},lT_p+t_i), \tag{37}$$

with real symmetric matrices $P_{i,m}\in\mathbb{R}^{n_x\times n_x}$, $i\in\mathcal{S}$, $m\in\mathcal{M}_i\triangleq\{0,1,\ldots,M_i\}$, $M_i\in\mathbb{N}$, and

$$P_{1,0}=\sum_{m=0}^{M_i}P_{S,m}, \tag{38}$$

$$P_{i,0}=\sum_{m=0}^{M_{i-1}}P_{i-1,m},\ i=2,3,\ldots,S, \tag{39}$$

$$\sum_{i=1}^{S}\sum_{m=1}^{M_i}P_{i,m}=0. \tag{40}$$

According to [14], (38)–(40) guarantee the continuity of $\mathcal{P}(t)$, $\forall t\geq 0$. Hence, $\mathcal{P}(t)$ is a continuous and Dini-differentiable T_p-periodic piecewise matrix polynomial function for $t\geq 0$. The variation of $\mathcal{P}(t)$ over the i-th subinterval of each period is described by an M_i-th degree matrix polynomial established by $P_{i,m}$, $i\in\mathcal{S}$, $m\in\mathcal{M}_i$.

To guarantee $\mathcal{P}(t)>0$ while minimizing the bounding region of $\mathcal{R}_x$, one may use either a scalar upper bound $\varepsilon>0$ or a real symmetric matrix $\hat{P}$ to characterize the region. Consider the following inequality:

$$\begin{bmatrix}\hat{P} & I\\ I & \epsilon I\end{bmatrix}\geq 0, \tag{41}$$

where $\epsilon=\varepsilon^{-1}>0$, and $\hat{P}>0$. Based on Lemma 3, a set of constraints is proposed as follows:

$$\sum_{m=0}^{r}\frac{\binom{r}{m}}{\binom{M_i}{m}}P_{i,m}-\hat{P}\geq 0,\ r=0,1,\ldots,M_i,\ i\in\mathcal{S}. \tag{42}$$

Alternatively, according to the existing result in Lemma 2, another set of constraints is proposed as follows:

$$\sum_{m=0}^{r}P_{i,m}-\hat{P}\geq 0,\ r=0,1,\ldots,M_i,\ i\in\mathcal{S}. \tag{43}$$

Apply Schur complement equivalence to (41) and combine the resulting inequality with (42) or (43). Either set of constraints (41) and (42), or (41) and (43), enables that for all $i\in\mathcal{S}$,

$$\mathcal{P}(t)=\mathcal{P}_i(t)\geq\hat{P}\geq\varepsilon I>0, \tag{44}$$

which ensures that for all $t\geq 0$,

$$\varepsilon x^T(t)x(t)\leq x^T(t)\hat{P}x(t)\leq x^T(t)\mathcal{P}(t)x(t)\leq 1, \tag{45}$$

satisfying (9). The minimization of bounding region for $\mathcal{R}_x$ can thus be transformed to an optimization problem of minimizing ϵ, which controls the upper-bound radius of a "bounding ball" for the system state. To shrink the shape of bounding region as an ellipsoid $E(\hat{P})$ rather than a ball defined by $\varepsilon x^T(t)x(t)\leq 1$, one may choose to minimize $-\ln(\det(\hat{P}))$ that reduces the sum of the eigenvalues of $\hat{P}$.

C. Tractable conditions for optimization

Using Lemma 3, one obtains the following theorem in terms of tractable LMIs, concerning the stability and bounding region of reachable set for PPPS (3).

Theorem 2: Consider PPPS (3) with $N_i\geq 1$, $M_i\geq 1$, $i\in\mathcal{S}$, and peak-bounded disturbance $\omega(t)$ satisfying (2). Given scalars $\alpha_i>0$, $i\in\mathcal{S}$, the system is asymptotically stable with reachable set $\mathcal{R}_x$ bounded by $\mathscr{R}$ satisfying (9), if there exist scalar $\epsilon>0$, matrix $\hat{P}>0$, and symmetric matrices $P_{i,m}$, $i\in\mathcal{S}$, $m\in\mathcal{M}_i$, such that conditions (38)–(41), condition (42) or (43), and the following inequalities hold:

$$\sum_{q=0}^{k}\frac{\binom{k}{q}}{\binom{M_i+N_i}{q}}\Theta_{i,q}<0,\ k=0,1,\ldots,M_i+N_i, \tag{46}$$

where

$$\Theta_{i,k}=\begin{bmatrix}\Delta_{i,k} & \Upsilon_{i,k}\\ \Upsilon_{i,k}^T & \Phi_{i,k}\end{bmatrix}, \tag{47}$$

and

$$\Delta_{i,0}=\alpha_i P_{i,0}+\frac{1}{T_i}P_{i,1}+\mathbf{sym}(P_{i,0}A_{i,0}), \tag{48}$$

$$\Delta_{i,k}=\alpha_i P_{i,k}+\frac{k+1}{T_i}P_{i,k+1}+\sum_{\substack{j+m=k\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}\mathbf{sym}\left(P_{i,m}A_{i,j}\right),$$
$$k=1,2,\ldots,M_i-1, \tag{49}$$

$$\Delta_{i,M_i}=\alpha_i P_{i,M_i}+\sum_{\substack{j+m=M_i\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}\mathbf{sym}\left(P_{i,m}A_{i,j}\right), \tag{50}$$

$$\Delta_{i,k}=\sum_{\substack{j+m=k\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}\mathbf{sym}\left(P_{i,m}A_{i,j}\right),\ k=M_i+1,\ldots,M_i+N_i; \tag{51}$$

$$\Upsilon_{i,0}=P_{i,0}\Omega_{i,0}, \tag{52}$$

$$\Upsilon_{i,k}=\sum_{\substack{j+m=k\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}P_{i,m}\Omega_{i,j},\ k=1,2,\ldots,M_i+N_i; \tag{53}$$

$$\Phi_{i,0}=-\frac{\alpha_i}{\bar{\omega}}I, \tag{54}$$

$$\Phi_{i,k}=0,\ k=1,2,\ldots,M_i+N_i. \tag{55}$$

Proof: For $t\in[lT_p+t_{i-1},lT_p+t_i)$, $i\in\mathcal{S}$, it follows that

$$\begin{aligned}
&\mathcal{D}^+V_i(t)+\alpha_iV_i(t)-\frac{\alpha_i}{\bar{\omega}}\omega^T(t)\omega(t)\\
&=x^T(t)\Bigg\{\mathbf{sym}\Bigg(\Big(\sum_{m=0}^{M_i}\sigma_i^m(t)P_{i,m}\Big)\Big(\sum_{j=0}^{N_i}\sigma_i^j(t)A_{i,j}\Big)\Bigg)\\
&\quad+\sum_{m=1}^{M_i}\frac{m}{T_i}\sigma_i^{m-1}(t)P_{i,m}+\alpha_i\Big(\sum_{m=0}^{M_i}\sigma_i^m(t)P_{i,m}\Big)\Bigg\}x(t)\\
&\quad+\omega^T(t)\Big(\sum_{j=0}^{N_i}\sigma_i^j(t)\Omega_{i,j}^T\Big)\Big(\sum_{m=0}^{M_i}\sigma_i^m(t)P_{i,m}\Big)x(t)\\
&\quad+x^T(t)\Big(\sum_{m=0}^{M_i}\sigma_i^m(t)P_{i,m}\Big)\Big(\sum_{j=0}^{N_i}\sigma_i^j(t)\Omega_{i,j}\Big)\omega(t)-\frac{\alpha_i}{\bar{\omega}}\omega^T(t)\omega(t)\\
&=x^T(t)\Bigg\{\sum_{k=0}^{N_i+M_i}\sigma_i^k(t)\sum_{\substack{j+m=k\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}\mathbf{sym}\left(P_{i,m}A_{i,j}\right)\\
&\quad+\sum_{k=0}^{M_i-1}\frac{k+1}{T_i}\sigma_i^k(t)P_{i,k+1}+\alpha_i\Big(\sum_{k=0}^{M_i}\sigma_i^k(t)P_{i,k}\Big)\Bigg\}x(t)\\
&\quad+\omega^T(t)\left(\sum_{k=0}^{N_i+M_i}\sigma_i^k(t)\sum_{\substack{j+m=k\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}\Omega_{i,j}^TP_{i,m}\right)x(t)\\
&\quad+x^T(t)\left(\sum_{k=0}^{N_i+M_i}\sigma_i^k(t)\sum_{\substack{j+m=k\\ j\in\mathcal{N}_i,m\in\mathcal{M}_i}}P_{i,m}\Omega_{i,j}\right)\omega(t)\\
&\quad-\frac{\alpha_i}{\bar{\omega}}\omega^T(t)\omega(t)\\
&=\xi^T(t)\left(\Theta_{i,0}+\sigma_i(t)\Theta_{i,1}+\cdots+\sigma_i^{N_i+M_i}(t)\Theta_{i,N_i+M_i}\right)\xi(t)\\
&=\xi^T(t)p_i(\sigma_i(t))\xi(t),
\end{aligned} \tag{56}$$

where $\xi(t)=\left[x^T(t),\omega^T(t)\right]^T$, and matrix polynomial

$$p_i(\sigma_i(t))=\sum_{k=0}^{M_i+N_i}\sigma_i^k(t)\Theta_{i,k} \tag{57}$$

with matrices $\Theta_{i,k}$, $i\in\mathcal{S}$, $k=0,1,\ldots,M_i+N_i$, satisfying (47)–(55), and $\sigma_i(t)\in[0,1)$, $i\in\mathcal{S}$. Over the i-th subinterval, $p_i(\sigma_i(t))$ in (57) can be expanded to a Bernstein matrix polynomial form in $\sigma_i(t)$:

$$p_i(\sigma_i(t))=\sum_{k=0}^{M_i+N_i}B_{i,k}(\sigma_i(t))\Lambda_{i,k}, \tag{58}$$

where

$$B_{i,k}(\sigma_i(t))=\binom{M_i+N_i}{k}\sigma_i^k(t)(1-\sigma_i(t))^{M_i+N_i-k}, \tag{59}$$

and the corresponding Bernstein coefficient matrices $\Lambda_{i,k}$, $i=1,2,\ldots,S$, $k=0,1,\ldots,M_i+N_i$, are obtained by matrices $\Theta_{i,0},\Theta_{i,1},\ldots,\Theta_{i,k}$:

$$\Lambda_{i,k}=\sum_{q=0}^{k}\frac{\binom{k}{q}}{\binom{M_i+N_i}{q}}\Theta_{i,q},\ k=0,1,\ldots,M_i+N_i. \tag{60}$$

When conditions (38)–(40), condition (42) or (43) and the inequalities in (46) hold, one has

$$p_i(\sigma_i(t))<0,\ i\in\mathcal{S}, \tag{61}$$

which implies that $\mathcal{D}^+V_i(t)+\alpha_iV_i(t)-\frac{\alpha_i}{\bar{\omega}}\omega^T(t)\omega(t)<0$, $i\in\mathcal{S}$. By Lemma 3, system (3) is asymptotically stable and under zero initial conditions, $V(t)=x^T(t)\mathcal{P}(t)x(t)\leq 1$. By (41) and either (42) or (43) one can guarantee that reachable set $\mathcal{R}_x$ is bounded by an ellipsoid determined by $\hat{P}$. The proof is complete. □

Based on Lemma 2, one obtains an alternative theorem using different constraints.

Theorem 3: Consider PPPS (3) with peak-bounded disturbance $\omega(t)$ satisfying (2). Given scalars $\alpha_i>0$, $i\in\mathcal{S}$, the system is asymptotically stable with reachable set $\mathcal{R}_x$ bounded by $\mathscr{R}$ satisfying (9), if there exist exist scalar $\epsilon>0$, matrix $\hat{P}>0$, and matrices $P_{i,m}$, $i\in\mathcal{S}$, $m\in\mathcal{M}_i$, such that conditions (38)–(40), condition (42) or (43), and the following inequalities hold:

$$\sum_{q=0}^{k}\Theta_{i,q}<0,\ k=0,1,\ldots,M_i+N_i, \tag{62}$$

where $\Theta_{i,k}<0$, $k=0,1,\ldots,M_i+N_i$, are described by (47)–(55).

Theorem 3 can be proved by following similar procedures in the proof of Theorem 2, with (46) replaced by (62) to guarantee the negativity of $p_i(\sigma_i(t))$, $i\in\mathcal{S}$.

Combine Theorem 2 and Theorem 3 with $\varepsilon=\epsilon^{-1}$. Given a scalar $\mu\in\{0,1\}$, the optimization problem of reachable set estimation can be solved subjected to four sets of constraints as follows:

$$\text{Minimize}\quad(1-\mu)\epsilon-\mu\ln(\det(\hat{P}))\quad\text{subject to}$$

$$\begin{cases} \text{Case 1: (38)–(40), (42), (46)} \\ \text{Case 2: (38)–(40), (43), (46)} \\ \text{Case 3: (38)–(40), (42), (62)} \\ \text{Case 4: (38)–(40), (43), (62)} \end{cases}$$

Therefore, the bounding region of $\mathcal{R}_x$ can be minimized and visualized by a ball $\varepsilon x^T(t)x(t) \leq 1$ established by $\varepsilon = \epsilon^{-1}$, or by an ellipsoid $x^T(t)\hat{P}x(t) \leq 1$ established by $\hat{P}$. The constraints in Cases 1–4 of the proposed optimization problem are with different conservatism in theory. Another factor that may affect the conservatism is that one can choose either $M_i \leq N_i$ or $M_i > N_i$ for the i-th subsystem of PPPS (3), which will be further discussed in the next section.

Remark 5: The above objective function of optimization covers two commonly used objectives, namely minimizing ϵ ($\mu = 0$) or minimizing $-\ln(\det(\hat{P}))$ ($\mu = 1$). In most studies on reachable set estimation, ϵ is used as a convenient index for comparing the conservatism in results [27], while $-\ln(\det(\hat{P}))$ is considered if a tighter estimation of bounding region is desirable [26], [29].

Remark 6: The proposed approach is applicable to estimating reachable sets for dynamic systems that can be represented in periodic piecewise forms, such as mechanical vibration systems [10] and power converter systems [31], which may include coupled interval time-varying parameters with high powers. Estimated results can further facilitate the monitoring, control and verification of practical systems [24].

IV. Illustrative Example

In this section, a numerical example is used to validate the proposed approach. Consider a PPPS involving 3 subsystems with $N_i = 2$, $i = 1, 2, 3$, fundamental period $T_p = 2.5$, and values of subsystem dwell time as $T_1 = 0.8$, $T_2 = 0.7$, $T_3 = 1$ in appropriate time unit. As described in (3)–(5), the relevant matrices are given as follows:

$$\begin{aligned}
&A_{1,0} = \begin{bmatrix} -3 & 1 \\ -1 & -2 \end{bmatrix},\ A_{1,1} = \begin{bmatrix} -1 & 0.2 \\ 0 & -2 \end{bmatrix},\ A_{1,2} = \begin{bmatrix} -1.5 & 0 \\ 0.1 & -1 \end{bmatrix}, \\
&A_{2,0} = \begin{bmatrix} -5 & 1 \\ -2 & -4 \end{bmatrix},\ A_{2,1} = \begin{bmatrix} -1 & 0.2 \\ 0 & -0.5 \end{bmatrix},\ A_{2,2} = \begin{bmatrix} -2 & 0.1 \\ 0.3 & -2 \end{bmatrix}, \\
&A_{3,0} = \begin{bmatrix} -4 & 1 \\ -1 & -3 \end{bmatrix},\ A_{3,1} = \begin{bmatrix} -4 & 0 \\ 0.2 & -1 \end{bmatrix},\ A_{3,2} = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix}, \\
&\Omega_{1,0} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \Omega_{1,1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \Omega_{1,2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \\
&\Omega_{2,0} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ \Omega_{2,1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \Omega_{2,2} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \\
&\Omega_{3,0} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},\ \Omega_{3,1} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},\ \Omega_{3,2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned} \tag{63}$$

Given $\alpha_1 = 1.5$, $\alpha_2 = 1.6$, $\alpha_3 = 1.8$ and disturbance signal $\omega(t) = 1.5\sin(3t)$ with $\bar{\omega} = 2.25$, one considers $M_i = 1, 2, 3$, $i \in \mathcal{S}$, for reachable set estimation, respectively. The simulation results are achieved using the MATLAB solver SeDuMi with the YALMIP surface [32].

A. Comparisons of upper bound ε

From Theorem 1, it is known that (42) and (46) can help enlarge the feasible sets compared to (43) and (62), respectively.

TABLE II: Values of upper bound ε ($\mu = 0$, $i = 1, 2, 3$)

Constraints	$M_i = 1 < N_i$	$M_i = 2 = N_i$	$M_i = 3 > N_i$
Case 1	0.3225	0.3407	0.3417
Case 2	0.3225	0.3405	0.3409
Case 3	0.2661	0.2786	0.2850
Case 4	0.2661	0.2696	0.2701

Consider the upper bound ε obtained by minimizing ϵ with $\mu = 0$, the results of four cases are listed in **Table II**. From the table, it can be observed that Case 1 using (42) and (46) which are both based on Lemma 3 achieves the largest values of ε, implying the least conservatism among all the cases. The conservatism in the results of ε is increasing from Case 1 to Case 4, which is consistent to the analysis in Section III.A.

Moreover, the conservatism decreases from $M_i = 1$ to $M_i = 3$ for all the cases. It indicates that a higher degree of polynomial function $\mathcal{P}_i(t)$ can help reduce the conservatism in bounding region optimization when $M_i \leq N_i$, while the effect may be less obvious when $M_i > N_i$, $i \in \mathcal{S}$.

A special case in **Table II** is $M_i = 1$, where the values of ε for Case 1 and Case 2 (also, Case 3 and Case 4) are the same, since conditions (42) and (43) share the same formulation at this time. The values of ε for Case 1 and Case 2, which are larger than those for Case 3 and Case 4, further demonstrate superiority of the proposed Bernstein polynomial-based reachable set estimation approach for PPPSs.

B. Comparisons of bounding ellipsoids

For $\mu = 0$ and $\mu = 1$, the bounding ellipsoids of reachable set $\mathcal{R}_x$ obtained by $M_i = 1, 2, 3$, $i \in \mathcal{S}$, are compared under different cases. The comparative results of Case 1 and Case 3 are shown in **Fig. 1**, while the results of Case 2 and Case 4 are shown in **Fig. 2**. It can be observed that the bounding ellipsoids for Case 1 (resp. Case 2) are smaller than those for Case 3 (resp. Case 4), showing the lower conservatism in reachable set estimation achieved by condition (46) based on Lemma 3, compared to condition (62) based on Lemma 2. The lower conservatism is due to the larger feasible set provided by Lemma 3, as discussed in Theorem 1.

Therefore, the proposed criteria using the Bernstein polynomial approach based on Lemma 3 can contribute to less conservative results in the optimization of reachable set bounding regions for PPPSs. One may balance the conservatism in reachable set estimation by choosing the desirable constraints based on Lemmas 2–3.

V. Conclusions and Future Work

This paper first uses a Bernstein polynomial approach to deal with the reachable set estimation problem for a class of periodic piecewise polynomial systems (PPPSs). Utilizing the properties of Bernstein polynomials, a useful lemma is presented (Lemma 3), and its advantage over the existing method is proved (Theorem 1). Two tractable sufficient conditions are provided in terms of LMIs (Theorem 2 and Theorem 3). By integrating the proposed lemmas and the existing theory on matrix polynomials, the optimization of bounding region for reachable set can be solved subject to four sets of constraints.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2020.3037041, IEEE Transactions on Automatic Control

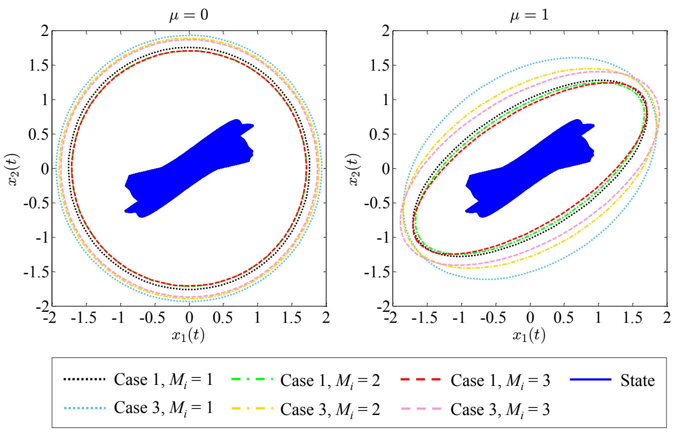

Fig. 1: Bounding ellipsoids of reachable sets for Cases 1, 3

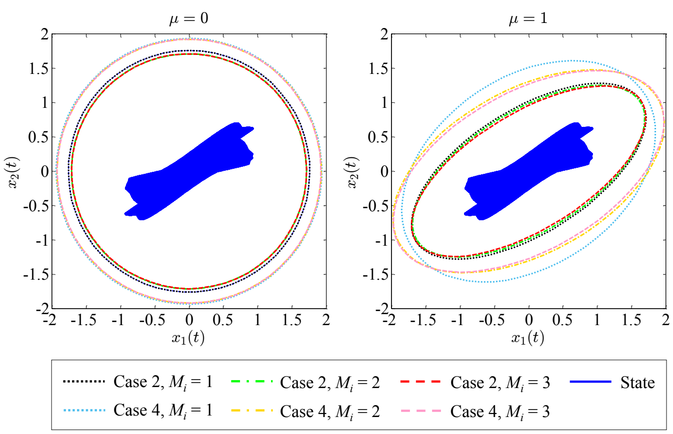

Fig. 2: Bounding ellipsoids of reachable sets for Cases 2, 4

The effectiveness of the proposed approach has been validated and visualized using an illustrative example. Based on the obtained comparative results, it can be concluded that the constraints obtained based on Lemma 3 provide lower conservatism in reachable set estimation, showing the superiority of Bernstein polynomials in interval polynomial systems. In the future work, the proposed approach will be further extended to the relevant performance analysis, synthesis and their practical applications.

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