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Consensus of Positive Networked Systems on Directed Graphs

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***Abstract*—This article addresses the distributed consensus problem for identical continuous-time positive linear systems with state-feedback control. Existing works of such a problem mainly focus on the case where the networked communication topologies are of either undirected and incomplete graphs or strongly connected directed graphs. On the other hand, in this work, the communication topologies of the networked system are described by directed graphs each containing a spanning tree, which is a more general and new scenario due to the interplay between the eigenvalues of the Laplacian matrix and the controller gains. Specifically, the problem involves complex eigenvalues, the Hurwitzness of complex matrices, and positivity constraints, which make analysis difficult in the Laplacian matrix. First, a necessary and sufficient condition for the consensus analysis of directed networked systems with positivity constraints is given, by using positive systems theory and graph theory. Unlike the general Riccati design methods that involve solving an algebraic Riccati equation (ARE), a condition represented by an algebraic Riccati inequality (ARI) is obtained for the existence of a solution. Subsequently, an equivalent condition, which corresponds to the consensus design condition, is derived, and a semidefinite programming algorithm is developed. It is shown that, when a protocol is solved by the algorithm for the networked system on a specific communication graph, there exists a set of graphs such that the positive consensus problem can be solved as well.**

***Index Terms*—Algebraic Riccati inequality (ARI), directed graphs, networked systems, positive consensus, positive systems.**

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NOMENCLATURE

$\mathbb{R}$	Set of real numbers.
$\mathbb{C}$	Set of complex numbers.
$\mathbb{R}^{m\times n}$	$m \times n$ matrix with real elements.
$\mathbb{C}^{m\times n}$	$m \times n$ matrix with complex elements.
$X > Y$	Real matrix $X - Y$ is positive definite.
I	Identity matrix with appropriate dimension.
$A \otimes B$	Kronecker product of matrices A and B.
$[A]_{ij}$	ith row and the jth column element of A.
$A \succeq B$	Matrix $A - B \succeq 0$.
$A \succ B$	Matrix $A - B \succ 0$.
$A \in \mathbb{M}^n$	Matrix $A \in \mathbb{R}^{n\times n}$ is Metzler.
$\mathbf{1}_n$	$[1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$.
$\mathbf{0}_n$	$[0, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^n$.
$\mathrm{Re}(\lambda)$	Real part of $\lambda \in \mathbb{C}$.
A^*	Hermitian transpose of matrix $A \in \mathbb{C}^{m\times n}$.
$\mu(A)$	Spectral abscissa of matrix A.
T	Transpose operation of matrix.
$\mathscr{V}$	Set of nodes: $\{v_1, v_2, \ldots, v_N\}$.
$\mathscr{E}$	Subset of $\mathscr{V} \times \mathscr{V}$.
$\mathscr{G}$	Graph: $(\mathscr{V}, \mathscr{E})$.
$\mathscr{N}_i$	Neighbor set of agent i: $\{v_j \in \mathscr{V} : (v_j, v_i) \in \mathscr{E}\}$.

I. INTRODUCTION

RECENT years have witnessed an increasing interest in cooperative control for networked systems, due to its enormous amounts of potential applications in electric power systems, unmanned mobile robots, intelligent transportation systems, and distributed sensor networks [1]. Cooperative control in networked systems for accomplishing designated missions usually has several different topics, including flocking [2], formation [3], and consensus/synchronization [4], [5]. To achieve the effective cooperative control of networked systems, the consensus issue is a fundamental challenge. The object of consensus is to reach an agreement of interest for all the agents by designing some distributed control protocols, which has been extensively studied recently [6]–[9]. In [6], the consensus problem for networked systems with switching topologies has been investigated by proposing a common quadratic Lyapunov function. By using the high-gain observer design approach, the observer-based coordinated control was achieved for the networked systems with input saturation [7]. In [8], the problem of average consensus has been investigated for time-varying directed networks of multiagent systems with limited bandwidth communication. In [9], consensus control

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for multiagent systems with a distributed event-triggered state-feedback protocol has been studied. Then, the consensus issue has been analyzed in the presence of denial-of-service attacks by event/self-triggered communication schemes in [10]. When a networked system has multiple leaders, a containment control problem arises, which means a set of agents arrive at the convex hull formed by the leaders. In [11], necessary and sufficient conditions have been proposed for the containment control of high-order multiagent systems by observer-type protocols. For more reports with regard to consensus of networked systems, one may refer to [3], [12]–[15].

The consensus of networked systems with positivity constraints, called *positive consensus*, is a new and challenging issue raised recently in the field of research [16], [17]. In positive consensus, the agents are described by a special kind of systems called *positive systems* [18]. Positive systems are systems whose describing variables can only take positive or at least nonnegative values. Since a lot of natural phenomena and engineering processes can be modeled by means of positive systems [19], [20], many problems of this topic have been extensively studied in recent years for single-positive systems [21]–[25] and multiple-positive systems connected in various forms [26]–[28]. In [26], necessary and sufficient synchronization conditions have been proposed for the nonnegative edge synchronization of networked systems on undirected graphs based on neighbors' output information. Then, via output feedback protocols, sufficient conditions have been given for the positive consensus of uncertain networked systems without utilizing the information of algebraic connectivity in [27]. Recently, the ℓ_1-gain performance analysis and distributed finite-time filter design have been investigated for positive systems over a sensor network on undirected graphs [28], where each sensor shared its measurement with its neighboring sensors possibly subject to random communication link failure and deception attacks. Though the positive consensus problem (**PCP**) for networked systems has been studied in the case where agents communicate with each other by either undirected [16], [28] and connected graphs [29] or strongly connected directed graphs [30], much work still needs to be done to provide a clearer picture of such a problem.

Differently from the existing works [16], [30], this article investigates the **PCP** in a more general setting that the agents of networked systems communicate with each other via directed graphs each containing a spanning tree. Notice that connected undirected graphs or strongly connected directed graphs are special cases of directed graphs containing a spanning tree. However, this fact leads to quite a different situation where the problem involves complex eigenvalues of the Laplacian matrix, the Hurwitzness of complex matrices, and positivity constraints. Due to a couple of tricky issues that have to be tackled, one cannot straightforwardly generalize the existing approaches to solve this problem, which has motivated our work. The main results and contributions of this work can be summarized as follows.

1) Necessary and/or sufficient conditions of positive consensus analysis and design are derived, and an effective semidefinite programming algorithm is developed for the solution.
2) Compared with the existing works where the agents communicate with each other by either undirected and connected graphs (only for single-input agents) [16], (for multi-input agents) [29], or strongly connected directed graphs [30], **PCP** has been extended to the case where the agents communicate with each other via directed graphs containing a spanning tree. Therefore, the above previous results become very special cases of our results (this will be shown clearly via an illustrative example in the simulation).
3) Another advantage over the existing works in [17], [30], and [31] is that the protocol solved by our algorithm is robust to the graph topology, and positive consensus can be achieved for a set of graphs.

The rest of this article is structured as follows. Some preliminary knowledge is reviewed in Section II. The main results of **PCP**, including positive consensus analysis and design conditions, as well as a semidefinite programming algorithm, are given in Section III. In Section IV, numerical examples with comparative results are provided. Finally, the conclusion is drawn in Section V.

II. Preliminaries

A directed edge from i to j is denoted as an ordered pair $(\upsilon_i, \upsilon_j) \in \mathscr{E}$, which means the child node j can directly receive information from the parent node i. An adjacency matrix of graph $\mathscr{G}$ with order N is an $N \times N$ matrix $\mathscr{A}$ defined as $a_{ij} > 0$ if $(\upsilon_j, \upsilon_i) \in \mathscr{E}$, but 0 otherwise. a_{ij} is the weight for edge $(\upsilon_j, \upsilon_i) \in \mathscr{E}$, and it is set equal to 1 if the weights of graph are irrelevant. Assume that there are no repeated edges and no self-loops, that is, $a_{ii} = 0\ \forall i \in \mathscr{I}$ with $\mathscr{I} = \{1, 2, \ldots, N\}$. The Laplacian matrix L of graph $\mathscr{G}$ of order N is an $N \times N$ matrix defined as $[L]_{ii} = \sum_{\upsilon_j \in \mathscr{N}_i} a_{ij}$ and $[L]_{ij} = -a_{ij}$ for any $i \neq j$. A sequence of successive edges in the form $\{(\upsilon_i, \upsilon_k), (\upsilon_k, \upsilon_l), \ldots, (\upsilon_m, \upsilon_j)\}$ is a directed path from node i to node j.

III. Main Results

A directed graph is said to contain a spanning tree if there is a node called the root such that there is a directed path from the root to any other nodes in the graph. For directed graphs, there are some important results [31], [32]: 0 is an eigenvalue of Laplacian matrix L with its right eigenvector $\mathbf{1}_N$, and all nonzero eigenvalues have positive real parts. Furthermore, 0 is a simple eigenvalue of Laplacian matrix L if and only if graph $\mathscr{G}$ contains a spanning tree.

Consider N agents, distributed on a directed graph $\mathscr{G}$, with identical continuous-time positive linear dynamics as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathscr{I} \tag{1}$$

where $x_i(t) := [x_{i1},\ x_{i2}, \ldots, x_{ir}]^{\mathrm{T}} \in \mathbb{R}^r$ is the state and $u_i(t) \in \mathbb{R}^m$ is the control input. System (1) is a multi-input positive linear system of any orders, and there is no stability assumption. Intuitively, it is said to be a continuous-time positive linear system if its state is nonnegative for any nonnegative initial state and nonnegative input [18]–[20]. Since (1) is a positive linear system, $A \in \mathbb{R}^{r\times r}$ is a Metzler matrix, and $B \in \mathbb{R}^{r\times m}$ is a nonnegative matrix. This is because

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matrix A is Metzler and matrix B is nonnegative is a necessary and sufficient condition guaranteeing (1) to be positive [18]–[20]. Throughout this article, it is assumed that (A, B) is stabilizable. Consider the distributed state-feedback control protocol [33]

$$u_i(t) = K \sum_{\upsilon_j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)), \quad i \in \mathcal{I}. \tag{2}$$

Also, define

$$x(t) := \left[x_1^{\mathrm{T}}(t),\ x_2^{\mathrm{T}}(t), \ldots,\ x_N^{\mathrm{T}}(t)\right]^{\mathrm{T}}.$$

Then, the overall closed-loop system is represented by

$$\dot{x}(t) = \mathbf{A}x(t) \tag{3}$$

where $\mathbf{A} = I_n \otimes A - (L \otimes BK)$. In this article, we study the consensus problem of positive networked systems on directed graphs containing a spanning tree, which is defined as follows:

PCP: For any nonnegative initial value, find a gain matrix K such that the consensus of the network is achieved, while their state trajectories remain in the nonnegative orthant.

For **PCP**, it is assumed that the communication topology of networked system is represented by a directed graph $\mathcal{G}$ containing a spanning tree. Then, the eigenvalues of L are, in general, complex numbers [31], which can be denoted by λ_i, $\forall i \in \mathcal{I}$ and ordered as $0 = \lambda_1 \prec \mathrm{Re}(\lambda_2) \preceq \cdots \preceq \mathrm{Re}(\lambda_N)$. Following a similar line in [16], we are investigating the positive consensus of networked systems on directed graphs. For ease of illustration, define $l_{\max} := \max([L]_{ii})$, $\forall i \in \mathcal{I}$. By expanding $\mathbf{A}$, we have

$$\mathbf{A} = \begin{bmatrix} A - \sum_{j=1}^{N} a_{1j}BK & a_{12}BK & \ldots & a_{1N}BK \\ a_{21}BK & A - \sum_{j=1}^{n} a_{2j}BK & \ldots & a_{2N}BK \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1}BK & a_{N2}BK & \ldots & A - \sum_{j=1}^{N} a_{Nj}BK \end{bmatrix}. \tag{4}$$

It is obvious that system (3) is positive if and only if $\mathbf{A}$ is Metzler [18]–[20]. By observing (4), it follows that $\mathbf{A}$ is Metzler if and only if $A - l_{\max}BK$ is Metlzer and $BK \succeq 0$ since $a_{ij} \succeq 0$ $\forall i, j \in \mathcal{I}$, which gives that $BK \succeq 0$ and $A - l_{\max}BK$ is Metzler. It is well known that the system can reach consensus if and only if $A_i := A - \lambda_i BK$, $i \in \mathcal{I}\backslash\{1\}$, are Hurwitz [33], [34]. The discussion has led to the following positive consensus analysis condition for **PCP**.

Proposition 1: For a directed graph containing a spanning tree with λ_i, $i \in \mathcal{I}$, as the eigenvalues of its Laplacian matrix, **PCP** is solvable by the given K if and only if the following conditions hold.

1) $BK \succeq 0$.
2) $A - l_{\max}BK$ is Metzler.
3) $A - \lambda_i BK$, $i \in \mathcal{I}\backslash\{1\}$, are Hurwitz.

Remark 1: The positive consensus analysis of single-input networked systems on an undirected and incomplete graph has been given in [16] where condition 3), i.e., $A - \lambda_i BK$, $i \in \mathcal{I}\backslash\{1\}$, are Hurwitz, involves real eigenvalues, which helps to derive some nice analytical results for consensus design. However, since, in the case of directed graphs, condition 3) involves, in general, complex eigenvalues, that is, λ_i, $i \in \mathcal{I}\backslash\{1\}$, may be complex, the consensus design results in [16] cannot be straightforwardly extended, and hence, the positive consensus design problem becomes more challenging. Notice that it does not involve complex eigenvalues in the results of [30] either.

Due to the infinite gain margin robustness property, the Riccati design method has been used for cooperative tracking control (leader–follower consensus) design of networked linear systems on strongly connected directed graphs [35] and directed graphs containing a spanning tree [36], respectively. It has some advantages for cooperative tracking control design, including [34]: 1) the decoupling from feedback controller design with the details of the graph topology and 2) the robustness to the graph topology. In this article, the idea of the Riccati design method is also used to solve the **PCP**. Different from the general Riccati design method that involves solving an ARE, our results require solving an algebraic Riccati inequality (ARI) since this equivalent form can provide some flexibility for parameterizing the controller gain. Also, some special properties regarding the Riccati design method still hold in our results that are given as follows.

Theorem 1: For a directed graph containing a spanning tree with λ_i, $i \in \mathcal{I}$, as the eigenvalues of its Laplacian matrix, **PCP** is solvable if there exist real matrices $P > 0$ and $S > 0$ such that the following conditions hold.

1) $BSB^{\mathrm{T}}P \succeq 0$.
2) $A - l_{\max}BSB^{\mathrm{T}}P$ is Metzler.
3) $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P < 0$.

Under the conditions, $K = SB^{\mathrm{T}}P$.

Proof: Letting $K = R^{-1}B^{\mathrm{T}}P = SB^{\mathrm{T}}P$ with $S = R^{-1} > 0$ and $P > 0$, conditions 1) and 2) are equivalent to those in Proposition 1. Notice that condition 3), i.e., $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P < 0$, is equivalent to that there exists a real matrix $Q > 0$ such that an ARE: $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P + Q = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBR^{-1}B^{\mathrm{T}}P + Q = 0$ holds with a unique solution $P > 0$. Then, straightforward computation gives the Lyapunov equation $\Phi(\lambda_i) := (A - \lambda_i BK)^*P + P(A - \lambda_i BK) = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_i) PB\ R^{-1}B^{\mathrm{T}}P$, $i \in \mathcal{I}\backslash\{1\}$. We have $\Phi(\lambda_2) = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBR^{-1}\ B^{\mathrm{T}}P = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P = -Q < 0$. Since $0 = \lambda_1 \prec \mathrm{Re}(\lambda_2) \preceq \cdots \preceq \mathrm{Re}(\lambda_n)$, we have $\Phi(\lambda_i) = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_i)PBR^{-1}B^{\mathrm{T}}P = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBR^{-1}B^{\mathrm{T}}P - 2(\mathrm{Re}(\lambda_i) - \mathrm{Re}(\lambda_2)) PBR^{-1}B^{\mathrm{T}}P = -Q - 2(\mathrm{Re}(\lambda_i) - \mathrm{Re}(\lambda_2))K^{\mathrm{T}}RK < 0$, $i \in \mathcal{I}\backslash\{1\}$. By the Lyapunov theory [37], we have that $A - \lambda_i BK$, $i \in \mathcal{I}\backslash\{1\}$, are Hurwitz if ARI: condition 3) $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P < 0$ holds. The proof is completed. □

Remark 2: It is worth pointing output that there exist $P > 0$, $R > 0$, and $Q > 0$ such that the algebraic Riccati equation

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(ARE) $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBR^{-1}B^{\mathrm{T}}P + Q = 0$ holds if and only if (A, B) is stabilizable via state feedback [34].

The Riccati design for cooperative tracking control requires solving an ARE with respect to two given real matrices $Q > 0$ and $R > 0$ ($S = R^{-1}$) and choosing a coupling gain appropriately [34]. However, such a design framework cannot assure the positivity of networked systems since $Q > 0$ and $R > 0$ are arbitrarily given. A key problem arises: *how to find a pair* (P, S) *such that the consensus and positivity of networked systems can be achieved*? This motivates us to present the following theorem for developing an algorithm.

Theorem 2: For a directed graph containing a spanning tree with λ_i, $i \in \mathscr{I}$, as the eigenvalues of its Laplacian matrix, ***PCP*** is solvable if there exist matrices $P > 0$, $S > 0$, and X such that the following conditions hold.

1) $BSB^{\mathrm{T}}P \succeq 0$.
2) $A - l_{\max}BSB^{\mathrm{T}}P$ is Metzler.
3) $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}X^{\mathrm{T}} - 2\mathrm{Re}(\lambda_2)XBSB^{\mathrm{T}}P + 2\mathrm{Re}(\lambda_2)XBSB^{\mathrm{T}}X^{\mathrm{T}} < 0$.

Under the conditions, $K = SB^{\mathrm{T}}P$.

Proof: Since condition 3) $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}X^{\mathrm{T}} - 2\mathrm{Re}(\lambda_2)XBSB^{\mathrm{T}}P + 2\mathrm{Re}(\lambda_2)XBSB^{\mathrm{T}}X^{\mathrm{T}} = A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P + 2\mathrm{Re}(\lambda_2)(X - P)BSB^{\mathrm{T}}(X - P)^{\mathrm{T}} < 0$ and $(X - P)BSB^{\mathrm{T}}(X - P)^{\mathrm{T}} \geq 0$, we have $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P < 0$ (notice $Q > 0$ is implicitly found). This means that condition 3) of Theorem 2 leads to condition 3) of Theorem 1.

If condition 3) of Theorem 1, that is, $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P < 0$, holds, it is obvious that there exists a matrix $X = P$ such that $A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P + 2\mathrm{Re}(\lambda_2)(X - P)BSB^{\mathrm{T}}(X - P)^{\mathrm{T}} < 0$ holds. It means that condition 3) of Theorem 1 leads to condition 3) of Theorem 2. Based on the discussion above, one can see that Theorems 1 and 2 are equivalent. The proof is completed. □

Theorem 2 is an equivalent condition of Theorem 1. To find a pair (P, S), a heuristic iterative algorithm based on Theorems 1 and 2 is developed in **Algorithm PCPDG** (**PCP** with Directed Graphs).

Algorithm PCPDG:

Step 1: Set $k = 1$, $\upsilon = 1$, and $\epsilon^{(0)} = 0$. For a given $S^{(1)} = I$, find a matrix $P^{(1)} = U^{-1} > 0$ such that the LMI: $UA^{\mathrm{T}} + AU - 2\mathrm{Re}(\lambda_2)BS^{(1)}B^{\mathrm{T}} < 0$ holds.

Step 2: Fix $X = P^{(k)}$ and $S = S^{(k)}$, and minimize $\epsilon^{(l)}$ with respect to $P > 0$

$$\text{s.t.} \begin{cases} BSB^{\mathrm{T}}P \succeq 0 \\ A - l_{\max}BSB^{\mathrm{T}}P \in \mathbb{M}^r \\ \Gamma(P, X, S) < \epsilon^{(\upsilon)}I \end{cases}$$

where the matrix function $\Gamma(P, X, S)$ is defined as $\Gamma(P, X, S) := A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}X^{\mathrm{T}} - 2\mathrm{Re}(\lambda_2)XB\ SB^{\mathrm{T}}P + 2\mathrm{Re}(\lambda_2)XBSB^{\mathrm{T}}X^{\mathrm{T}}$.

Step 3: If $\epsilon^{(\upsilon)} \preceq 0$, a controller is obtained as $K = SB^{\mathrm{T}}P$. **STOP.** otherwise, go to next step.

Step 4: If $|\epsilon^{(\upsilon)} - \epsilon^{(\upsilon-1)}|/\epsilon^{(\upsilon)} \prec \theta$, where θ is a prescribed tolerance, then this algorithm fails to find the desired solution. **STOP.** otherwise, set $k = k + 1$, $\upsilon = \upsilon + 1$, update $P^{(k)}$ as $P^{(k)} = P$, and then go to next step.

Step 5: Fix $P = P^{(k)}$, and minimize $\epsilon^{(\upsilon)}$ with respect to $S > 0$

$$\text{s.t.} \begin{cases} BSB^{\mathrm{T}}P \succeq 0 \\ A - l_{\max}BSB^{\mathrm{T}}P \in \mathbb{M}^r \\ \Omega(P, S) < \epsilon^{(\upsilon)}I \end{cases}$$

where the matrix function $\Omega(P, S)$ is defined as $\Omega(P, S) := A^{\mathrm{T}}P + PA - 2\mathrm{Re}(\lambda_2)PBSB^{\mathrm{T}}P$.

Step 6: If $\epsilon^{(\upsilon)} \preceq 0$, a controller is obtained as $K = SB^{\mathrm{T}}P$. **STOP.** otherwise, go to next step.

Step 7: If $|\epsilon^{(\upsilon)} - \epsilon^{(\upsilon-1)}|/\epsilon^{(\upsilon)} \prec \theta$, where θ is a prescribed tolerance, then this algorithm fails to find the desired solution. **STOP.** otherwise, update $S^{(k)}$ as $S^{(k)} = S$, set $\upsilon = \upsilon + 1$, and then go to Step 2.

Remark 3: Notice that $\Gamma(P, X, S) = \Omega(P, S) + 2\mathrm{Re}(\lambda_2)(X - P)BSB^{\mathrm{T}}(X - P)^{\mathrm{T}}$. For $\Gamma(P^{(k+1)}, P^{(k)}, S^{(k)}) < \epsilon^{(\upsilon)}I$, $\Omega(P^{(k+1)}, S^{(k+1)}) < \epsilon^{(\upsilon+1)}I$, $\Gamma(P^{(k+2)}, P^{(k+1)}$, and $S^{(k+1)}) < \epsilon^{(\upsilon+2)}I$, we have $\mu(\Gamma(P^{(k+2)}, P^{(k+1)}, S^{(k+1)})) \leq \mu(\Omega(P^{(k+1)}, S^{(k+1)})) \leq \mu(\Gamma(P^{(k+1)}, P^{(k)}, S^{(k)}))$, and $\epsilon^{(\upsilon+2)} \preceq \epsilon^{(\upsilon+1)} \preceq \epsilon^{(\upsilon)}$ $\forall k$, $\upsilon \succeq 1$. It means that **Algorithm PCPDG** generates a sequence of matrices $\{P^{(k)}, S^{(k)}\}_{k=1}^{\bar{k}}$ such that $\{\epsilon^{(\upsilon)}\}_{\upsilon=1}^{\bar{\upsilon}}$ decreases monotonically. When $\epsilon^{(\upsilon)}$ decreases to a nonpositive number, a feasible solution $\{P, S\}$ is obtained. The tolerance θ is usually a sufficiently small positive number used as the stopping criterion of the algorithm. Based on the experience of simulation, it could be chosen between $1 \times \mathrm{e}^{-4}$ and $1 \times \mathrm{e}^{-5}$. **Algorithm PCPDG** belongs to the type of sequential semidefinite programming algorithms, which appears in many synthesis problems in the field of control.

Using the concept of consensus region [32], [36], one can evaluate the performance of consensus control protocols to show how consensusability depends on structural parameters of the communication graphs.

Definition 1: The consensus region of a given protocol (2) is a complex region: $\mathbb{S} \triangleq \{s \in \mathbb{C} \mid A - sBK \text{ is Hurwitz}\}$.

It is shown that, for protocol (2) with the Riccati design control gain, the consensus region is unbounded [36]. According to [36], in our approach, an unbounded consensus region is $\bar{\mathbb{S}} = \{\alpha + \beta \mathrm{i} \mid \alpha \in [\mathrm{Re}(\bar{\lambda}_2), +\infty),\ \beta \in (-\infty, +\infty)\}$ for a given communication graph $\bar{\mathscr{G}}$ with $\lambda_2 = \bar{\lambda}_2$. However, for positive networked systems, the consensusability and positivity are two important properties that should be concerned. Based on Proposition 1 and Definition 1, the positive consensus can be achieved if $\lambda_i \in \bar{\mathbb{S}}$, $i \in \mathscr{I}\backslash\{1\}$, and $\mathbf{A}$ is Metzler. Then, one can directly obtain the following proposition.

Proposition 2: For a given communication graph $\bar{\mathscr{G}}$ with $l_{\max} = \bar{l}_{\max}$, a controller K is obtained by **Algorithm PCPDG**, and then, **PCP** is also solvable for agents on a set of graphs: $\{\mathscr{G} \mid \lambda_i \in \bar{\mathbb{S}},\ i \in \mathscr{I}\backslash\{1\},\ l_{\max} \preceq \bar{l}_{\max}\}$.

It can be proved by following the lines of the analysis in Proposition 1 and the proof in Theorem 1.

Remark 4: In our work, as the positive system is considered, one has to investigate the positivity and Hurwitzness issues at the same time. The "positive consensus region" in this article consists of two parts: the Metzler region and the Hurwitz region. The Metzler region is used to analyze the positivity of multiagent systems, and the Hurwitz region is used to

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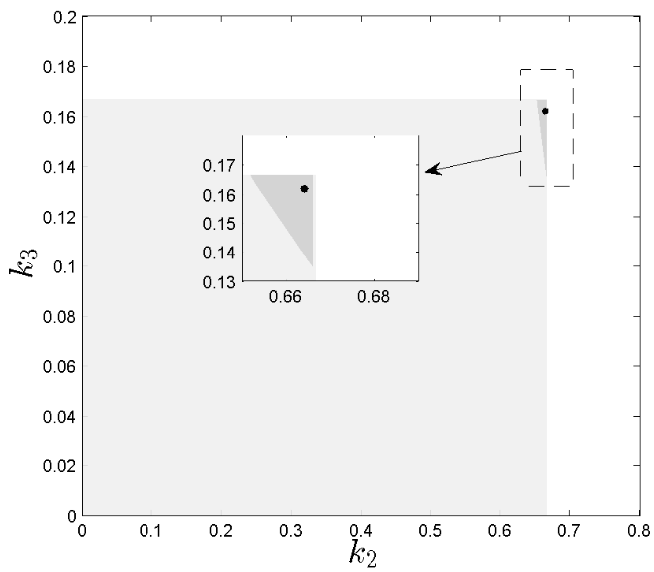

Fig. 1. Metzler and Hurwitz regions of A_2 and A_3.

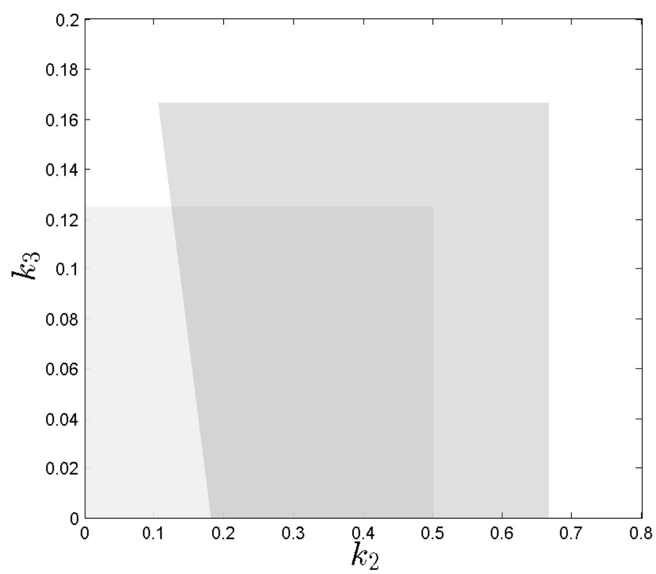

Fig. 2. Metzler and Hurwitz regions of A_4.

study the consensus of agents. Therefore, the consensus region proposed for general linear systems without considering the positivity [32], [36] cannot be used to analyze the consensus problem in our work.

IV. Numerical Simulations

In order to illustrate the efficacy of the proposed approaches, simulations on three examples with some comparative results are presented in this section.

A. Example 1

In this example, we give a comparison of our approach and those proposed in [17], [30], and [31]. It is worth pointing out that the undirected single-input agents, the undirected multi-input agents, and the strongly connected and balanced directed multi-input agents are considered in [16], [29], and [30], respectively. Consider the positive networked system in (1) with four identical positive linear systems whose dynamics is represented by

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & -2.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Since connected undirected graphs are special cases of directed graphs, our approach can also be used to solve **PCP** with undirected graphs. In this example, the agents communicate with each other via an undirected graph with the following Laplacian matrix:

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

where $l_{\max} = 3$ and the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$, and $\lambda_4 = 4$. Define $K := [k_1,\ k_2,\ k_3] \in \mathbb{R}^{1\times 3}$. Since the agents are single-input positive systems, the controller gain is required to be nonnegative due to condition 1) in Theorem 1. Then, condition 2) in Theorem 1 requires that $A - l_{\max}BK$ is Metzler implying $K_{\min} \preceq K \preceq K_{\max}$, where $K_{\min} = [0\ 0\ 0]$ and $K_{\max} = [0\ 2/3\ 1/6]$. The feasible solution regions of matrices A_2, A_3 and A_4 are shown in Figs. 1 and 2 where the Metzler regions are highlighted in light gray and the Hurwitz regions are expressed in medium gray. Notice that the Metzler and Hurwitz regions of A_2, A_3, and A_4 have some overlapping parts highlighted in dark gray such that they are Metzler–Hurwitz. Unfortunately, it is found that their Metzler–Hurwitz regions (that is, the dark gray color regions in Figs. 1 and 2) are separated. In other words, one cannot find a K such that A_2, A_3, and A_4 are all Metzler–Hurwitz. Due to this fact, the approaches proposed in [16] cannot solve **PCP** since they require A_2, A_3, and A_4 are all Metzler–Hurwitz, which is unnecessary in our approach. However, it is known that the feasible solution region for this problem is actually the dark gray region shown in Fig. 1 since its solutions satisfy the conditions in Theorem 1. Though the area of this region is very small, as shown in the figure, by using **Algorithm PCPDG**, a feasible solution is found successfully as

$$K = [0 \quad 0.66412 \quad 0.16181]$$

which is shown by the black dot in Fig. 1. Using such a controller, we have

$$A_2 = A_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -0.32824 & 0.67638 \\ 0 & 1.3359 & -2.9618 \end{bmatrix} \in \mathbb{M}^3$$

$$A_4 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4.313 & -0.29448 \\ 0 & -0.65648 & -3.4472 \end{bmatrix} \notin \mathbb{M}^3$$

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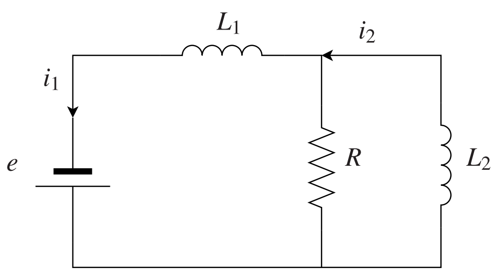

Fig. 3. Positive electric circuit model.

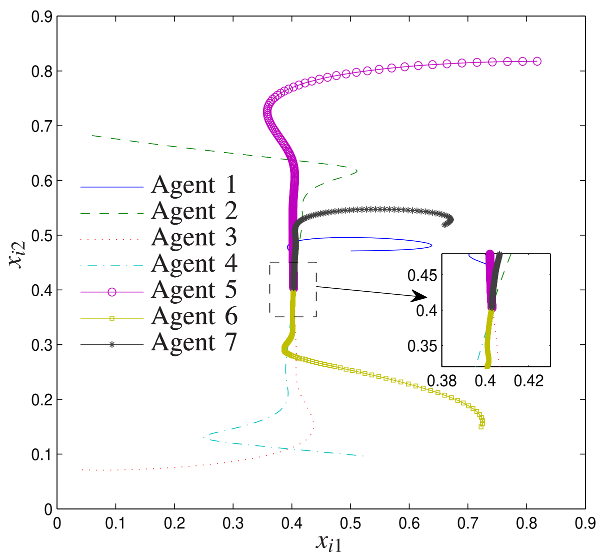

Fig. 4. Consensus result of the positive networked system with controller (7).

whose eigenvalues are $\{-1, -0.020991, -3.2691\}$ and $\{-1, -4.4971, -3.2631\}$, respectively. Also

$$BK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.3282 & 0.32362 \\ 0 & 0.66412 & 0.16181 \end{bmatrix} \succeq 0$$

$$A - l_{\max} BK = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2.9847 & 0.02914 \\ 0 & 0.00764 & -3.2854 \end{bmatrix} \in \mathbb{M}^3.$$

According to Proposition 1, **PCP** is solved. In addition, the approach proposed in [29] is also used to solve the problem, and a feasible solution is obtained as follows:

$$K = [0 \quad 0.66415 \quad 0.16391].$$

Therefore, both the approach proposed in this work and that in [29] have solved the problem of this example successfully.

B. Example 2

In this example, we compare our approach with those proposed in [17], [30], and [31] again. Consider an electrical network model consisting of multiple positive electrical circuits [30], as shown in Fig. 3, with seven agents. By Kirchhoff's voltage law, we have

$$\begin{cases} e(t) = L_1 \dfrac{\mathrm{d}i_1(t)}{\mathrm{d}t} + R(i_1(t) - i_2(t)) \\ R(i_1(t) - i_2(t)) = L_2 \dfrac{\mathrm{d}i_2(t)}{\mathrm{d}t}. \end{cases} \tag{5}$$

Choosing $i_1(t)$ and $i_2(t)$ as the two state variables and $e(t)$ as the control input yields the system in (1) with the following system matrices:

$$A = \begin{bmatrix} -\dfrac{R_1}{L_1} & \dfrac{R_1}{L_1} \\ \dfrac{R_1}{L_2} & -\dfrac{R_1}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{L_1} \\ 0 \end{bmatrix}.$$

The values of the parameters are chosen as $R_1 = 1\ \Omega$ and $L_1 = L_2 = 1$ H.

Example 2.1: In [30], the agents communicate through a strongly connected and balanced directed graph, and the associated Laplacian matrix is given as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

In this case, we have $l_{\max} = 1$ and $\lambda_2 = 0.3765 - 0.7818\mathrm{j}$. It is shown that the problem of such case can be solved by the approach in [30]. By using **Algorithm PCPDG**, a feasible solution is found as

$$K = [7.6685 \quad 0.50579].$$

Example 2.2: We consider another communication topology that contains a spanning tree but is not strongly connected and balanced. The associated Laplacian matrix is given as follows:

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{6}$$

Since it is not strongly connected and balanced, the approach proposed in [30] does not work for **PCP**. In this case, we have $l_{\max} = 2$ and $\lambda_2 = 0.69098 - 0.95106\mathrm{j}$. By using **Algorithm PCPDG**, a feasible solution is found as

$$K = [7.6523 \quad 0.25427]. \tag{7}$$

With the controller gain, the consensus result is shown in Fig. 4. Therefore, our approach has solved the problem of the two cases successfully, while the approach in [30] fails to give a solution of the second case. Since the approaches proposed in [16] and [29] have assumed that the communication topologies of agents are undirected, they cannot solve the problem of the two cases.

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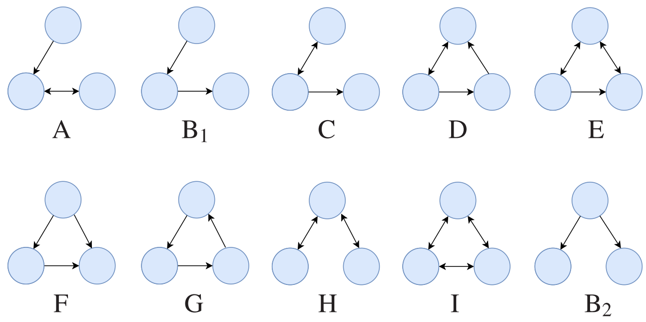

Fig. 5. Communication topologies of the three-agent network.

TABLE I
FEATURES OF LAPLACIAN MATRICES

Type	A	B_1/B_2	C	D	E	F	G	H	I
Number	6	6/3	6	6	6	6	2	3	1
λ_2	0.38197	1	1	2	2	1	$1.5-0.86603j$	1	3
λ_3	2.618	1	2	2	3	2	$1.5+0.86603j$	3	3
$l_{\max}$	2	1	1	2	2	2	1	2	2

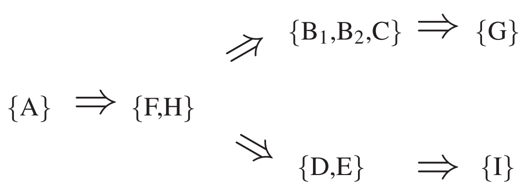

Fig. 6. Implication of different protocols for **PCP** using the features of different graphs in Fig. 5.

C. Example 3

In this example, we consider a positive networked system with three multi-input agents, and the system matrices of each agent are represented as follows:

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 1.8 & -2 & 0 \\ 1 & 0 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If we label the agents with 1–3 and set the weights of edges to 1, there are 45 communication topologies in total, which can be divided into ten sets, as shown in Fig. 5. The features of their Laplacian matrices, including eigenvalues and $l_{\max}$, are summarized in Table I. Among them, we can see that the graphs of Type H and Type I are undirected [16], which constitute 8.9% of all communication topologies; the graphs of Type G are strongly connected and balanced directed [30], which constitute 4.4% of all communication topologies. Therefore, they are only two very special cases among them. Notice that the percentage goes down quickly as the number of agents increases.

According to Proposition 2, the implication of different protocols for **PCP** using the features of the graphs in Fig. 5 is shown in Fig. 6. Specifically, choosing the graphs of Type A as the topology of agents can give a robust protocol by our algorithm such that **PCP** is solvable with a set of graphs, which includes all the graphs in Fig. 5. Choosing the graphs of

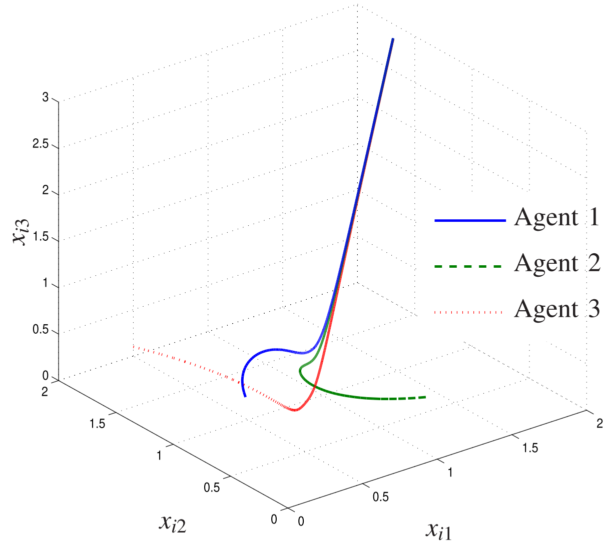

Fig. 7. Consensus result of agents on the graph of Type G with controller (8).

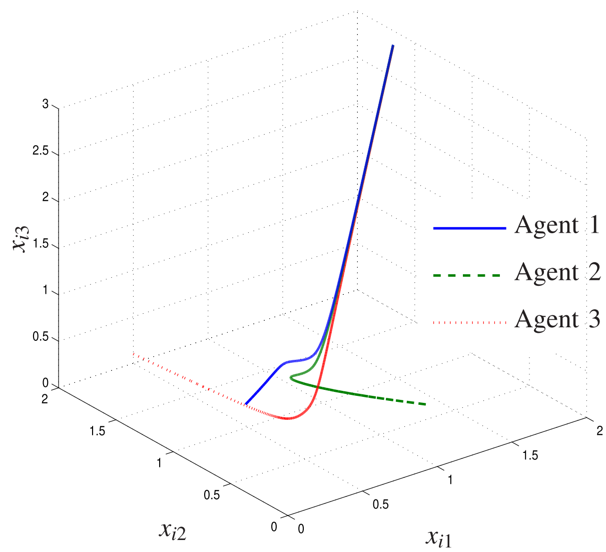

Fig. 8. Consensus result of agents on the graph of Type I with controller (8).

Type B_1/B_2/C can give a protocol such that the solvability of **PCP** with the graphs of Type G is guaranteed. The graphs of Type G are strongly connected, balanced, and directed [30], which is a small set of all the graphs in Fig. 5. In this example, we choose the graphs of Type A as the communication topology of agents. By using **Algorithm PCPDG**, a feasible solution is found as

$$K = \begin{bmatrix} 9.8549 & 0.59498 & 0.43384 \\ 0.59498 & 7.8054 & 0 \end{bmatrix}. \tag{8}$$

Using such a controller and for the graphs of Type A, we have

$$A_2 = \begin{bmatrix} -6.7643 & 1.7727 & 0.8343 \\ 1.5727 & -4.9814 & 0 \\ 1.0000 & 0 & -0.5000 \end{bmatrix} \in \mathbb{M}^3$$

$$A_3 = \begin{bmatrix} -28.8001 & 0.4423 & -0.1358 \\ 0.2423 & -22.4345 & 0 \\ 1.0000 & 0 & -0.5000 \end{bmatrix} \notin \mathbb{M}^3$$

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whose eigenvalues are $\{-7.8498, -4.0396, -0.3563\}$ and $\{-28.8121, -22.4177, -0.5048\}$, respectively. Also

$$BK = \begin{bmatrix} 9.8549 & 0.5950 & 0.4338 \\ 0.5950 & 7.8054 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0$$

$$A - l_{\max} BK = \begin{bmatrix} -22.7098 & 0.8100 & 0.1323 \\ 0.6100 & -17.6108 & 0 \\ 1.0000 & 0 & -0.5000 \end{bmatrix} \in \mathbb{M}^3.$$

According to Proposition 1, **PCP** is solved. An unbounded consensus region is $\bar{\mathbb{S}} = \{\alpha + \beta \mathrm{i} \mid \alpha \in [0.38197, +\infty), \beta \in (-\infty, +\infty)\}$ for protocol (8). We have verified that the agents with all the graphs in Fig. 5 using protocol (8) can achieve positive consensus. Due to the limitation of length, we only show the consensus results with the graphs of Types G and I in Figs. 7 and 8, respectively. It is worth pointing out that protocol (8) can solve **PCP** for a different number of agents with different communication topologies if their corresponding eigenvalues $\lambda_i \in \bar{\mathbb{S}}$, $i \in \mathscr{I} \backslash \{1\}$, and $l_{\max} \preceq 2$.

V. Conclusion

This article has addressed the **PCP** for identical positive linear systems on directed graphs each containing a spanning tree. A necessary and sufficient condition of positive consensus analysis for networked systems has been given. Then, two positive consensus design conditions have been derived by employing the Riccati design method. Based on the consensus design conditions, a semidefinite programming algorithm has been developed. It has been shown that positive consensus can be achieved for a set of graphs with the protocol solved by our algorithm for agents on a specific communication graph. Simulations on three examples with some comparative results have been presented to show the superiority of the proposed approach.

It is worth pointing out that the proposed framework for designing positive consensus controllers can be straightforwardly generalized to the case where observer-type output-feedback controllers are used. Furthermore, one can solve the leader–follower consensus problem or the containment control problem for positive networked systems by such a framework as well. Our future works will focus on the positive finite-time/specified-time consensus problems of positive multiagent systems [38]–[40].

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