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Further Improvements on Non-Negative Edge Consensus of Networked Systems

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***Abstract*—In this article, the non-negative edge consensus problem is addressed for positive networked systems with undirected graphs using state-feedback protocols. In contrast to existing results, the major contributions of this work included: 1) significantly improved criteria of consequentiality and non-negativity, therefore leading to a linear programming approach and 2) necessary and sufficient criteria giving rise to a semidefinite programming approach. Specifically, an improved upper bound is given for the maximum eigenvalue of the Laplacian matrix and the (out-) in-degree of the degree matrix, and an improved consensuability and non-negativevity condition is obtained. The sufficient condition presented only requires the number of edges of a nodal network without the connection topology. Also, with the introduction of slack matrix variables, two equivalent conditions of consensuability and non-negativevity are obtained. In the conditions, the system matrices, controller gain, as well as Lyapunov matrices are separated, which is helpful for parameterization. Based on the results, a semidefinite programming algorithm for the controller is readily developed. Finally, a comprehensive analytical and numerical comparison of three illustrative examples is conducted to show that the proposed results are less conservative than the existing work.**

***Index Terms*—Networked systems, non-negative edge consensus, positive linear system, undirected graphs.**

I. Introduction

A NETWORKED system, which is composed of multiple subsystems communicating with each other through some pairwise connections, aims at accomplishing various control objects. In recent years, we have witnessed a widespread interest in cooperative control for networked systems due to two main observations: 1) a lot of surprising behaviors or phenomena in nature, for instance, cooperative transport of food by a group of ants and predator avoidance of a school of fishes and 2) all kinds of potential and real-life applications, such as carrying out tasks through some cooperative means in the applications of unmanned aerial vehicles [21] and mobile robots [17]. Most of these results have focused on the evolution of the nodes in networked systems and considered the edges as their connection [4], [5], [13], [34]–[38], while a few studies focus on the edge dynamics of the complex network since the edges can very often be modeled as dynamic systems [16], [20], [22]. It is known that the edge variables of a complex network are non-negative in many situations. For instance, the data transmission of communication networks and the vehicle flows of traffic networks are usually quantified by some non-negative or positive values. These kinds of networked systems can be modelled as a special type of system with positive dynamics, that is, the system state is always non-negative for any non-negative initial state and non-negative input [10], [14], [15], [18], [19], [29]. Motivated by these, the non-negative edge consensus problem (*NECP*) for continuous-time networked systems was raised and investigated recently by Wang *et al.* [30]–[32] for the first time. Two major contributions were achieved: 1) the peculiar nature between nodal networks and edge networks was revealed analytically and 2) some elegant sufficient conditions based on the relationship between them were derived to achieve the desired non-negative edge consensus. Moreover, a different yet related problem for the systems, which is called semiglobal observer-based consensus, was studied in [23]. In addition, Su *et al.* [24], [26]–[28] investigated the *NECP* for discrete-time networked systems with observer-based protocols and/or input constraints. In the *NECP*, the communication of the original "node" networked system was modeled by a nodal graph while the communication of the "edge" networked system was modeled by a line graph [8], [16]. Particularly, we focus on the edge networked system and investigate the consensus problem among the edges with the consideration of non-negativevity in this work. As indicated by the analysis in [32], to solve the *NECP*, one can first obtain a line graph based on the mapping of a nodal graph, and then formulate the *NECP* as the general consensus problem of an edge networked system with non-negativevity constraints.

Since all existing results for non-negative edge consensus are sufficient conditions, improved conditions having less or no conservatism are always desirable. In this article, the *NECP* is revisited and the improvement has been made as follows.

1) Significantly enhanced consensuability analysis and synthesis results that are less conservative than the existing works are proposed.
2) Necessary and sufficient consensuability analysis and synthesis conditions are proposed along with slack matrix variables.

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3) Effective and efficient convex programming algorithms are developed based on the theoretical results.

II. Preliminaries

The symbol $\mathbb{R}$ is used to denote the set of real numbers, $\mathbb{R}^n$ is used to denote the n dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ is used to denote the set of $m \times n$ matrix for which all entries belong to $\mathbb{R}$. Throughout this article, for real symmetric matrices X and Y, the notation $X \geq Y$, (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). I denotes the identity matrix with appropriate dimension. $A \bigotimes B$ denotes the Kronecker product of matrices A and B. $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ denotes the set of natural numbers from 1 to n. For a matrix $A \in \mathbb{R}^{m\times n}$, $[A]_{ij}$ denotes the element located at the ith row and the jth column. $A \succeq 0$ (respectively, $A \succ 0$) means that for all i and j, $[A]_{ij} \succeq 0$ (respectively, $[A]_{ij} \succ 0$). The notation $A \succeq B$ (respectively, $A \succ B$) means that $A - B \succeq 0$ (respectively, $A - B \succ 0$). Matrix $A \in \mathbb{R}^{n\times n}$ is called Metzler, if all of its off-diagonal elements are non-negative, which is represented by $A \in \mathbb{M}^n$. The spectral abscissa, that is, the maximum among the real part of the eigenvalues of matrix A, is represented by $\alpha(A)$. The symbol $*$ is used to denote a matrix which can be inferred by symmetry. The superscript T denotes the transpose of a matrix. Matrices are assumed to have compatible dimensions for algebraic operations they are is not explicitly stated.

Graphs are commonly used to represent the sensing, communication, and other interaction topologies in networked systems. In the following, we assume that the topology of the nodal graph is represented by an undirected, connected graph $\bar{\mathscr{G}} = (\bar{\mathscr{V}}, \bar{\mathscr{E}})$, where $\bar{\mathscr{V}} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ is the set of nodes, and $\bar{\mathscr{E}} \subset \bar{\mathscr{V}} \times \bar{\mathscr{V}}$ is the set of unordered pairs of nodes, called edges. To represent the communication among the edges of a nodal network, the corresponding line graph should be constructed. Since the line graph of an undirected, connected nodal graph is also connected, in the following, we assume that the communication among edges of a nodal network is represented by an undirected, connected line graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{v_1, v_2, \ldots, v_m\}$ is the set of nodes and $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ is the set of unordered pairs of nodes, called edges. Two nodes v_i and v_j are adjacent if $(v_i, v_j) \in \mathscr{E}$. The adjacency matrix $\mathscr{A} \in \mathbb{R}^{m\times m}$ of an undirected, connected line graph $\mathscr{G}$ is defined as $[\mathscr{A}]_{ij} = [\mathscr{A}]_{ji} = 1$ if $(v_i, v_j) \in \mathscr{E}$ but 0 otherwise. The set of neighbors of node i is denoted by $\mathscr{N}_i = \{j \in \mathscr{V} : (v_i, v_j) \in \mathscr{E}\}$. The degree (in-degree) matrix $\mathscr{D}$ of line graph $\mathscr{G}$ is defined as $\mathscr{D} = \text{diag}(d_1, d_2, \ldots, d_m)$, where $d_i = \sum_{j=1}^{m} [\mathscr{A}]_{ij}$, $i \in \mathbb{Z}_m$. The Laplacian matrix is defined as $\mathscr{L} = \mathscr{D} - \mathscr{A}$. The description of the nodal graph has a similar definition as the line graph. If the undirected line graph $\mathscr{G}$ is connected, the Laplacian matrix $\mathscr{L}$ has a simple eigenvalue 0 and all the other eigenvalues are positive and real, which can always be sorted in nondecreasing order as $0 = \lambda_1 \preceq \lambda_2 \preceq \cdots \preceq \lambda_m$ [7], [33]. Define $d_{\max} := \max\{d_i\}$, $i \in \mathbb{Z}_m$. $\mathbf{1}_n$ denotes $[1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$. Regarding the details of how to derive the line graph from an undirected nodal graph, one can refer to [25] and [32].

III. Main Results

Consider a nodal network with a nodal graph whose edges are identical continuous-time positive linear systems [1], [6], [9] as follows:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathbb{Z}_m \tag{1}$$

distributed on a line graph $\mathscr{G}$, where $x_i(t) := [x_{i1}, x_{i2}, \ldots, x_{ir}]^{\mathrm{T}} \in \mathbb{R}^r$ denotes the state and $u_i(t) \in \mathbb{R}^m$ denotes the control input. System (1) is a multi-input positive linear system of any order. Generally speaking, system (1) is said to be a continuous-time positive linear system if the state is non-negative for any non-negative initial state and non-negative input [1], [6], [9]. Also, since (1) is a positive linear system, the system matrix $A \in \mathbb{R}^{r\times r}$ is Metzler, and the system matrix $B \in \mathbb{R}^{r\times m}$ is non-negative. It is worth mentioning that system matrix A is Metzler and system matrix B is non-negative which is a necessary and sufficient condition for (1) to be positive [1], [6], [9]. Throughout this article, it is assumed that (A, B) is stabilizable. Consider the commonly used state-feedback protocol for consensus

$$u_i(t) = K \sum_{v_j \in \mathscr{N}_i} [\mathscr{A}]_{ij}(x_j(t) - x_i(t)), \quad i \in \mathbb{Z}_m. \tag{2}$$

One can define

$$X(t) := [x_1^{\mathrm{T}}(t),\ x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$$

and obtain an edge networked system as

$$\dot{X}(t) = \mathbf{A}X(t) \tag{3}$$

where $\mathbf{A} = I_m \otimes A - (L \otimes BK)$.

On the basis of the aforementioned definitions, the following non-negative consensus problem for the edge networked system (3) is studied.

Non-Negative Edge Consensus Problem: Find a protocol (2) such that, for any non-negative $X(0)$, $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0\ \forall i, j \in \mathbb{Z}_m$, and $X(t) \succeq 0$ for $t \succeq 0$.

For *NECP*, similar to [32], it is assumed that the communication topology among edges is represented by a line graph that is undirected and connected. In their work, a detailed analysis result was developed by using the information of the number of nodes and edges of a nodal graph, and a sufficient condition was obtained. According to the line in the analysis of their work using positive systems theory and graph theory, we notice that *NECP* is solvable if and only if $\mathbf{A}$ is Metzler and $A_i := A - \lambda_i BK$, $i \in \mathbb{Z}_m\backslash\{1\}$, are Hurwitz. This directly derives the necessary and sufficient analysis condition for *NECP* as follows.

Proposition 1: Consider the edge networked system in (3) with an undirected, connected line graph, *NECP* is solvable if and only if the following conditions hold: 1) matrix BK is non-negative; 2) matrix $A - d_{\max}BK$ is Metzler; and 3) matrices $A_iP_i + P_i^{\mathrm{T}}A_i^{\mathrm{T}} < 0$, $P_i > 0$, $i \in \mathbb{Z}_m\backslash\{1\}$.

For the comparison purpose, a sufficient analysis condition is also provided in an equivalent form as follows.

Lemma 1 [32, Th. 1]: Consider the edge networked system in (3) with an undirected, connected line graph, *NECP* is solvable if the following conditions hold: 1) matrix BK is non-negative; 2) matrix $A-(4m^2-2m)BK$ is Metzler; and 3) matrix $A-(4/(m^2-m))BK$ is Hurwitz where m denotes the number of edges.

In Lemma 1, since an upper bound of $d_{\max}$ and λ_m is estimated by $4m^2-2m$, and a lower bound of λ_2 is estimated by $4/(m^2-m)$, one has $d_{\max} \prec 4m^2-2m$, $\lambda_m \preceq 4m^2-2m$, and $\lambda_2 \succeq 4/(m^2-m)$. With $BK \succeq 0$, one has $A-(4/(m^2-m))BK \succeq A_2 \succeq A_3 \succeq \ldots \succeq A_m \succ A-(4m^2-2m)BK$ and $A-d_{\max}BK \succeq A-(4m^2-2m)BK$. Then, based on the property of Metzler matrices (see [32, Lemma 8]), one has $A-d_{\max}BK$ is Metzler and $A-\lambda_i BK$, $i \in \mathbb{Z}_m\backslash\{1\}$, are Metzler and Hurwitz if the conditions in Lemma 1 hold, which indicates that *NECP* is solved according to Proposition 1.

The Laplacian matrix of an undirected, connected graph with m nodes has the following fact that $d_{\max} \preceq m-1 \prec m$ and $\lambda_m \preceq m$ [2]. Notice that $4m^2-2m-m = m(4m-3) \succ 0$ for $m \succeq 1$, a better upper bound of $d_{\max}$ and λ_m should be m rather than $4m^2-2m$ and, thus, an improved result of Lemma 1 without using the global information of a graph can be obtained as follows.

Proposition 2: Consider the edge networked system in (3) with an undirected, connected line graph, *NECP* is solvable if the following conditions hold: 1) matrix BK is non-negative; 2) matrix $A-mBK$ is Metzler; and 3) matrix $A-(4/(m^2-m))BK$ is Hurwitz, where m denotes the number of edges.

Remark 1: The gap between the two upper bounds diverges at a rate equal to $m(4m-3)$ for $m \succeq 1$ as m increases.

Since the asymptotic stability of continuous-time positive linear systems admits a linear co-positive Lyapunov function [6], a non-negative edge consensus design condition represented in the form of linear programming can be obtained in the following proposition.

Proposition 3: Considering the edge networked system in (3) with an undirected, connected line graph, *NECP* is solvable if there exist a diagonal matrix $D > 0$ and a matrix U such that the following linear program holds: 1) BU is non-negative; 2) $AD-mBU$ is Metzler; and 3) $(AD-(4/(m^2-m))BU)\mathbf{1}_r$ is negative where m denotes the number of edges. When these conditions hold, $K = UD^{-1}$.

Proof: Notice that $K = UD^{-1}$ and $D > 0$ imply that $U = KD$. Substituting it into the conditions, one can see that conditions 1) and 2) are equivalent to those in Proposition 2 since matrix $D > 0$ is diagonal. Also, condition 3) becomes $(A-(4/(m^2-m))BK)D\mathbf{1}_r \prec 0$. According to the asymptotic stability condition of positive linear systems [6], one can conclude that $A-(4/(m^2-m))BK$ is Hurwitz. This completes the proof. ■

All the results in [32] and our Propositions 2 and 3 are sufficient conditions for the existence of a solution. However, a necessary and sufficient condition is always desirable for the solvability of *NECP*. In the following, we will focus on Proposition 1 which is a necessary and sufficient condition, and give some equivalent conditions of it for the solvability of *NECP*. Two novel equivalent descriptions of Proposition 1 are obtained with the introduction of slack matrix variables.

Theorem 1: Considering the edge networked system in (3) with an undirected, connected line graph, *NECP* is solvable if there exist matrices $P_i > 0$, G_i, H_i, $i \in \mathbb{Z}_m\backslash\{1\}$, and a diagonal matrix $Q > 0$ such that the following conditions hold: 1) matrix BKQ is non-negative; 2) matrix $AQ-d_{\max}BKQ$ is Metzler; and 3)

$$\Sigma_{1i} := \begin{bmatrix} \bar{\Sigma}_{1i} & P_i^{\mathrm{T}} - G_i^{\mathrm{T}} + AH_i^{\mathrm{T}} & BKQ - \lambda_i G_i^{\mathrm{T}} \\ * & -H_i - H_i^{\mathrm{T}} & -\lambda_i H_i \\ * & * & -Q \end{bmatrix} < 0$$
$$i \in \mathbb{Z}_m\backslash\{1\} \tag{4}$$

where $\bar{\Sigma}_{1i} = AG_i + G_i^{\mathrm{T}}A^{\mathrm{T}} - BKQK^{\mathrm{T}}B^{\mathrm{T}}$.

Proof: Notice that $A_i = A - \lambda_i BK$, $i \in \mathbb{Z}_m\backslash\{1\}$ has been defined previously. Obviously, since $Q > 0$ is diagonal, conditions 1) and 2) are equivalent to those in Proposition 1. Then, we need to prove the equivalence of conditions 3).

Sufficiency: Let

$$T_1 := \begin{bmatrix} I & 0 & BK \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Premultiplying and postmultiplying Σ_{1i} by T_1 and T_1^{T}, respectively, gives

$$\begin{aligned}\Sigma_{2i} &:= T_1 \Sigma_{1i} T_1^{\mathrm{T}} \\ &= \begin{bmatrix} A_iG_i + G_i^{\mathrm{T}}A_i^{\mathrm{T}} & P_i - G_i^{\mathrm{T}} + A_iH_i^{\mathrm{T}} & -\lambda_i P_i \\ * & -H_i - H_i^{\mathrm{T}} & 0 \\ * & * & -Q \end{bmatrix} < 0 \\ & i \in \mathbb{Z}_m\backslash\{1\}.\end{aligned}$$

Let

$$T_{2i} = \begin{bmatrix} I & A_i & 0 \end{bmatrix}, \quad i \in \mathbb{Z}_m\backslash\{1\}.$$

Premultiplying and postmultiplying Σ_{2i} by T_{2i} and T_{2i}^{T}, respectively, gives

$$A_iP_i + P_iA_i^{\mathrm{T}} < 0, \quad i \in \mathbb{Z}_m\backslash\{1\}$$

which is equivalent to condition 3) in Proposition 1.

Necessity: If condition 3) in Proposition 1 holds, one can always find a diagonal matrix $\bar{Q} > 0$ and a sufficiently large scalar $c \succ 0$ such that

$$A_iP_i + P_iA_i^{\mathrm{T}} + \frac{\lambda_i^2}{c}P_i\bar{Q}^{-1}P_i < 0, \; P_i > 0, \; i \in \mathbb{Z}_m\backslash\{1\}. \tag{5}$$

Letting $Q = c\bar{Q}$, then (5) becomes

$$A_iP_i + P_iA_i^{\mathrm{T}} + \lambda_i^2 P_iQ^{-1}P_i < 0, \; P_i > 0, \; i \in \mathbb{Z}_m\backslash\{1\}. \tag{6}$$

If (6) holds, there must exist matrices $\bar{H}_i > 0$, $i \in \mathbb{Z}_m\backslash\{1\}$, and a sufficiently small scalar $d > 0$ such that

$$\begin{cases} -d\bar{H}_i - d^2\bar{H}_i^{\mathrm{T}}A_i(A_iP_i + P_iA_i^{\mathrm{T}} + \lambda_i^2P_iQ^{-1}P_i)^{-1}A_i^{\mathrm{T}}\bar{H}_i < 0 \\ P_i > 0, \quad i \in \mathbb{Z}_m\backslash\{1\} \end{cases} \tag{7}$$

hold. Selecting $G_i = P_i$ and $H_i = d\bar{H}_i$, (7) becomes

$$\begin{cases} -H_i - H_i^{\mathrm{T}}A_i(A_iG_i + G_i^{\mathrm{T}}A_i^{\mathrm{T}} + \lambda_i^2P_iQ^{-1}P_i)^{-1}A_i^{\mathrm{T}}H_i < 0 \\ P_i > 0, \quad i \in \mathbb{Z}_m\backslash\{1\}. \end{cases} \tag{8}$$

By Schur complement equivalence, (8) is equivalent to $\Sigma_{2i} < 0$, $i \in \mathbb{Z}_m\backslash\{1\}$, which has further implied that $\Sigma_{1i} = T_1^{-1}\Sigma_{2i}T_1^{-\mathrm{T}} < 0$, $i \in \mathbb{Z}_m\backslash\{1\}$, hold.

In summary, if

$$A_iP_i + P_iA_i^{\mathrm{T}} < 0, \quad P_i > 0, \quad i \in \mathbb{Z}_m\backslash\{1\}$$

hold, one can always find matrices $G_i = P_i > 0$, $H_i > 0$, $i \in \mathbb{Z}_m\backslash\{1\}$, and a diagonal matrix $Q > 0$ such that $\Sigma_{1i} < 0$, $i \in \mathbb{Z}_m\backslash\{1\}$, hold. This completes the proof. ■

The most important characteristic in Theorem 1 is that controller K has been further separated from Lyapunov matrices P_i thanks to the slack matrix variables. However, by observing (4), one can see that it remains a nonlinear matrix inequality problem, which is not easy to solve. This motivates us to give another equivalent condition for the solution in the following.

Theorem 2: Consider the edge networked system in (3) with an undirected, connected line graph, *NECP* is solvable if there exist matrices $P_i > 0$, G_i, H_i, $i \in \mathbb{Z}_m\backslash\{1\}$, S, and a diagonal matrix $Q > 0$ such that the following conditions hold: 1) matrix BS is non-negative; 2) matrix $AQ - d_{\max}BS$ is Metzler; and 3)

$$\Pi_i := \begin{bmatrix} \bar{\Pi}_i & P_i - G_i^{\mathrm{T}} + AH_i^{\mathrm{T}} & BS - \lambda_i G_i^{\mathrm{T}} \\ * & -H_i - H_i^{\mathrm{T}} & -\lambda_i H_i \\ * & * & -Q \end{bmatrix} < 0, \; i \in \mathbb{Z}_m\backslash\{1\} \tag{9}$$

where $\bar{\Pi}_i = AG_i + G_i^{\mathrm{T}}A^{\mathrm{T}} - BSM^{\mathrm{T}}B^{\mathrm{T}} - BMS^{\mathrm{T}}B^{\mathrm{T}} + BMQM^{\mathrm{T}}B^{\mathrm{T}}$. When these conditions hold, $K = SQ^{-1}$.

Proof: Notice that $K = SQ^{-1}$ gives rise to $S = KQ$ and we substitute it into the conditions. One can see that conditions 1) and 2) are equivalent to those in Theorem 1. We need to prove the equivalence of condition 3) in the following.

Sufficiency: It follows from $\bar{\Pi}_i = AG_i + G_i^{\mathrm{T}}A^{\mathrm{T}} - BSM^{\mathrm{T}}B^{\mathrm{T}} - BMS^{\mathrm{T}}B^{\mathrm{T}} + BMQM^{\mathrm{T}}B^{\mathrm{T}} = AG_i + G_i^{\mathrm{T}}A^{\mathrm{T}} - BKQK^{\mathrm{T}}B^{\mathrm{T}} + B(K-M)Q(K-M)^{\mathrm{T}}B^{\mathrm{T}} = \bar{\Sigma}_{1i} + B(K-M)Q(K-M)^{\mathrm{T}}B^{\mathrm{T}}$ and $B(K-M)Q(K-M)^{\mathrm{T}}B^{\mathrm{T}} \geq 0$ that $\Sigma_{1i} \leq \Pi_i < 0$, $i \in \mathbb{Z}_m\backslash\{1\}$.

Necessity: If (4) holds, there must exist a matrix $M = K$ such that $-BKQK^{\mathrm{T}}B^{\mathrm{T}} = -BKQM^{\mathrm{T}}B^{\mathrm{T}} - BMQK^{\mathrm{T}}B^{\mathrm{T}} + BMQM^{\mathrm{T}}B^{\mathrm{T}}$ and, thus, (4) gives rise to (9) due to $S = KQ$. This completes the proof. ■

Theorems 1 and 2 are two equivalent conditions of Proposition 1. Notice that $\Pi_i < 0$, $i \in \mathbb{Z}_m\backslash\{1\}$, are nonlinear matrix inequalities with respect to the variables to be solved in Theorem 2. However, if matrix M is fixed, they all become linear matrix inequalities that are convex. By defining $\Pi_i < \epsilon I$, $i \in \mathbb{Z}_m\backslash\{1\}$, and minimizing ϵ with respect to M, one can see from the proof in Theorem 2 that when $M = K$, the minimal ϵ is obtained. Therefore, one can fix M and minimize ϵ, and then update M as K. Repeating such a procedure, a heuristic iterative algorithm based on Theorem 2 is developed in Algorithm 1 [non-negative edge consensus problem with undirected graphs (*NECPUG*)].

Remark 2: In step 1, a matrix $M^{(1)}$, which guarantees the edge consensus of networked systems without the requirement of non-negativevity, is found for initializing the algorithm. Such a process can be easily realized since a lot of consensus approaches for general linear systems have been developed [11], [33].

Algorithm 1 NECPUG

Step 1: Set $k = 1$ and $\epsilon^{(0)} = 0$. Find an $M^{(1)}$ such that $A - \lambda_i BM^{(1)}$, $i \in \mathbb{Z}_m\backslash\{1\}$, are Hurwitz.

Step 2: Fix $M = M^{(k)}$, minimize $\epsilon^{(k)}$ s.t. $\{BS \succeq 0,\ AQ - d_{\max}BS \in \mathbb{M}^r,\ \Pi_i < \epsilon^{(k)}I,\ i \in \mathbb{Z}_m\backslash\{1\}\}$ with respect to $P_i > 0$, G_i, H_i, S, and a diagonal matrix $Q > 0$, $i \in \mathbb{Z}_m\backslash\{1\}$. If $\epsilon^{(k)} \preceq 0$, $K = SQ^{-1}$. *STOP*. Otherwise, go to next step.

Step 3: If $|\epsilon^{(k)} - \epsilon^{(k-1)}|/\epsilon^{(k)} \prec \theta$, where θ is a prescribed tolerance, then this algorithm fails to find the desired solution. *STOP*. Otherwise, set $k = k + 1$, update $M^{(k)} = SQ^{-1}$, then go to step 2.

Remark 3: Generally speaking, the major advantage of our approach lies in: 1) the slack matrix variable Q in our approach is not only independent of the Lyapunov matrices P_i but also able to parametrize the controller gain to preserve the non-negativevity without introducing any conservatism and 2) the introduced slack matrix variables G_i, H_i, and Q are expected to provide some additional flexibility for improving the iterative computation of the algorithm [3], [12].

Remark 4: If the topologies of the edge networked system are described by directed graphs each containing a spanning tree, the *NECP* would become more complicated due to the interplay between the eigenvalues of the Laplacian matrix and the controller gains. Specifically, the problem would involve complex eigenvalues in general, the Hurwitzness of complex matrices, as well as non-negativevity constraints, which make the analysis of the Laplacian matrix difficult. Another issue that should be addressed is the relationship between nodal graph and line graph guaranteeing the non-negative edge consensus while the nodal graph is directed. Once these issues are tackled appropriately, we believe that our methods could be extended to solve the *NECP* defined on directed graphs.

IV. Illustrative Examples

This section gives three illustrative examples to compare our proposed approaches with the existing one in [32].

A. Example 1

In this example, we will show that the solution set of our approach is a strict superset of that provided by the existing approach. Notice that in the existing work [32], all the edge dynamic systems in the simulations are positive linear systems that are marginally stable, having one zero eigenvalue and one negative eigenvalue. Let us take the system (28) from [32, Sec. IV] the following example:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

whose spectrum are $\{-2,\ 0\}$ and the communication topology is represented by the graph in [32, Fig. 2]. Obviously, this system is stabilizable. It is worth mentioning that in both Lemma 1 and Proposition 2, conditions 1) and 2) guarantee the nonnegaivity while condition 3) is concerned with the consensuability. Notice that the conditions 1) and 3) in Lemma 1 and Proposition 2 are identical, so it suffices to analyze their conditions 2). The Laplacian matrix of an undirected, connected graph with m nodes has the following fact that $d_{\max} \preceq m-1 \prec m$ [2]. Also, we have that $4m^2-2m-m=m(4m-3)\succ 0$ for $m\succeq 1$. Hence, we can conclude that $4m^2-2m\succ m\succ d_{\max}$ for $m\succeq 1$. Notice that the condition 1) matrix BK is non-negative in Lemma 1 and Propositions 1 and 2. By observing condition 2) of Proposition 1 (which is a necessary and sufficient condition for nonenagtive edge consensus), and comparing the conditions 2) of Lemma 1 and Proposition 2, we conclude that $A-(4m^2-2m)BK\succ A-mBK\succ A-d_{\max}BK$ are all Metzler for $m\succeq 1$. This is also equivalent to

$$\min_{i\neq j,[BK]_{ij}\neq 0}\frac{[A]_{ij}}{[BK]_{ij}}\succeq 4m^2-2m\succeq m\succeq d_{\max}\succ 0 \quad (10)$$

for $i,j\in\mathbb{Z}_m$ and $m\succeq 1$. From (10), we can see that a tighter upper bound of $d_{\max}$ should be m rather than $4m^2-2m$ since the conditions from Proposition 2 can provide a larger solution set than from Lemma 1. In other words, the set of solutions by our approach is a strict superset of that provided by Lemma 1. Moreover, the advantage increases linearly with m. In this example, since $m=16$ and $d_{\max}=8$, we have that $4m^2-2m=992\succ m=16\succ d_{\max}=8$, which is consistent with the theoretical analysis.

By observing the conditions in Lemma 1 and Proposition 2, we know that only condition (ii) between them is different and the other two are identical. One is that $A-992BK$ is Metzler and the other one is that $A-16BK$ is Metzler. The "Metzler region" represents the set of controller K such that the non-negativevity of the edge networked system is preserved. Letting $K=[k_1,\ k_2]$ and noticing the condition (i) $BK\succeq 0$, the Metzler region corresponding to $A-992BK$ can be expressed as

$$\begin{cases}0\preceq k_1\preceq \frac{1}{992}\\ 0\preceq k_2\preceq \frac{1}{992}.\end{cases}$$

The Metzler region corresponding to $A-16BK$ can be expressed as

$$\begin{cases}0\preceq k_1\preceq \frac{1}{16}\\ 0\preceq k_2\preceq \frac{1}{16}.\end{cases}$$

It can be clearly seen that the latter one from Propositon 2 in this article is a strict superset of the former one from Lemma 1. Indeed, the area of the gain parameter region (Proposition 2) is 3844 times that in the existing approach (Lemma 1). This is also consistent with the previous analysis that a tighter upper bound of $d_{\max}$ should be m rather than $4m^2-2m$. In other words, the results proposed in Propositions 2 are significantly less conservative than Lemma 1.

Besides, based on the co-positive Lyapunov function [7], Proposition 3, which is an equivalent condition of Proposition 2, is obtained in the form of linear programming. We can use the linear program in Proposition 3 to solve the controller directly. Solving the linear program in Proposition 3 using MATLAB R2014a gives a feasible solution as

$$K=\begin{bmatrix}0.06 & 0.06\end{bmatrix}. \quad (11)$$

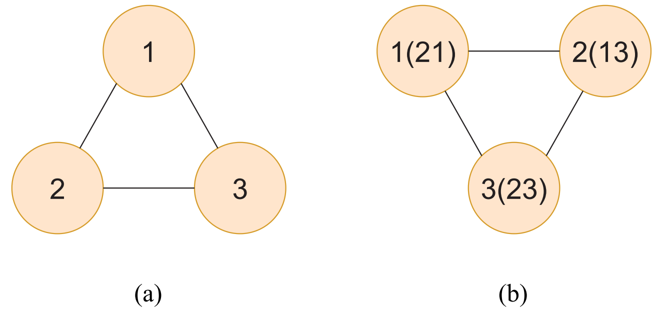


Fig. 1. (a) Nodal graph. (b) Line graph.

Substituting (11) into Lemma 1, we found that

$$A-992BK=\begin{bmatrix}-60.520 & -58.520\\ -58.520 & -60.520\end{bmatrix}$$

is not Metlzer. On the contrary, substituting (11) into Proposition 2, we found that

$$A-16BK=\begin{bmatrix}-1.96 & 0.04\\ 0.04 & -1.96\end{bmatrix}$$

is Metlzer. The solution in [32] is also presented here as follows:

$$K=\begin{bmatrix}0.001 & 0.001\end{bmatrix}. \quad (12)$$

With controller (12), it can be verified that both

$$A-992BK=\begin{bmatrix}-1.992 & 0.0008\\ 0.0008 & -1.992\end{bmatrix}$$

and

$$A-16BK=\begin{bmatrix}-1.0160 & 0.9840\\ 0.9840 & -1.0160\end{bmatrix}$$

are all Metzler. Therefore, Proposition 2 (or 3) has provided a controller (11) not belonging to the solution set of Lemma 1, while Lemma 1 has provided a controller (12) belonging to the solution sets of both Proposition 2 (or 3) and Lemma 1. The results could further indicate that Proposition 2 (or 3) provides more feasible solutions than Lemma 1, which is also consistent with the previous theoretical analysis.

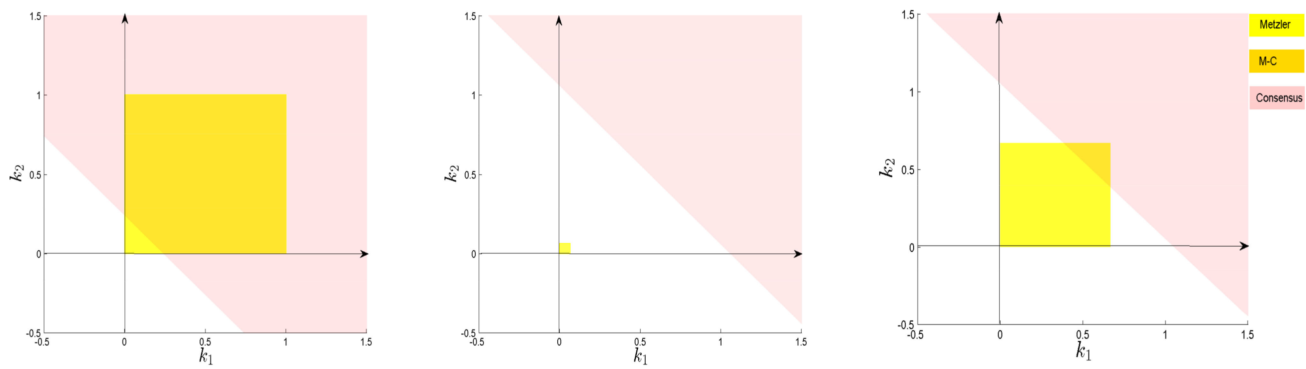
B. Example 2

In Example 1, we consider the case of marginally stable agents. In order to compare our approaches with the existing work thoroughly, in the following sections, we consider the case of unstable agents.

To show the effectiveness of the approaches, we consider an edge networked system whose edges are unstable with system matrices. To intuitively compare the conservatism of Propositions 1 and 2 as well as Lemma 1, we consider a positive edge networked system in (3) and each edge dynamic system has the following system matrices:

$$A=\begin{bmatrix}-1.3 & 2\\ 2 & -1.3\end{bmatrix},\quad B=\begin{bmatrix}1\\ 1\end{bmatrix}$$

whose spectrum is $\{-3.3,\ 0.7\}$. Obviously, this system is stabilizable. The nodal graph and line graph, each comprising three nodes and three edges, are shown in Fig. 1 from which



(a) (b) (c)

Fig. 2. Metzler and consensus regions. (a) Proposition 1. (b) Lemma 1 [31, Th. 1]. (c) Proposition 2.

we can see that they are identical. The Laplacian matrix of the line graph is as follows:

$$\mathscr{L} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Then, we have $d_{\max} = 2$, $\lambda_2 = \lambda_3 = 3$. Lemma 1 gives an upper bound of $d_{\max}$ and λ_m as $4m^2 - 2m = 30$ while Proposition 2 gives $m = 3$. Let $K = [k_1, k_2]$. The "consensus region" represents the set of K such that the consensus of edge networked system is achieved. To illustrate the conservatism of these analysis results for *NECP*, the Metzler and consensus regions of Propositions 1 and 2 and Lemma 1 are plotted in Fig. 2. Consensus region represents the set of K such that the consensus of edge networked system is achieved. Specifically, K in the Metzler region expressed by yellow can guarantee the non-negativevity. Using the Routh–Hurwitz stability criterion, it is found that the consensus region of Proposition 1 is determined by the following linear inequality:

$$k_1 + k_2 \succ \frac{7}{30}$$

while that of Lemma 1 and Proposition 2 is determined by the linear inequality

$$k_1 + k_2 \succ \frac{21}{20}.$$

It can be seen that the consensus regions of them are unbounded. If K is located at the consensus region expressed by red, then edge consensus is achieved. The M-C region, which is a common region of consensus and Metzler regions, expressed by orange, is exactly the feasible solution region of *NECP*. From Fig. 2(b), we can see that its Metzler region is a strict superset of the other ones, and the feasible solution region is empty. Apparently, this situation would get worse and worse as the number of edges increases. Therefore, the existing approaches, which are based on the analysis condition in Lemma 1, cannot give a solution for *NECP*. Fortunately, since both Propositions 1 and 2 have their feasible solution regions expressed by orange in Fig. 2(a) and (c), they can be used to derive some consensuability and non-negativevity synthesis approaches that have less or no conservatism for the solvability of *NECP*. According to the discussions on the conservatism of Propositions 1 and 2, we know that Propositions 1 and 2 have their feasible solution regions expressed by orange in Fig. 2(a) and (c). Therefore, we can use the approach of Proposition 3 (based on Proposition 2), and Algorithm *NECPUG* (based on Proposition 1) to solve *NECP*. Solving the linear program in Proposition 3 using MATLAB R2014a gives a feasible solution as

$$K = \begin{bmatrix} 0.6125 & 0.6039 \end{bmatrix}. \tag{13}$$

With controller (13), we have

$$\begin{aligned} BK &= \begin{bmatrix} 0.6125 & 0.6039 \\ 0.6125 & 0.6039 \end{bmatrix} \succeq 0 \\ A_2 = A_3 &= A - mBK \\ &= \begin{bmatrix} -3.1375 & 0.1883 \\ 0.1625 & -3.1117 \end{bmatrix} \in \mathbb{M}^2 \\ A - (4/(m^2 - m))BK &= \begin{bmatrix} -1.7083 & 1.5974 \\ 1.5917 & -1.7026 \end{bmatrix} \in \mathbb{M}^2 \end{aligned}$$

whose eigenvalues are $\{-0.11093, -3.3\}$. The conditions in Proposition 2 have been satisfied.

By Algorithm *NECPUG*, a feasible solution is obtained as

$$K = \begin{bmatrix} 0.8134 & 0.1158 \end{bmatrix}. \tag{14}$$

With controller (14), we have

$$\begin{aligned} BK &= \begin{bmatrix} 0.8134 & 0.1158 \\ 0.8134 & 0.1158 \end{bmatrix} \succeq 0 \\ A - d_{\max}BK &= \begin{bmatrix} -2.9268 & 1.7684 \\ 0.3732 & -1.5316 \end{bmatrix} \in \mathbb{M}^2 \\ A_2 = A_3 &= \begin{bmatrix} -3.7402 & 1.6526 \\ -0.4402 & -1.6474 \end{bmatrix} \notin \mathbb{M}^2 \end{aligned}$$

whose eigenvalues are $\{-2.0876, -3.3\}$. The conditions in Proposition 1 have been satisfied. Let the initial conditions of Agents 1–3, respectively, be

$$\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} x_{21}(0) \\ x_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}, \quad \begin{bmatrix} x_{31}(0) \\ x_{32}(0) \end{bmatrix} = \begin{bmatrix} 10 \\ 50 \end{bmatrix}. \tag{15}$$

The non-negative edge consensus results using controllers (13) and (14) are shown in Figs. 3 and 4 where the small

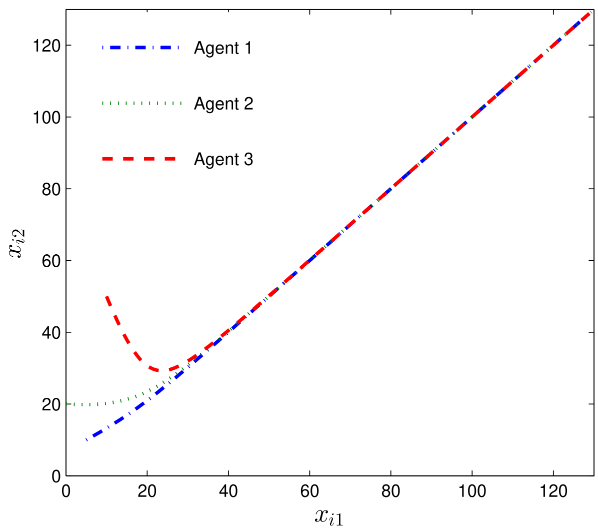


Fig. 3. Non-negative edge consensus using controller (13).

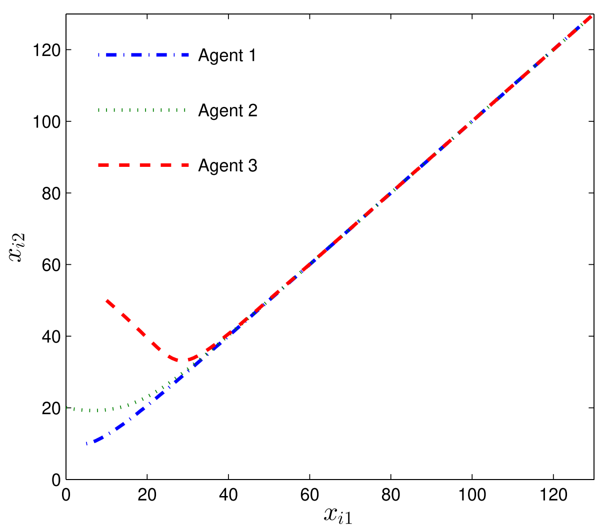


Fig. 4. Non-negative edge consensus using controller (14).

circle denotes the initial trajectory point of each agent. From the two figures, we can see that the edge consensus has been achieved and the states of edges remained non-negative.

In this example, the existing approach provided an empty solution region since their theoretical results are more conservative than ours, which has been shown in Fig. 2(b). Notice that controller (13) shows that A_2 and A_3 are Metzler while those given by controller (14) are not.

Through Proposition 3, represented by linear programming, our proposed approach is not just a sufficient condition for solvability but also an efficient one that is more than the semidefinite programming approach, from a computational perspective.

This is because A_i, $i \in \mathbb{Z}_m \backslash \{1\}$, being Metzler, are necessary in Propositions 2 and 3, which are sufficient conditions for the solvability of *NECP*. Although the linear programming approach of Proposition 3 is a sufficient condition for the solvability, however, it is generally more efficient than

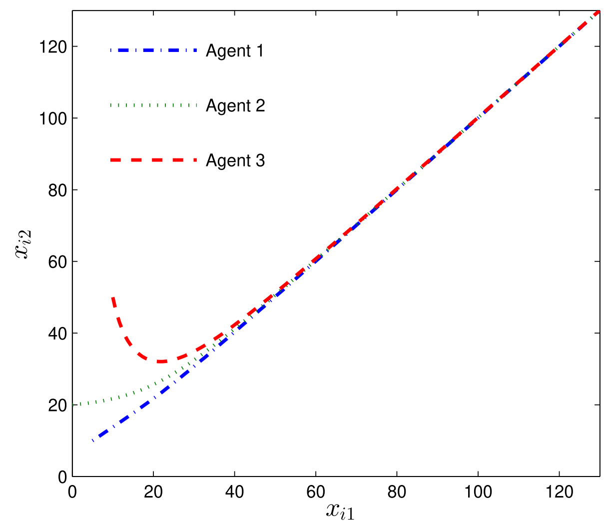


Fig. 5. Non-negative edge consensus using controller (16).

the semidefinite programming approach from a computational point of view.

C. Example 3

Consider a positive edge networked system in (3) and each edge dynamic system has the following system matrices:

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

whose eigenvalues are $\{-3,\ 1\}$. Obviously, this system is stabilizable and has one unstable pole 1. Assume that it has a similar communication topology as that in case 1.

Using the approach in Proposition 3 and the existing work, no feasible solution has been found. Then, we use Algorithm *NECPUG*, and a feasible solution is obtained as

$$K = \begin{bmatrix} 0.5398 & 0.9126 \end{bmatrix}. \tag{16}$$

With controller (16), we have

$$BK = \begin{bmatrix} 0.5398 & 0.9126 \\ 0.5398 & 0.9126 \end{bmatrix} \succeq 0$$

$$A - d_{\max} BK = \begin{bmatrix} -2.0796 & 0.1748 \\ 0.9204 & -2.8252 \end{bmatrix} \in \mathbb{M}^2$$

$$A_2 = A_3 = \begin{bmatrix} -2.6194 & -0.7378 \\ 0.3806 & -3.7378 \end{bmatrix} \notin \mathbb{M}^2$$

whose eigenvalues are $\{-3,\ -3.3572\}$. The conditions in Proposition 1 have been satisfied. In this example, Algorithm *NECPUG* has solved it successfully since it has no conservatism. Similarly, we let the initial conditions of Agents 1–3 be (15). The non-negative edge consensus result using controller (16) is shown in Fig. 5 where the small circle denotes the initial trajectory point of each agent. From the figure, we can see that the edge consensus has been achieved and the states of edges remained non-negative.

A summary of the three illustrative examples is given as follows.

1) The first example has shown that, though both our approach and the existing approach can provide feasible

solutions, the set of solutions by our approach is a strict superset of that provided by the existing approach.

2) In the second example, an analytical and numerical comparison of solution regions by two different approaches was carried out. We can see that the feasible solution region by the existing approach is empty while our approaches can solve the problem, which indicates our approaches are less conservative.

3) From the third example, it has been shown that Algorithm *NECPUG* has solved the problem successfully since it is developed via the necessary and sufficient conditions of Theorem 2 (which is derived on the basis of Proposition 1 and Theorem 1).

V. Conclusion

In this article, the *NECP* has been addressed and solved for positive networked systems with undirected graphs using state-feedback protocols. An improved upper bound has been given for the maximum eigenvalue of the Laplacian matrix and the (out) in-degree of the degree matrix. This can lead to an improved consensuability and non-negativevity condition. In addition, by introducing some slack matrix variables, two necessary and sufficient conditions of consensuability and non-negativevity have been given such that the system matrices, controller gain, as well as Lyapunov matrices are separated. The conditions can lead to a semidefinite programming algorithm for solvability. The proposed results have been verified with comparisons via three illustrative examples.

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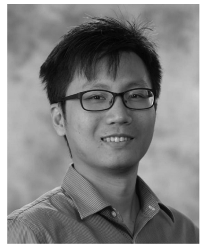

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