Necessary and Sufficient Conditions on Consensus of General Fractional-Order Multi-agent Systems over Directed Networks

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Abstract—This paper tackles the consensus problem for a group of multi-agent systems communicating over a directed network. Our networked system consists of general fractional-order dynamic systems with order $\alpha \in [1, 2)$ in continuous time. This presents a more challenging issue than the previous research on simple single/double fractional-order integrators. Specifically, we investigate the consensus issue for agents described by fractional-order systems with general linear dynamics, which presents a challenge due to the limited applicability of existing tools for integer-order models. To address this, we utilize spectral graph theory and fractional-order systems theory to derive several equivalent conditions without conservatism for such systems where agents communicate through directed graphs. We develop a tractable convex programming algorithm for controller design based on the obtained results. We then demonstrate the effectiveness of our proposed approach through simulations on higher-order dynamic systems and fractional-order circuits.

Index Terms—Cooperative control, consensus problem, directed graphs, networked fractional-order systems

1 Introduction

1.1 Background

The study of collective behaviours in networked multi-agent systems has attracted significant attention in the last decade. This interest stems from the inspiration drawn from natural phenomena, such as flocking in birds and swarming in insects, as well as the wide range of engineering applications, including the formation of multi-robot systems and unmanned vehicles. The consensus problem, which requires agreement among all agents using only local information, is the focus of collective behaviours among agents. Extensive research has been conducted on the consensus problem for agents with integer-order dynamics, such as first/second-order models [15], [24], [25] and higher-order models [7], [20], [26]. However, numerous natural phenomena and engineering applications are more accurately characterized by non-integer-order/fractional-order dynamics [1], [8], [22]. This observation has motivated our investigation of the consensus issue in networked systems with fractional-order dynamics.

1.2 Related Work

In recent years, significant effort has been devoted to addressing the consensus issue in networked fractional-order systems. The issue was first addressed in [2], where agents consisted of fractional-order single integrators and interesting convergence speed and fractional-order results were revealed. Shen *et al*. investigated fractional-order single integrators with time delay and proposed necessary and sufficient conditions for consensus in [17], [18]. Yu *et al*. explored the leader-following tracking consensus of double integrators and uncovered interesting results of the Laplacian matrix when the fractional order was less than 2 [30]. Liu *et al*. investigated the consensus issue for positive fractional-order and integer-order multi-agent systems on directed graphs [10], [12]. Su *et al*. proposed necessary and sufficient conditions for consensus of fractional-order single/double integrators via sampled-data control [21], [28], [31]. Gong *et al*. studied the problem of fault-tolerant consensus control for heterogeneous nonlinear fractional-order integrators in [6]. More recently, Chen *et al*. addressed the consensus problem in networks of linear fractional-order systems with order $\alpha \in (0, 1)$ over directed graphs [3], and Ye *et al*. investigated the consensus issue of networked systems in which the agents have order $\alpha \in (1, 2)$ and communicate through undirected graphs [27]. However, limited progress has been made in solving the problem of consensus in networked multi-agent systems with agents that have an order of $\alpha \in (1, 2)$ and directed graphs. As noted in [5], achieving consensus for agents with order $\alpha \in (1, 2)$ and directed graphs is challenging due to the significant differences in stability conditions compared to integer-order systems [13], [14], [23]. Therefore, investigating this problem is expected to provide a more comprehensive and innovative understanding of cooperative networked systems.

1.3 Contribution

This paper addresses the consensus issue for a class of networked multi-agent systems with general fractional-order

- *This work is supported in part by the University of Macau (SRG2022-00054-FST), and in part by the Research Grants Council (RGC) of Hong Kong (17205919, 17207020 and 17205721).*
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This article has been accepted for publication in IEEE Transactions on Network Science and Engineering. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TNSE.2023.3301015

linear models, in which the order $\alpha \in (1, 2)$ and the topology are *directed*, to fill the literature gap in the field of cooperative networked systems. The major contributions of this work in comparison to previous studies [3], [5], [11], [27], [29] are as follows: First, a novel stability characterization is derived for complex fractional-order linear systems (see Lemma 1); Second, a consensus analysis condition without conservatism is proposed; Third, a tractable convex programming algorithm is developed for consensus synthesis of networked systems with order $\alpha \in (1, 2)$ and directed graphs.

Notations: We use $\mathbb{R}$ (or $\mathbb{C}$) to denote the set of real (or complex) numbers. The symbol j ($j^2 = -1$) represents the imaginary unit. The Hermitian transpose (or conjugate transpose) of a complex matrix $\mathcal{X}$ is denoted by $\mathcal{X}^*$. For two real symmetric matrices $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{n\times n}$, we use $\mathcal{X} > \mathcal{Y}$ (respectively, $\mathcal{X} \geq \mathcal{Y}$) to indicate that $\mathcal{X} - \mathcal{Y}$ is positive definite (respectively, positive semidefinite). We use the symbol $\mathcal{X} \succ \mathcal{Y}$ (respectively, $\mathcal{X} \succeq \mathcal{Y}$) to represent that $\mathcal{X} - \mathcal{Y}$ is positive (respectively, nonnegative) for any real matrices $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{m\times n}$. We use the symbol $\mathcal{X} \in \mathbb{H}$ to represent that matrix $\mathcal{X}$ is Hurwitz. The real part of a complex matrix $\mathcal{X} \in \mathbb{C}^{p\times p}$ is denoted by $\mathcal{R}(\mathcal{X})$, and its imaginary part is represented by $\mathcal{I}(\mathcal{X})$. We use the symbol $\arg(a)$ to represent the argument of complex number a. We use $\mathbb{N}^+$ to represent the set of positive integers. The symbol $\mathbb{N}_k^p$ is used to denote a sequence of positive integers as $k, k+1, \ldots, p$. Unless explicitly defined otherwise, we assume that all matrices' dimensions in this paper are compatible.

2 PRELIMINARIES

2.1 Problem Fundamentals

The results presented in [4], [16] are crucial in the analysis of consensus problems in networked systems with general fractional-order models and directed topologies.

2.1.1 Fundamentals of Fractional-Order Calculus

Assuming that $f(t)$ is a continuous function, we define its Caputo fractional derivative and integral of order $\alpha \in (n-1, n), n \in \mathbb{N}^+$ in the following:

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} f^{(n)}(\tau) \mathrm{d}\tau$$

and

$$\mathcal{I}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau$$

where $f^{(n)}(\cdot)$ represents the n-th order derivative of $f(\cdot)$ and $\Gamma(\cdot)$ denotes the Gamma function:

$$\Gamma(\alpha) = \int_{0}^{t} e^{-t} t^{\alpha-1} \mathrm{d}t.$$

Since we are going to discuss the linear fractional-order systems, without loss of generality, the lower limit of the fractional integrals and derivatives is assumed $t_0 = 0$ in the sequel.

Let us consider a complex fractional-order linear model as follows:

$$\mathcal{D}^\alpha x(t) = \check{\mathcal{A}} x(t) + \check{\mathcal{B}} u(t),\ \alpha \in [1, 2) \tag{1}$$

where complex matrices $\check{\mathcal{A}}$ and $\check{\mathcal{B}}$ are of appropriate dimensions, and the complex system's state and control input are denoted as $x(t)$ and $u(t)$, respectively. Moreover, we derive some useful results of system (1) and summarize them as follows.

Lemma 1. The asymptotic stability of FOS (1) (with zero input) is achieved if and only if one of the below conditions, that are equivalent, holds:
1) Assuming that $\check{\mathcal{A}}$'s eigenvalues are $\lambda_i(\check{\mathcal{A}})$, $i \in \mathbb{N}_1^p$, the following condition is satisfied: $|\arg(\lambda_i(\check{\mathcal{A}}))| > \alpha\pi/2$;
2) Define $\check{\mathcal{A}}_{\mathrm{r}} = \sin(\alpha\pi/2)\mathcal{R}(\check{\mathcal{A}}) - \cos(\alpha\pi/2)\mathcal{I}(\check{\mathcal{A}})$ and $\check{\mathcal{A}}_{\mathrm{i}} = \sin(\alpha\pi/2)\mathcal{I}(\check{\mathcal{A}}) + \cos(\alpha\pi/2)\mathcal{R}(\check{\mathcal{A}})$, then the Hurwitzness of the following real matrix is guaranteed:

$$\begin{bmatrix} \check{\mathcal{A}}_{\mathrm{r}} & \check{\mathcal{A}}_{\mathrm{i}} \\ -\check{\mathcal{A}}_{\mathrm{i}} & \check{\mathcal{A}}_{\mathrm{r}} \end{bmatrix}. \tag{2}$$

Proof. Please refer to Section 6.1. □

2.1.2 Graph Theory

One can use a graph to depict the topology of a networked system. When all the edges in a graph are directed from one node to another, it is known as a directed graph. Thus, undirected graphs are generally regarded as special cases of directed graphs. In this work, we assume that the agents of a networked system communicate via a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that is directed, and $\mathcal{V} := \{1, 2, \ldots, N\}$ is used to represent the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is employed to denote the edge set. It is assumed that, $b, c \in \mathcal{V}$, $(b, c) \in \mathcal{E}$ means that agent c can acquire the full state information of agent b. We define the path of graph $\mathcal{G}$ as a sequence $\{b, c, d, \ldots, g, h, l\}$ of which the successive tuples $(b, c), (c, d), \ldots, (g, h), (h, l) \in \mathcal{E}$. In this paper, it is assumed that the graph $\mathcal{G}$ has a spanning tree, which means that there always exist a root $b \in \mathcal{V}$ and a path starting from b to any other node $c \in \mathcal{V}$. The adjacency matrix of graph $\mathcal{G}$ is defined and represented by an $N \times N$ matrix Λ in which $[\Lambda]_{bc} = 1$ in the case that $(c, b) \in \mathcal{E}$ and $[\Lambda]_{bc} = 0$ otherwise. In addition, graph $\mathcal{G}$ is assumed to have no self-loops. The neighbor set of any node $b \in \mathcal{V}$ is defined as $\aleph_b := \{c \in \mathcal{V} : (c, b) \in \mathcal{E}\}$. For graph $\mathcal{G}$, we can define its Laplacian matrix using an $N \times N$ matrix L for $b, c \in \mathcal{V}$ as follows:

$$[L]_{bc} = \begin{cases} \sum_{m=1}^{N} [\Lambda]_{bm} & \text{if } b = c \\ -[\Lambda]_{bc} & \text{if } b \neq c \end{cases} \tag{3}$$

If the Laplacian matrix L contains a spanning tree, it usually has complex eigenvalues which are represented by λ_k, $k \in \mathbb{N}_1^N$. Then we can order them as $0 = \mathcal{R}(\lambda_1(L)) < \mathcal{R}(\lambda_2(L)) \leq \ldots \leq \mathcal{R}(\lambda_N(L))$.

2.2 Problem Setting

Let us consider a networked system constructed by N agents that are identical and connected through a directed graph. The dynamics of each agent is modelled as the following state-space form,

$$\mathcal{D}^\alpha x_k(t) = \mathcal{A} x_k(t) + \mathcal{B} u_k(t),\ \ k \in \mathbb{N}_1^N \tag{4}$$

where agent k's order $\alpha \in [1, 2)$, $x_k(t) := [x_{k1}, x_{k2}, \ldots, x_{kp}]^{\mathrm{T}} \in \mathbb{R}^p$ represents agent k's state, and $u_k(t) \in \mathbb{R}^r$ represents agent k's input. In addition, the pair $(\mathcal{A}, \mathcal{B})$ is assumed stabilizable.

$$\Phi_k := \begin{bmatrix} \mathcal{A}\sin(\alpha\pi/2) - \theta_k \mathcal{B}K & \mathcal{A}\cos(\alpha\pi/2) - \gamma_k \mathcal{B}K \\ -\mathcal{A}\cos(\alpha\pi/2) + \gamma_k \mathcal{B}K & \mathcal{A}\sin(\alpha\pi/2) - \theta_k \mathcal{B}K \end{bmatrix},$$
$$\theta_k := \sin(\alpha\pi/2)\mathcal{R}(\lambda_k(L)) - \cos(\alpha\pi/2)\mathcal{I}(\lambda_k(L)),$$
$$\gamma_k := \sin(\alpha\pi/2)\mathcal{I}(\lambda_k(L)) + \cos(\alpha\pi/2)\mathcal{R}(\lambda_k(L)).$$

A state-feedback law is employed in this paper:

$$u_k(t) = K \sum_{l=1}^{N} [\Lambda]_{kl}(x_l - x_k), \quad k \in \mathbb{N}_1^N \tag{5}$$

where K is the control protocol that we need to find. For clear illustration, the overall system's state $x(t) := [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pN}$. Then one can represent the whole networked system in (4) as follows:

$$\mathcal{D}^\alpha x(t) = \Omega x(t) \tag{6}$$

where $\Omega = \mathbb{I}_N \otimes \mathcal{A} - L \otimes \mathcal{B}K$.

In this article, we address the consensus problem for a system in which agents are represented by general fractional-order linear models, and communicate through directed topologies. Based on the dynamic system descriptions presented earlier, we formulate the problem we aim to solve as follows.

Problem CFNS (Consensus of Fractional-order Networked System): Find a state-feedback gain K of (5) that solves the consensus of (4), that is, for any $x_k(0)$, $k \in \mathbb{N}_1^N$, $\lim_{t\to\infty}(x_l(t) - x_k(t)) = 0,\ \forall l, k \in \mathbb{N}_1^N$.

3 Main Results

We derive some necessary and sufficient conditions for the analysis and synthesis of **Problem CFNS** in this section by employing graph theory and fractional-order systems theory.

Theorem 1. For $\alpha \in [1, 2)$, **Problem CFNS** has a feasible solution K, is equivalent to one of the following two equivalent conditions:
1) $|\arg(\lambda_i(\mathcal{A} - \lambda_k(L)\mathcal{B}K))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^p$ and $k \in \mathbb{N}_2^N$;
2) Matrix Φ_k is Hurwitz (definition of matrix Φ_k is placed at the top of this page).

Proof. Define $e_k(t) = \sum_{l=1}^{N}[\Lambda]_{kl}(x_l - x_k),\ k \in \mathbb{N}_1^N$, and $e(t) = [e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pN}$, we have

$$e(t) = -(L \otimes I_p)x(t), \tag{7}$$

and

$$\mathcal{D}^\alpha e(t) = (I_N \otimes \mathcal{A} - L \otimes \mathcal{B}K)e(t). \tag{8}$$

Because it is assumed that the agents communicate through a graph which contains a spanning tree, we can utilize the property of Laplacian matrix [21], [27], [28], [33] and find that one can always find a coordinate transformation and the whole system (8) can be reduced to the $N-1$ systems:

$$\mathcal{D}^\alpha \epsilon_k(t) = \mathcal{A}_k \epsilon_k(t),\ k \in \mathbb{N}_2^N \tag{9}$$

where $\mathcal{A}_k := \mathcal{A} - \lambda_k(L)\mathcal{B}K = \mathcal{A} - \mathcal{R}(\lambda_k(L))\mathcal{B}K - \mathcal{I}(\lambda_k(L))\mathcal{B}K$. Notice that $\lim_{t\to\infty} e_k(t) = 0, k \in \mathbb{N}_1^N \Leftrightarrow \lim_{t\to\infty} \epsilon_k(t) = 0, k \in \mathbb{N}_2^N$. Consequently, to achieve the consensus of system (8), one can equivalently solve the stabilization problem of the $N-1$ systems in (9). Condition 1) is thus readily obtained by Lemma 1. Moreover, through some matrix manipulations, one can conclude that the $N-1$ systems in (9) are stable if and only if the condition 2), that is, $\Phi_k, k \in \mathbb{N}_2^N$, are Hurwitz matrices, is satisfied. This completes the proof. □

Remark 1. Theorem 1 presents an equivalent condition for the consensus analysis of networked systems with general fractional-order models and directed topologies, which was not previously available. This result is significant because it enables the development of synthesis conditions and numerical algorithms for **Problem CFNS**. Furthermore, it is worth noting that Φ_k is a real matrix, which enhances the practical usefulness of this condition.

By employing the useful results concluded in Theorem 1, an *equivalent condition* on the consensus synthesis of agents is derived for **Problem CFNS** as follows.

Theorem 2. For $\alpha \in [1, 2)$, **Problem CFNS** has a feasible solution K if and only if matrices $\mathcal{P}_k > 0$ or $\mathcal{Q}_k > 0$ ($k \in \mathbb{N}_2^N$), M and

$$\mathcal{K} := \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \tag{10}$$

satisfy one of the following equivalent conditions:
1) $U_k(\mathcal{P}_k, \mathcal{K}, M) :=$

$$\begin{bmatrix} \hat{\mathcal{A}}^{\mathrm{T}}\mathcal{P}_k + \mathcal{P}_k^{\mathrm{T}}\hat{\mathcal{A}} - \mathcal{K}^{\mathrm{T}}\mathcal{K} + \Theta_1 & \# \\ \hat{\mathcal{B}}_k^{\mathrm{T}}\mathcal{P}_k - \mathcal{K} & -\mathbb{I} \end{bmatrix} < 0; \tag{11}$$

2) $W_k(\mathcal{Q}_k, \mathcal{K}, M) :=$

$$\begin{bmatrix} \hat{\mathcal{A}}\mathcal{Q}_k + \mathcal{Q}_k^{\mathrm{T}}\hat{\mathcal{A}}^{\mathrm{T}} - \hat{\mathcal{B}}_k\mathcal{K}\mathcal{K}^{\mathrm{T}}\hat{\mathcal{B}}_k^{\mathrm{T}} + \hat{\mathcal{B}}_k\Theta_2\hat{\mathcal{B}}_k^{\mathrm{T}} & \# \\ \mathcal{K}^{\mathrm{T}}\mathcal{B}_k^{\mathrm{T}} - \mathcal{Q}_k & -\mathbb{I} \end{bmatrix} < 0 \tag{12}$$

where $\Theta_1 := (\mathcal{K} - M)^{\mathrm{T}}(\mathcal{K} - M)$ and $\Theta_2 := (\mathcal{K} - M)(\mathcal{K} - M)^{\mathrm{T}}$,

$$\hat{\mathcal{A}} := \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{bmatrix} \times \begin{bmatrix} \sin(\alpha\pi/2)I_p & \cos(\alpha\pi/2)I_p \\ -\cos(\alpha\pi/2)I_p & \sin(\alpha\pi/2)I_p \end{bmatrix},$$
$$\hat{\mathcal{B}}_k := \begin{bmatrix} \theta_k\mathcal{B} & \gamma_k\mathcal{B} \\ -\gamma_k\mathcal{B} & \theta_k\mathcal{B} \end{bmatrix}.$$

Proof. It is noted that, to solve **Problem CFNS**, we need to find a controller K such that matrices $\Phi_k = \hat{\mathcal{A}} - \hat{\mathcal{B}}_k\mathcal{K}$ ($k \in \mathbb{N}_2^N$) are all Hurwitz and $\mathcal{K}$ must have the representation in (10). By Lyapunov's stability theory, the condition that matrices $\Phi_k = \hat{\mathcal{A}} - \hat{\mathcal{B}}_k\mathcal{K}$, $k \in \mathbb{N}_2^N$, are all Hurwitz, is equivalent to that any of the following conditions holds:

$$\Phi_k^{\mathrm{T}}\mathcal{P}_k + \mathcal{P}_k\Phi_k < 0,\ \ \mathcal{P}_k > 0, \tag{13}$$

or

$$\mathcal{Q}_k\Phi_k^{\mathrm{T}} + \Phi_k\mathcal{Q}_k < 0,\ \ \mathcal{Q}_k > 0. \tag{14}$$

In the following, we will show the equivalence of (11) and (13). Define a nonsingular matrix as

$$\mathcal{T} = \begin{bmatrix} I & 0 \\ -\mathcal{K} & I \end{bmatrix}.$$

Performing a similarity transformation to U_k yields

$$\Psi_k := \mathcal{T}^{\mathrm{T}} U_k \mathcal{T} = \begin{bmatrix} \Phi_k^{\mathrm{T}} \mathcal{P}_k + \mathcal{P}_k \Phi_k + (\mathcal{K}-M)^{\mathrm{T}}(\mathcal{K}-M) & \# \\ \hat{\mathcal{B}}_k^{\mathrm{T}} \mathcal{P}_k & -I \end{bmatrix} < 0 \tag{15}$$

for $k = 2, 3, \ldots, N$. This implies that $\Phi_k^{\mathrm{T}} \mathcal{P}_k + \mathcal{P}_k \Phi_k + (\mathcal{K} - M)^{\mathrm{T}}(\mathcal{K} - M) < 0$, or equivalently, $\Phi_k^{\mathrm{T}} \mathcal{P}_k + \mathcal{P}_k \Phi_k < -(\mathcal{K} - M)^{\mathrm{T}}(\mathcal{K} - M) \leq 0$ since $(\mathcal{K} - M)^{\mathrm{T}}(\mathcal{K} - M) \geq 0$. This completes the sufficiency part. Assuming that $\Phi_k^{\mathrm{T}} \hat{\mathcal{P}}_k + \hat{\mathcal{P}}_k \Phi_k < 0$ holds and letting $M = \mathcal{K}$, then one can find a set of scalar a_k, $k = 2, 3, \ldots, N$ that are sufficiently small such that

$$-I - a_k \hat{\mathcal{B}}_k^{\mathrm{T}} \hat{\mathcal{P}}_k (\Phi_k^{\mathrm{T}} \hat{\mathcal{P}}_k + \hat{\mathcal{P}}_k \Phi_k + (\mathcal{K}-M)^{\mathrm{T}}(\mathcal{K}-M))^{-1} \hat{\mathcal{P}}_k \hat{\mathcal{B}}_k < 0.$$

Letting $\mathcal{P}_k = a_k \hat{\mathcal{P}}_k$ and noticing that $(\mathcal{K}-M)^{\mathrm{T}}(\mathcal{K}-M) = 0$, then (11) becomes

$$-I - \hat{\mathcal{B}}_k^{\mathrm{T}} \mathcal{P}_k (\Phi_k^{\mathrm{T}} \mathcal{P}_k + \mathcal{P}_k \Phi_k + (\mathcal{K}-M)^{\mathrm{T}}(\mathcal{K}-M))^{-1} \mathcal{P}_k \hat{\mathcal{B}}_k < 0.$$

Using the property of Schur complement equivalence and simple matrix operations, it follows that

$$U_k = \mathcal{T}^{-\mathrm{T}} \Psi_k \mathcal{T}^{-1} < 0$$

for $k \in \mathbb{N}_2^N$. This completes the necessity part. Therefore, (13) and (11) are equivalent. To show the equivalence of (12) and (14), one can define a nonsingular matrix:

$$\mathcal{S}_k = \begin{bmatrix} I & 0 \\ \mathcal{K}^{\mathrm{T}} \hat{\mathcal{B}}_k^{\mathrm{T}} & I \end{bmatrix}.$$

The rest of this proof follows similarly as above, and thus is omitted here. The whole proof is completed. ☐ In order to obtain the $\mathcal{K}$ as in (10), we can partition it as

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}$$

where $\mathcal{K}_{11} \in \mathbb{R}^{r \times p}$ and $\mathcal{K}_{22} \in \mathbb{R}^{r \times p}$. Then define a new variable associated with $\mathcal{K}$ as follows:

$$\mathcal{F} = \begin{bmatrix} \mathcal{K}_{22} & 0 \\ 0 & \mathcal{K}_{11} \end{bmatrix}. \tag{17}$$

Notice that when $\|\mathcal{K} - \mathcal{F}\|_2^2 = 0$, the representation of $\mathcal{K}$ is successfully obtained as (10). This motivates us to develop a convex programming algorithm in the following for solving a feasible $\mathcal{K}$ satisfying the conditions concluded in Theorem 2.

Algorithm CFNS:

Step 1. Initialize: $i = 1$, $\epsilon^{(0)} = 0$, $\delta = 0$, $M^{(1)}$ (ensuring $\Phi_k = \mathcal{A} - \mathcal{B}_k M^{(1)}$, $k \in \mathbb{N}_2^N$ are Hurwitz).
Step 2. Fix $M = M^{(i)}$, minimize $\epsilon^{(i)}$

$$\text{s.t.} \begin{cases} U_k < 0, \ (k \in \mathbb{N}_2^N), \\ \begin{bmatrix} -\epsilon^{(i)} I & (\mathcal{K} - \mathcal{F})^{\mathrm{T}} \\ \mathcal{K} - \mathcal{F} & -I \end{bmatrix} < 0, \mathrm{w.r.t.}\{\mathcal{P}_k > 0, \mathcal{K}, \mathcal{F}\}. \end{cases}$$

If $\epsilon^{(i)} \prec \eta$ (η is a prescribed tolerance), a feasible $\mathcal{K}$ is found. **STOP**. Otherwise, go to next step.
Step 3. If $|\epsilon^{(i)} - \delta|/\epsilon^{(i)} \prec \theta$, **STOP**. Otherwise, go to next step.
Step 4. If $|\epsilon^{(i)} - \epsilon^{(i-1)}|/\epsilon^{(i)} \prec \theta$, set $\delta = \epsilon^{(i)}$, $i = i+1$, update $M^{(i)} = \mathcal{K}$, go to **Step 5**. Otherwise, set $i = i+1$, update $M^{(i)} = \mathcal{K}$, then go to **Step 2**.
Step 5. Fix $M = M^{(i)}$, minimize $\epsilon^{(i)}$

$$\text{s.t.} \begin{cases} W_k < 0, \ (k \in \mathbb{N}_2^N), \\ \begin{bmatrix} -\epsilon^{(i)} I & (\mathcal{K} - \mathcal{F})^{\mathrm{T}} \\ \mathcal{K} - \mathcal{F} & -I \end{bmatrix} < 0, \mathrm{w.r.t.}\{\mathcal{Q}_k > 0, \mathcal{K}, \mathcal{F}\}. \end{cases}$$

If $\epsilon^{(i)} \prec \eta$, a feasible $\mathcal{K}$ is found. **STOP**. Otherwise, go to next step.
Step 6. If $|\epsilon^{(i)} - \delta|/\epsilon^{(i)} \prec \theta$, **STOP**. Otherwise, go to next step.
Step 7. If $|\epsilon^{(i)} - \epsilon^{(i-1)}|/\epsilon^{(i)} \prec \theta$, set $\delta = \epsilon^{(i)}$, $i = i+1$, update $M^{(i)} = \mathcal{K}$, go to **Step 2**. Otherwise, set $i = i+1$, update $M^{(i)} = \mathcal{K}$, then go to **Step 5**.

Remark 2. In Step 1 of the algorithm, one can obtain an $M^{(1)} = \mathcal{L}\mathcal{X}^{-1}$ by solving $\hat{\mathcal{A}}\mathcal{X} - \hat{\mathcal{B}}_k \mathcal{L} + (\hat{\mathcal{A}}\mathcal{X} - \hat{\mathcal{B}}_k \mathcal{L})^{\mathrm{T}} < 0$ with respect to $\mathcal{X} > 0$ and $\mathcal{L}$. It should be noted that, during the iterative process of algorithm, we have $\epsilon^{(i+1)} \preceq \epsilon^{(i)}$ for $i \succeq 2$, because one can always find $\mathcal{K} = \mathcal{K}^{(i)}$, $\mathcal{P}_k > 0$ or $\mathcal{Q}_k > 0$, and $\mathcal{F}^{(i)}$, $2 \leq k \leq N$ such that $\|\mathcal{K}^{(i)} - \mathcal{F}^{(i)}\|_2^2 \preceq \epsilon^{(i)} \preceq \|\mathcal{K}^{(i-1)} - \mathcal{F}^{(i-1)}\|_2^2 \preceq \epsilon^{(i-1)}$, $k \in \mathbb{N}_2^N$. This means that the errors decrease or stay the same as the algorithm progresses.

4 Illustrative Examples

In this section, we provide validation for the proposed results and algorithm in Section 3 by presenting three illustrative examples.

4.1 High-Order Integrator Dynamics

There are scarce results dedicated to the analysis of networks consisting of multiple higher-order fractional integrator models [19], [32], [34]. In this example, we consider a networked system (4) composed of four agents. The system matrix of each agent is given as follows:

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalues of matrix $\mathcal{A}$ are shown in Fig. 1, from which one can find that it is unstable when $\alpha = 1.3$, since its poles all locate at the origin. The communication topology of networked systems is denoted using $\mathcal{G}$ with the follwing Laplacian matrix:

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{18}$$

The eigenvalues of L are, respectively, $\lambda_1(L) = 0$, $\lambda_2(L) = 1.5 + 0.866j$, $\lambda_3(L) = 1.5 - 0.866j$ and $\lambda_4(L) = 2$. $\theta_2 = 1.7297$, $\theta_3 = 0.9434$, $\theta_4 = 1.7820$ $\gamma_2 = 0.0906$,

This article has been accepted for publication in IEEE Transactions on Network Science and Engineering. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TNSE.2023.3301015

$$K = \begin{bmatrix} 0.8687 & 11.9562 & 44.8001 & 82.0047 & 83.2621 & 46.7320 & 12.0127 \end{bmatrix}. \quad (16)$$

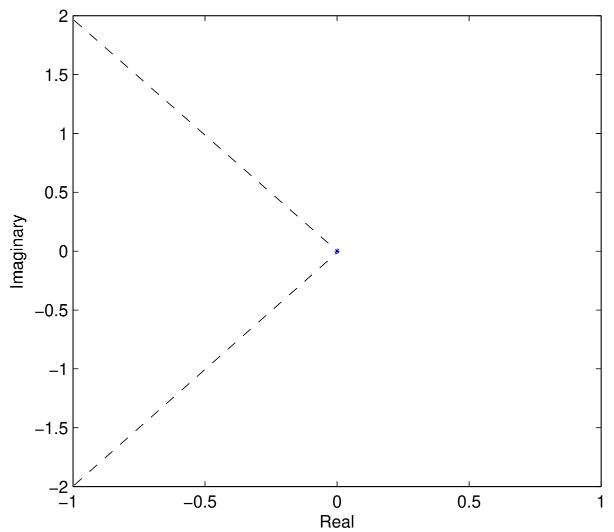

Fig. 1. Eigenvalues of matrix $\mathcal{A}$ in Example 1

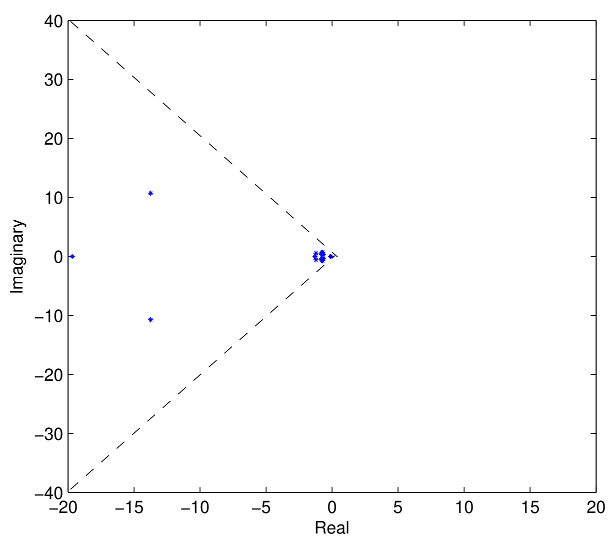

Fig. 2. Eigenvalues of matrices $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ with controller (16)

$\gamma_3 = -1.4526$, and $\gamma_4 = -0.9080$. Using Algorithm CFNS, we obtained a controller (16) (shown at the top of this page). The eigenvalues of $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ with controller (16) are shown in Fig. 2 from which we can see that all the eigenvalues locate at the stability region, and thus $|\arg(\lambda_i(\mathcal{A}_k))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^7$, $k \in \mathbb{N}_2^4$, are satisfied. Figs. 3 and 4 depict the state consensus evolution of agents with controller (16) (State components 6 and 7 of agents).

4.2 Fractional-Order Electric Circuit Model

Consider a networked system (4) composed of four agents over the directed graph $\mathcal{G}$, as in the previous example.

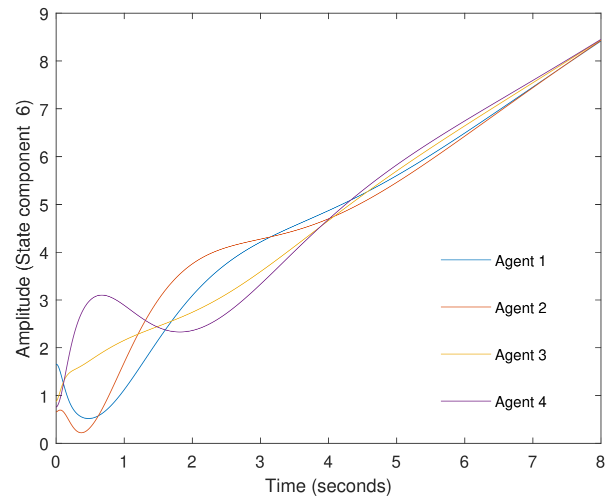

Fig. 3. Consensus of agents with controller (16) (State component 6)

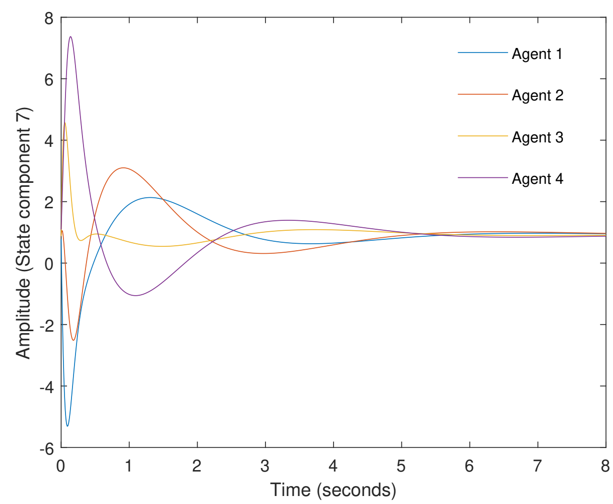

Fig. 4. Consensus of agents with controller (16) (State component 7)

Each agent in the system is represented by a fractional-order electric circuit (refer to Example 4.2 in [9]), with the following system matrices:

$$\mathcal{A} = \begin{bmatrix} -\frac{R_1+R_2}{L_1} & \frac{R_2}{L_1} & 0 & \frac{R_1}{L_1} \\ \frac{R_2}{L_2} & -\frac{R_2+R_3}{L_2} & \frac{R_3}{L_2} & 0 \\ 0 & \frac{R_3}{L_3} & -\frac{R_3}{L_3} & 0 \\ \frac{R_1}{L_4} & \frac{R_3}{L_4} & 0 & -\frac{R_1+R_3+R_4}{L_4} \end{bmatrix} \quad (19)$$

and

$$\mathcal{B} = \begin{bmatrix} \frac{1}{L_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{L_3} & 0 \\ 0 & \frac{1}{L_3} & \frac{1}{L_4} \end{bmatrix}.$$

The resistances $R_1 = R_2 = R_3 = 1$ ohm and $R_4 = 0$ ohm, and the inductances $L_1 = L_2 = L_3 = L_4 = 1$ henry. The

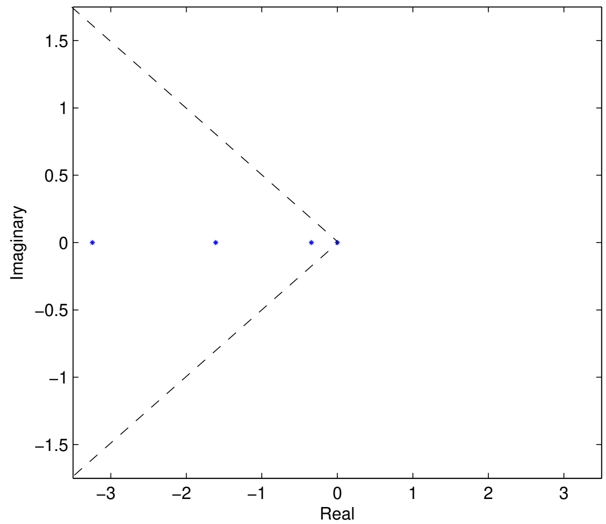

Fig. 5. Eigenvalues of matrix $\mathcal{A}$ in Example 2

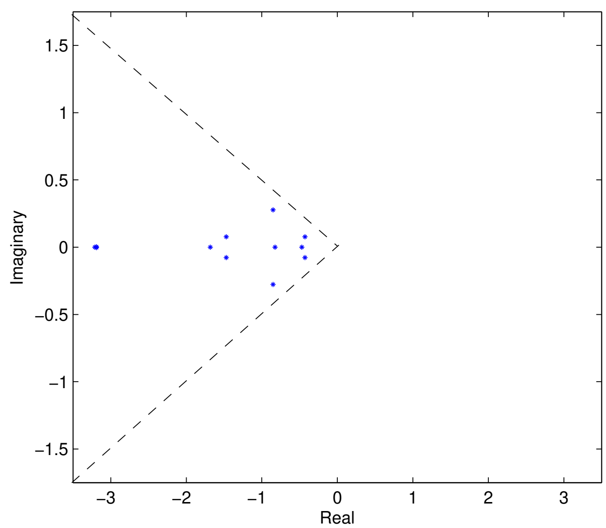

Fig. 6. Eigenvalues of matrices $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ with controller (20)

eigenvalues of matrix (19) are shown in Fig. 5, from which we can see that it is an unstable system because there is one pole at the origin. Letting $\alpha = 1.5$ and using Algorithm CFNS, we obtained a controller:

$$K = \begin{bmatrix} -0.1950 & 0.0675 & 0.1556 & 0.3651 \\ 0.1591 & 0.0485 & 0.1205 & 0.1604 \\ -2.1449 & -0.5499 & 0.9525 & 4.0882 \end{bmatrix}. \tag{20}$$

The eigenvalues of $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ with controller (20) are shown in Fig. 6, from which we know that all the eigenvalues locate at the stability region, and thus $|\arg(\lambda_i(\mathcal{A}_k))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^7$, $k \in \mathbb{N}_2^4$, are satisfied. To verify the robustness of the obtained controller in (20), the eigenvalues of $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ with controller (20) and uncertainty $\Delta R_2 \in [-0.5, 0.5]$ (100 samples) are shown in Fig. 7, from which we know that all the eigenvalues locate at the stability region, and thus $|\arg(\lambda_i(\mathcal{A}_k))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^7$, $k \in \mathbb{N}_2^4$, are satisfied.

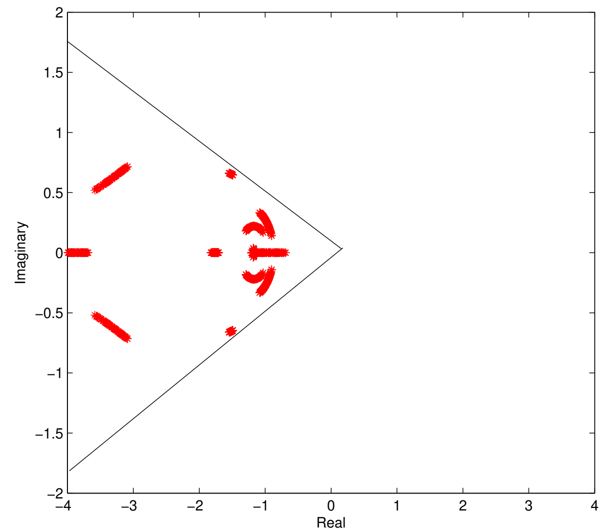

Fig. 7. Eigenvalues of matrices $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ with controller (20) and uncertainty ΔR_2

4.3 General Linear Dynamics

We consider a networked system (4) consisting of 4 agents over the directed graph $\mathcal{G}$ as in the previous example. The system matrices are represented by (21) (shown at the top of the next page). Fig. 8 shows the eigenvalues of the matrix $\mathcal{A}$, from which we can see that the system is unstable with $\alpha = 1.2$. Using Algorithm CFNS, we obtained the controller (22). Fig. 9 shows the eigenvalues of $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ with controller (16). We can see that all the eigenvalues are located in the stability region, and thus $|\arg(\lambda_i(\mathcal{A}_k))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^7$ and $k \in \mathbb{N}_2^4$ are satisfied.

5 Conclusions

In this paper, we have addressed the consensus problem of a class of networked systems over directed graphs, where agents are represented by general fractional-order linear dynamics with order $\alpha \in [1, 2)$. The goal of consensus control is to design distributed protocols for agents so that the entire dynamic system can achieve consensus. Using spectral graph theory and fractional-order systems theory, we have derived several equivalent conditions for the consensus analysis and synthesis of networked systems with fractional-order models. Simulation results on higher-order dynamic models and fractional-order circuit models have demonstrated the effectiveness of our proposed approach and algorithm. In the future, we will focus on the cooperative control problem of fractional-order linear models with order $\alpha \in (0, 1)$, since a necessary and sufficient condition for addressing this issue has not yet been developed.

$$\mathcal{A}=\begin{bmatrix} 0.9945 & -0.9785 & 0.3096 & 0.8609 & 1.4091 & -0.5797 & -0.3156 & 1.8295 \\ -1.5193 & -1.9179 & -0.2399 & 0.5682 & 0.0225 & 1.9880 & -1.2636 & -0.9387 \\ 0.1002 & 1.6947 & -0.9695 & -0.3238 & 0.5426 & -1.1033 & 0.9031 & 1.6983 \\ -0.6967 & 0.6148 & 1.0078 & -0.4370 & 1.8036 & 0.6098 & -0.5185 & -1.1049 \\ 0.1858 & 1.7305 & -1.0853 & 1.2646 & -0.2241 & 0.4200 & 1.3662 & -0.5057 \\ -0.4045 & -1.3460 & -1.7433 & -0.7303 & -1.7599 & -0.4510 & 0.9369 & -1.6500 \\ -0.3396 & 1.6844 & 1.0693 & 1.2582 & 1.4670 & -1.4313 & 0.2841 & 0.5605 \\ -1.2770 & 1.1786 & 0.6848 & 1.1563 & 0.5248 & -1.8995 & -1.2926 & -1.2775 \end{bmatrix}, \mathcal{B}=\begin{bmatrix} -1.8198 & 1.2022 \\ 0.8927 & 0.9834 \\ -0.6102 & 1.2525 \\ 0.6425 & -0.4668 \\ -0.4645 & 0.4691 \\ 0.5094 & 0.3020 \\ -1.9134 & 0.1202 \\ 1.6423 & -0.8997 \end{bmatrix}. \tag{21}$$

$$K=\begin{bmatrix} 17.6834 & -1.7922 & 6.5441 & 4.5133 & 9.2604 & -16.7184 & -15.8648 & 20.4279 \\ 28.4485 & -2.0377 & 4.3855 & 15.4354 & 23.0124 & -15.3470 & -19.7867 & 20.6598 \end{bmatrix}. \tag{22}$$

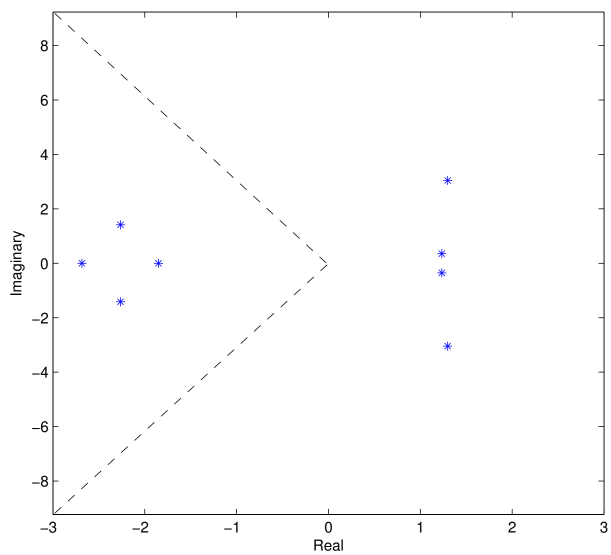

Fig. 8. Eigenvalues of matrix $\mathcal{A}$ in Example 3

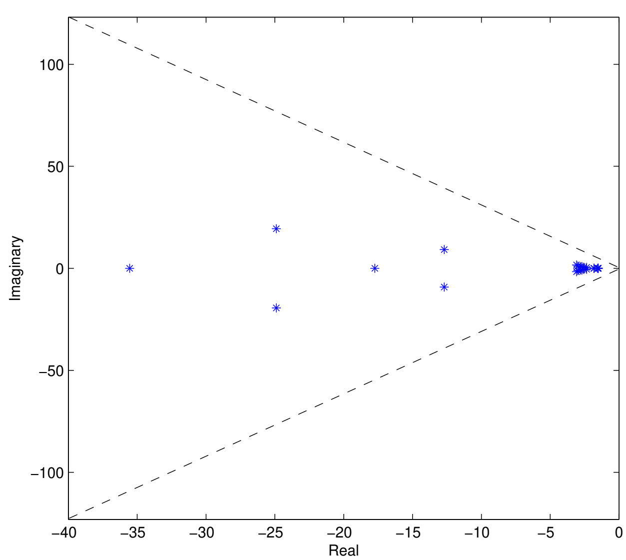

Fig. 9. Eigenvalues of matrices $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ with controller (22)

6 Appendix

6.1 Proof of Lemma 1

Let $\mathcal{R}(x(t)) \in \mathbb{R}^p$ be the real part of $x(t)$, and $\mathcal{I}(x(t)) \in \mathbb{R}^p$ be the imaginary part of $x(t)$, then it can be represented as $x(t) := \mathcal{R}(x(t)) + j\mathcal{I}(x(t))$. The FOS in (1) with $u(t) = 0$ can be rewritten as $\mathcal{D}^\alpha\mathcal{R}(x(t)) + \mathcal{D}^\alpha\mathcal{I}(x(t))j = \mathcal{R}(\hat{\mathcal{A}})\mathcal{R}(x(t)) - \mathcal{I}(\hat{\mathcal{A}})\mathcal{I}(x(t)) + \mathcal{I}(\hat{\mathcal{A}})\mathcal{R}(x(t))j + \mathcal{R}(\hat{\mathcal{A}})\mathcal{I}(x(t))j,\ \alpha \in [1, 2)$. To show it in a more clear manner, we derive it as:

$$\begin{bmatrix}\mathcal{D}^\alpha\mathcal{R}(x(t)) \\ \mathcal{D}^\alpha\mathcal{I}(x(t))\end{bmatrix} = \begin{bmatrix}\mathcal{R}(\check{\mathcal{A}}) & -\mathcal{I}(\check{\mathcal{A}}) \\ \mathcal{I}(\check{\mathcal{A}}) & \mathcal{R}(\check{\mathcal{A}})\end{bmatrix} \times \begin{bmatrix}\mathcal{R}(x(t)) \\ \mathcal{I}(x(t))\end{bmatrix}. \tag{23}$$

Hence, in the following analysis, we will use the real system in (23) to equivalently characterize the complex FOS in (1) with zero input.

By the Theorem 2 in [16], it is known that, the system in (23) is asymptotically stable, is equivalent to that $|\arg(\lambda_i(\check{\mathcal{A}}))| > \alpha\pi/2$, $i \in \mathbb{N}_1^p$, where

$$\mathbb{A} := \begin{bmatrix}\mathcal{R}(\check{\mathcal{A}}) & -\mathcal{I}(\check{\mathcal{A}}) \\ \mathcal{I}(\check{\mathcal{A}}) & \mathcal{R}(\check{\mathcal{A}})\end{bmatrix}.$$

Define

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix}-jI_p & -jI_p \\ I_p & -I_p\end{bmatrix},\quad V^* = \frac{1}{\sqrt{2}j}\begin{bmatrix}-I_p & jI_p \\ -I_p & -jI_p\end{bmatrix}.$$

The following result can be obtained using simple matrix manipulations:

$$V^*\mathbb{A}V = \begin{bmatrix}\check{\mathcal{A}} & 0 \\ 0 & \bar{\check{\mathcal{A}}}\end{bmatrix}$$

where $\bar{\check{\mathcal{A}}}$ is used to denote the conjugate of $\check{\mathcal{A}}$. Notice that $|\arg(\lambda_i(\check{\mathcal{A}}))| > \alpha\pi/2$ and $|\arg(\lambda_i(\bar{\check{\mathcal{A}}}))| > \alpha\pi/2$ are two equivalent conditions. Consequently, we have $|\arg(\lambda_i(\check{\mathcal{A}}))| > \alpha\pi/2$ is equivalent to $|\arg(\lambda_i(\bar{\check{\mathcal{A}}}))| > \alpha\pi/2$, which implies that the FOS in (1) (or (23)) with zero input achieves asymptotic stability, is equivalent to $|\arg(\lambda_i(\check{\mathcal{A}}))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^p$. This completes the proof for Condition 1).

In the following, we will give the proof for Condition 2). By the Theorem 3 in [16], it is known that the asymptotic stability of system (23) is achieved if and only if the Hurwitzness of the following matrix is guaranteed:

$$\tilde{\mathcal{A}} := \begin{bmatrix}\mathbb{A} & 0 \\ 0 & \mathbb{A}\end{bmatrix} \times \begin{bmatrix}\sin(\alpha\pi/2)I_{2p} & \cos(\alpha\pi/2)I_{2p} \\ -\cos(\alpha\pi/2)I_{2p} & \sin(\alpha\pi/2)I_{2p}\end{bmatrix}. \tag{24}$$

Define

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix}-jI_{2p} & -jI_{2p} \\ I_{2p} & -I_{2p}\end{bmatrix},\ T^* = \frac{1}{\sqrt{2}j}\begin{bmatrix}-I_{2p} & jI_{2p} \\ -I_{2p} & -jI_{2p}\end{bmatrix}.$$

The following result can be obtained using similarity transformation:

$$T^* \tilde{\mathcal{A}} T = \begin{bmatrix} w\mathbb{A} & 0 \\ 0 & \bar{w}\mathbb{A} \end{bmatrix}$$

where $w := \sin(\alpha\pi/2) + \cos(\alpha\pi/2)j$ and $\bar{w} := \sin(\alpha\pi/2) - \cos(\alpha\pi/2)j$. The Hurwitzness of matrix (24) is guaranteed if and only if the matrix $T^* \tilde{\mathcal{A}} T$ is Hurwitz. Notice that matrix $w\mathbb{A}$ and $\bar{w}\mathbb{A}$ are conjugate to each other, we have $w\mathbb{A}$ is Hurwitz if and only if $\bar{w}\mathbb{A}$ is Hurwitz. Therefore, $w\mathbb{A}$ is Hurwitz, is equivalent to that matrix $T^* \tilde{\mathbb{A}} T$ (or matrix (24)) is Hurwitz. Expanding $w\mathbb{A}$ as

$$w\mathbb{A} = \begin{bmatrix} w\mathcal{R}(\check{\mathcal{A}}) & -w\mathcal{I}(\check{\mathcal{A}}) \\ w\mathcal{I}(\check{\mathcal{A}}) & w\mathcal{R}(\check{\mathcal{A}}) \end{bmatrix}.$$

The following result can be obtained using similarity transformation:

$$V^* w\mathbb{A} V = \begin{bmatrix} w\check{\mathcal{A}} & 0 \\ 0 & w\bar{\check{\mathcal{A}}} \end{bmatrix}$$

where $\bar{\check{\mathcal{A}}}$ is the conjugate of $\check{\mathcal{A}}$. Notice that we have that $|\arg(\lambda_i(w\check{\mathcal{A}}))| > \alpha\pi/2$ and $|\arg(\lambda_i(w\bar{\check{\mathcal{A}}}))| > \alpha\pi/2$ are equivalent. Hence, it follows that $|\arg(\lambda_i(w\check{\mathcal{A}}))| > \alpha\pi/2$ and $|\arg(\lambda_i(w\bar{\check{\mathcal{A}}}))| > \alpha\pi/2$ are two equivalent conditions, which further indicates that the Hurwitzness of matrix (24) is guaranteed if and only if $|\arg(\lambda_i(w\check{\mathcal{A}}))| > \alpha\pi/2$ for $i \in \mathbb{N}_1^p$. Representing $w\check{\mathcal{A}}$ as $w\check{\mathcal{A}} = \sin(\alpha\pi/2)\mathcal{R}(\check{\mathcal{A}}) - \cos(\alpha\pi/2)\mathcal{I}(\check{\mathcal{A}}) + j\sin(\alpha\pi/2)\mathcal{I}(\check{\mathcal{A}}) + \cos(\alpha\pi/2)\mathcal{R}(\check{\mathcal{A}}) = \check{\mathcal{A}}_\mathrm{r} + j\check{\mathcal{A}}_\mathrm{i}$. Using the similarity transformation to matrix (2) again, we can show that matrix (2) is Hurwitz is equivalent to that matrix $w\check{\mathcal{A}}$ is Hurwitz. In conclusion, the FOS in (1) with zero input (or the system in (23)) is asymptotically stable, is equivalent to that the matrix in (2) is Hurwitz. This completes the proof. □

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This article has been accepted for publication in IEEE Transactions on Network Science and Engineering. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TNSE.2023.3301015

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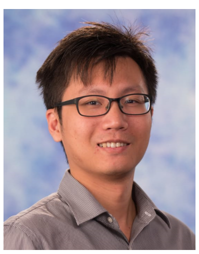

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