Nonnegative Consensus Tracking of Networked Systems With Convergence Rate Optimization

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***Abstract*— This article investigates the nonnegative consensus tracking problem for networked systems with a distributed static output-feedback (SOF) control protocol. The distributed SOF controller design for networked systems presents a more challenging issue compared with the distributed state-feedback controller design. The agents are described by multi-input multi-output (MIMO) positive dynamic systems which may contain uncertain parameters, and the interconnection among the followers is modeled using an undirected connected communication graph. By employing positive systems theory, a series of necessary and sufficient conditions governing the consensus of the nominal, as well as uncertain, networked positive systems, is developed. Semidefinite programming consensus design approaches are proposed for the convergence rate optimization of MIMO agents. In addition, by exploiting the positivity characteristic of the systems, a linear-programming-based design approach is also proposed for the convergence rate optimization of single-input multi-output (SIMO) agents. The proposed approaches and the corresponding theoretical results are validated by case studies.**

***Index Terms*— Linear programming, networked systems, nonnegative consensus tracking, positive systems, robust consensus, semidefinite programming.**

I. Introduction

IN the last decade, the coordination problem of networked systems has attracted increasing attention among researchers. The interest in this problem is mainly motivated by a large number of applications in various areas. For example, useful and extensive applications have been utilized in unmanned aerial vehicles [9], mobile robots [5], flocking [28], and sensor networks [25]. One critical issue during handling the coordination of networked systems is how to manage the agents such that they can reach an agreement by designing distributed control protocols with local information, which is called the consensus/synchronization problem [4], [29]. In leader-follower networked systems, the leader is usually unaffected by the followers but can guide their behaviors. The control objective of such systems can be realized easily via controlling the leader only. Hence, the leader-follower consensus strategy not only simplifies the design and implementation but also reduces the control cost or energy [34]. Most work on the consensus problem is focused on static consensus protocols using the full state information of agents. However, in practical applications, the full state information for controller design is generally not available, thus the output feedback design approaches for consensus of networked systems are desirable. Another practical issue that should be considered is uncertainty, which is inevitably present in system parameters for various unpredictable reasons.

Positive systems are dynamic systems whose state and output variables take nonnegative values consistently under given nonnegative initial conditions and inputs [6], [22], [23]. Such systems can be seen frequently and extensively utilized in a variety of fields, for example, heat exchangers, economics, industrial engineering involving chemical reactors, and storage systems [8]. In recent years, the research topic on positive systems arouses a significant amount of interest, see [1], [22] and the references therein. Indeed, different control problems of networked systems with positivity constraints have been studied in [26], [18], [32], [21], and [11] recently. Specifically, those networked systems consisting of single integrators or double integrators were typical positive ones [9], [26]. The integrator-based consensus algorithm was then applied to the emissions control for a fleet of Plug-in Hybrid Electric Vehicles (PHEVs) in [18]. The leaderless edge consensus problem with positivity constraints was studied in [32]. The observer and controller design problems were investigated for positive fractional-order networked systems in [30] and [11]. For the first time, we investigate the nonnegative consensus tracking problem of networked systems in this article.

It is known that the consensus problem of networked systems with static output-feedback (SOF) control is equivalent to the simultaneous SOF stabilization problem. However, the consensus of uncertain networked positive systems is difficult to be settled due to two main reasons. One is the SOF control issue which is generally NP-hard [7] due to the fact that it is a bilinear matrix inequality (BMI) problem [3]. The other issue to tackle is how to reach a consensus of

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agents while maintaining the positivity of the investigated system. In this article, such a challenging problem with the aforementioned issues will be considered and solved. The main contributions of this article are: 1) It is the first attempt to investigate the nonnegative consensus tracking problem of networked systems with SOF control. 2) Using positive systems theory, necessary and sufficient conditions for the consensus analysis of networked positive systems without or with uncertainties are derived. 3) Based on the consensusability analysis results, two types of consensus design approaches, that is, the semidefinite programming and the linear-programming-based approaches, are developed for computing the required controllers, as well as optimizing the convergence rates.

The rest of this article is organized as follows. Section II presents some mathematical preliminaries and defines the problem to be solved. In Section III, the positive leader-follower consensus conditions on the analysis and design of networked positive systems are proposed by semidefinite programming optimization approaches. In Section IV, the nonnegative consensus tracking in networked positive systems comprising single-input multi-output (SIMO) agents is achieved utilizing linear-programming-based optimization approaches. In Section V, numerical simulations are provided to verify the obtained results. Section VI summarizes this article with some remarks.

II. Preliminaries

A. Notations

The notations used in this article are standard. For any matrices $\mathbf{X}$ and $\mathbf{Y}$ that are real symmetric and of the same dimension, we let $\mathbf{X} \geq \mathbf{Y}$ (respectively, $\mathbf{X} > \mathbf{Y}$) represent that their difference $\mathbf{X} - \mathbf{Y}$ is positive semidefinite (respectively, positive definite). The transpose operation of the corresponding matrix is denoted by the superscript T. Matrices in algebraic operations without specific statements are defaulted to be dimension-compatible. The identity matrix of the appropriate dimension is denoted by I. We use symbol $|\cdot|$ to represent the Euclidean norm for vectors while for matrices, we use symbols $\|\cdot\|$ and $\|\cdot\|_1$ to denote the spectral norm and the 1-norm, respectively. $\mathbf{X}\otimes\mathbf{Y}$ denotes the Kronecker product of matrices $\mathbf{X}$ and $\mathbf{Y}$. For matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$, notation $[\mathbf{X}]_{ij}$ is to represent its element in the ith row and the jth column. Nonnegative matrix $\mathbf{X} \succeq 0$ (respectively, positive matrix $\mathbf{X} \succ 0$) indicates that for any combinations of i and j, the element $[\mathbf{X}]_{ij} \succeq 0$ (respectively, $[\mathbf{X}]_{ij} \succ 0$). We use notation $\mathbf{X} \succeq \mathbf{Y}$ (respectively, $\mathbf{X} \succ \mathbf{Y}$) to indicate that their difference satisfies $\mathbf{X} - \mathbf{Y} \succeq 0$ (respectively, $\mathbf{X} - \mathbf{Y} \succ 0$). Matrix $\mathbf{X} \in \mathbb{R}^{n\times n}$ is addressed as a Metzler matrix if all its off-diagonal components are nonnegative, which is symbolized as $\mathbf{X} \in \mathbb{M}^n$. The spectral abscissa of a matrix $\mathbf{X}$ is represented as $\alpha(\mathbf{X})$. The orthogonal complement $\mathbf{X}^\perp$ of any real matrix $\mathbf{X} \in \mathbb{R}^{m\times n}$ with $m \prec n$ is defined as the matrix with maximum column rank satisfying $\mathbf{X}\mathbf{X}^\perp = 0$ and $(\mathbf{X}^\perp)^{\mathrm{T}}\mathbf{X}^\perp = I$. We use the notation $\mathrm{diag}(X_1, X_2, \ldots, X_n)$ as meaning of the block diagonal matrix with $X_1, X_2, \ldots, X_n$ on the diagonal. $\mathbf{1}_n$ denotes $[1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n$, while $\mathbf{0}_n$ denotes $[0, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^n$.

B. Positive Systems Theory

We consider the continuous-time linear system as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^r$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denote the system state, input and measured output, respectively. $A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times m}$ and $C \in \mathbb{R}^{p\times r}$ denote the system matrices of appropriate dimensions. Since our object of this article is networked positive systems, several definitions and statements about the positivity concerning system (1) are listed as below [2], [8], [10].

Definition 1: System (1) is addressed as a continuous-time positive linear system if for any initial value $x(0) \succeq 0$ and input $u(t) \succeq 0$, the state trajectory $x(t) \succeq 0$, and the output $y(t) \succeq 0$ for any $t \succeq 0$.

Lemma 1: System (1) is positive under the necessary and sufficient conditions that matrix A is Metzler, moreover, matrices B and C perform nonnegative.

Lemma 2: For a given Metzler matrix A, it is Hurwitz if and only if ∃ a diagonal matrix $P > 0$ such that

$$A^{\mathrm{T}}P + PA < 0.$$

Lemma 3: For a given Metzler matrix A, it is Hurwitz if and only if one of the following two statements holds:

1) ∃ a column vector $p \succ 0$, such that $Ap \prec 0$;
2) ∃ a column vector $q \succ 0$, such that $q^{\mathrm{T}}A \prec 0$.

Lemma 4: For any two Metzler matrices $A_1, A_2 \in \mathbb{R}^{r\times r}$, if $A_1 \succeq A_2$, then $\alpha(A_1) \succeq \alpha(A_2)$.

C. Graph Theory

With regard to the follower systems, we employ an undirected graph $\mathscr{G} = (\mathscr{W}, \mathscr{F})$ which consists of a finite node set $\mathscr{W} = \{w_1, w_2, \ldots, w_n\}$ and an edge set $\mathscr{F} \subset \mathscr{W} \times \mathscr{W}$ for representation. If set $\mathscr{W}$ contains n nodes in total, the graph $\mathscr{G}$ will be n-order, the nodes in which can also be specifically labeled by integer i, $i \in \mathscr{I} = \{1, 2, \ldots, n\}$. Two nodes w_i and w_j are adjacent if $(w_i, w_j) \in \mathscr{F}$, which means that nodes i and j have interactions. An adjacency matrix J of graph $\mathscr{G}$ with order n is a square matrix defined as $[J]_{ij} = [J]_{ji} = 1$ if $(w_i, w_j) \in \mathscr{F}$, otherwise as 0. If $(w_i, w_j) \in \mathscr{F}$, then w_j is one of the neighbors of w_i. The set of w_i's neighbors is denoted by $\mathscr{N}_i = \{w_j \in \mathscr{W} : (w_i, w_j) \in \mathscr{F}\}$. The Laplacian matrix L of n-order graph $\mathscr{G}$ is an $n \times n$ matrix characterized by $[L]_{ii} = \sum_{v_j\in\mathscr{N}_i}[J]_{ij}$ and $[L]_{ij} = -[J]_{ij}$ for any $i \neq j$. In graph $\mathscr{G}$, a path with length $n-1$ equals an ordered sequence containing n different nodes $\{w^{(1)}, w^{(2)}, \ldots, w^{(n)}\}$ where $(w^{(i)}, w^{(i+1)}) \in \mathscr{F}$. The undirected graph $\mathscr{G}$ is called connected, if, for any two nodes, there always exists a path, otherwise called disconnected.

As to the leader–follower system, another graph $\bar{\mathscr{G}}$ associated with the system constructed by one leader denoted as w_0 as well as n followers are taken into consideration. In graph $\bar{\mathscr{G}}$, the leader node connects some of the n follower nodes (related to graph $\mathscr{G}$) by directed edges. Graph $\bar{\mathscr{G}}$ is referred to as "connected" if at least one follower node in each component of graph $\mathscr{G}$ is linked to the leader node by a directed edge, which means the follower node can sense the instruction from the leader. The leader adjacency matrix associated with graph $\bar{\mathscr{G}}$ is represented as $G := \text{diag}(g_1, g_2, \ldots, g_n)$, where $g_i = 1$ indicates node υ_i can sense the data from the leader and otherwise $g_i = 0$. Obviously, the Laplacian matrix $\mathscr{L}$ of graph $\bar{\mathscr{G}}$ is in the following form:

$$\mathscr{L} = \begin{bmatrix} 0 & \mathbf{0}_n^{\mathrm{T}} \\ -G\mathbf{1}_n & L+G \end{bmatrix}.$$

D. Problem Formulation

Consider a network system described by an n-agent communication graph, with identical positive linear dynamic systems as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad y_i(t) = Cx_i(t), \; i \in \mathscr{I} \tag{2}$$

where $x_i(t) := [x_{i1}, x_{i2}, \ldots, x_{ir}]^{\mathrm{T}} \in \mathbb{R}^r$ denotes the state, $u_i(t) \in \mathbb{R}^m$ denotes the system input, $y_i(t) \in \mathbb{R}^p$ denotes the measured output. Matrix $A \in \mathbb{R}^{r\times r}$ is Metzler, while matrices $B \in \mathbb{R}^{r\times m}$ and $C \in \mathbb{R}^{p\times r}$ are nonnegative. Notice that the system in (2) is an multi-input multi-output (MIMO) positive linear system of any orders and can be stable, marginally stable or unstable. The dynamics of the leader is represented by

$$\dot{x}_0(t) = Ax_0(t), \quad y_0(t) = Cx_0(t) \tag{3}$$

where $x_0(t) \in \mathbb{R}^r$ is the state and $y_0(t) \in \mathbb{R}^p$ is the measured output.

In nominal cases, system matrices A, B, and C are known. However, when dealing with the uncertain case, the specific parameters of system matrices A, B, and C are unknown, but fixed. One can consider the robust case that system matrices A, B and C have interval uncertainties, which can be expressed by

$$A_{\mathrm{m}} \preceq A \preceq A_{\mathrm{M}}, \quad B_{\mathrm{m}} \preceq B \preceq B_{\mathrm{M}}, \quad C_{\mathrm{m}} \preceq C \preceq C_{\mathrm{M}} \tag{4}$$

where $A_{\mathrm{m}} \in \mathbb{R}^{r\times r}$ is Metzler, $0 \preceq B_{\mathrm{m}} \in \mathbb{R}^{r\times m}$, and $0 \preceq C_{\mathrm{m}} \in \mathbb{R}^{p\times r}$. It is assumed that system (A, B, C) is stabilizable and detectable, and matrices B and C are full-rank throughout this article.

The distributed SOF protocol in [24] of the ith agent is

$$u_i(t) = K \sum_{\upsilon_j \in \mathscr{N}_i} [L]_{ij}\big(y_i(t) - y_j(t)\big) + g_i K\big(y_0(t) - y_i(t)\big) \tag{5}$$

where K is the SOF controller matrix to be designed. It is different from the state-feedback controller since it does not need the full state information of the closed-loop system, thus the design of K is more challenging compared with that in the state-feedback case. Furthermore, define $x(t) := [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{nr}$, $\hat{x}(t) := [x_0^{\mathrm{T}}(t), x_0^{\mathrm{T}}(t), \ldots, x_0^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{nr}$. Throughout this article, two assumptions are taken regarding the communication graph of agents: 1) graph $\mathscr{G}$ is connected, and 2) at least one follower node can sense the information of the leader. It is known that with assumptions 1) and 2), matrix $L+G$ is positive definite and hence, its eigenvalues are positive. In the following, we denote the eigenvalues of matrix $L+G$ as λ_i, $i \in \mathscr{I}$, with $\lambda_1 \preceq \lambda_2 \preceq \cdots \preceq \lambda_n$. By using the distributed SOF protocol in (5), the whole closed-loop system is represented by

$$\dot{x}_0(t) = Ax_0(t), \quad \dot{x}(t) = \mathbb{A}x(t) + \mathbb{B}\hat{x}(t) \tag{6}$$

where $\mathbb{A} = I_n \otimes A - (L+G)\otimes BKC$ and $\mathbb{B} = G \otimes BKC$. To show how the convergence rate of uncertain multiagent systems is guaranteed, define the error signals to be $z(t) := x(t) - \hat{x}(t)$ and $z_0(t) := x_0(t) - x_0(t)$. Then the leader-follower system in (6) can be represented in the sense of error as

$$\dot{z}(t) = \mathbb{A}z(t). \tag{7}$$

Thus, the exponential convergence rate is defined as follows.

Definition 2: The multiagent system in (2) and (3) achieves leader-follower consensus with exponential convergence rate $\sigma \succ 0$ if and only if $|z(t)| \preceq \kappa e^{-\sigma t}$, $t \succeq 0$, where $z(0)$ is the initial error and κ is a system-related constant.

Remark 1: Notice that an equivalent description of the condition in Definition 2 is $\alpha(\mathbb{A}) \prec -\sigma$ with $\sigma \succ 0$.

Obviously from (7) and Definition 2, the leader-follower consensus of the system in (2) and (3) is guaranteed with an exponential convergence rate $\sigma \succ 0$ if and only if the spectral abscissa $\alpha(\mathbb{A}) \prec -\sigma$. As known in [19], there exists a unitary matrix U such that

$$(U \otimes I_n)^{\mathrm{T}}\mathbb{A}(U \otimes I_n) = I_n \otimes A - \Lambda \otimes BKC \tag{8}$$

where

$$\Lambda = U^{\mathrm{T}}(L+G)U = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Expanding (8) yields

$$I_n \otimes A - \Lambda \otimes BKC = \begin{bmatrix} A-\lambda_1 BKC & 0 & \ldots & 0 \\ 0 & A-\lambda_2 BKC & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A-\lambda_n BKC \end{bmatrix}. \tag{9}$$

It follows from (9) that the spectral abscissa $\alpha(\mathbb{A}) \prec -\sigma$ holds if and only if $\alpha(A - \lambda_i BKC) \prec -\sigma$, $i \in \mathscr{I}$, hold, or equivalently, there exist matrices $P_i > 0$, $i \in \mathscr{I}$, such that $(A - \lambda_i BKC)^{\mathrm{T}} P_i + P_i(A - \lambda_i BKC) < -2\sigma P_i$. In this article, the positive leader-follower consensus of nominal and uncertain multiagent systems with a guaranteed convergence rate are studied, which are defined as follows:

Problem Nonnegative Consensus Tracking of Networked Systems (NCTNS): For any unknown initial $x(0) \succeq 0$, design the protocol in (5) such that the follower agents in (2) approach the leader in (3) asymptotically, that is, $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$, $\forall i \in \mathscr{I}$, with exponential convergence rate $\sigma \succ 0$ guaranteed, while the state trajectory of the multiagent system remains nonnegative for $t \succeq 0$.

Problem Robust Nonnegative Consensus Tracking of Networked Systems (RNCTNS): For any unknown initial $x(0) \succeq 0$, design the protocol in (5) such that the uncertain follower agents in (2) and (4), approach the uncertain leader in (3) and (4) asymptotically, that is, $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0$,

$\forall i \in \mathscr{I}$, with exponential convergence rate $\sigma \succ 0$ guaranteed, while the state trajectory of the uncertain multiagent system remains nonnegative for $t \succeq 0$.

III. Main Results

This section is devoted to the analysis and discussion of **Problem NCTNS** as well as **Problem RNCTNS**, for obtaining some necessary and sufficient conditions of consensus analysis and design. The corresponding numerical algorithms for the computation of consensus controllers are also developed.

A. Consensus Analysis

First, **Problem NCTNS** is considered. For ease of illustration, we define $l_{\max} := \max_{i \in \mathscr{I}} l_i$. According to Lemma 1, and observing the whole closed-loop system in (6) as well as noticing the fact that the leader-follower consensus is achieved if and only if all the matrices $A_i := A - \lambda_i BKC$, $i \in \mathscr{I}$, are Hurwitz [24], a necessary and sufficient condition for solving **Problem NCTNS** can be obtained as below.

Proposition 1: **Problem NCTNS** can be solved if and only if the following conditions are satisfied: 1) $BKC \succeq 0$, 2) $A - l_{\max}BKC$ is Metzler, and 3) $\alpha(A_i) \prec -\sigma$, $i \in \mathscr{I}$.

Now, we are going to discuss **Problem RNCTNS**. In the following of this article, we define $\beta := \max\{\lambda_n, l_{\max}\}$. By using the special characterizations of positive systems, we can obtain the condition to satisfy positive consensusability of **Problem RNCTNS**, the proof of which is also provided below.

Proposition 2: **Problem RNCTNS** can be solved if the following conditions hold: 1) $K \succeq 0$, 2) $A_{\rm m} - \beta B_{\rm M} K C_{\rm M}$ is Metzler, 3) $\alpha(A_{\rm M} - \lambda_1 B_{\rm m} K C_{\rm m}) \prec -\sigma$.

Proof: Since β is the larger value between λ_n and $l_{\max}$, moreover, 1) $K \succeq 0$ and $B_{\rm m}KC_{\rm m} \succeq 0$, we have the following facts: $A_{\rm M} - \lambda_i B_{\rm m}KC_{\rm m} \succeq A_{\rm m} - \lambda_i B_{\rm M}KC_{\rm M} \succeq A_{\rm m} - \lambda_{i+1} B_{\rm M}KC_{\rm M}$, $A_{\rm M} - \lambda_i B_{\rm m}KC_{\rm m} \succeq A_{\rm M} - \lambda_{i+1} B_{\rm m}KC_{\rm m} \succeq A_{\rm m} - \lambda_{i+1} B_{\rm M}KC_{\rm M} \succeq A_{\rm m} - \lambda_n B_{\rm M}KC_{\rm M} \succeq A_{\rm m} - \beta B_{\rm M}KC_{\rm M}$ and $A_{\rm m} - l_{\max} B_{\rm M}KC_{\rm M} \succeq A_{\rm m} - \beta B_{\rm M}KC_{\rm M}$. Also, we have $B_{\rm M}KC_{\rm M} \succeq BKC \succeq B_{\rm m}KC_{\rm m}$, $A_{\rm M} - \lambda_i B_{\rm m}KC_{\rm m} \succeq A - \lambda_i BKC \succeq A_{\rm m} - \lambda_i B_{\rm M}KC_{\rm M} \succeq A_{\rm m} - \lambda_n B_{\rm M}KC_{\rm M} \succeq A_{\rm m} - \beta B_{\rm M}KC_{\rm M}$, and $A_{\rm M} - l_{\max} B_{\rm m}KC_{\rm m} \succeq A - l_{\max}BKC \succeq A_{\rm m} - l_{\max} B_{\rm M}KC_{\rm M}$. Hence, if 2) $A_{\rm m} - \beta B_{\rm M}KC_{\rm M}$ is Metzler, then $A - l_{\max}BKC$ and $A - \lambda_i BKC$, $\forall i \in \mathscr{I}$, where $A_{\rm m} \preceq A \preceq A_{\rm M}$, $B_{\rm m} \preceq B \preceq B_{\rm M}$ and $C_{\rm m} \preceq C \preceq C_{\rm M}$, are all Metzler. In addition, it follows from the result in Lemma 4 that $A - \lambda_i BKC$, $\forall i \in \mathscr{I}$, are Hurwitz if $A_{\rm M} - \lambda_1 B_{\rm m}KC_{\rm m}$ is Hurwitz. Also, according to the aforementioned consensusability convergence rate analysis and Lemma 4, the convergence rate of uncertain networked positive systems is guaranteed if 3) $\alpha(A_{\rm M} - \lambda_1 B_{\rm m}KC_{\rm m}) \prec -\sigma$. Then based on the analysis process of Proposition 1, it can be seen that **Problem RNCTNS** can be solved if 1) to 3) hold. Therefore, the proof is completed. □

Remark 2: It follows from the proof of Proposition 2 that conditions 1) to 3) ensure matrices A_i, $\forall i \in \mathscr{I}$, $A - l_{\max}BKC$ and $A - \beta BKC$, where $A_{\rm m} \preceq A \preceq A_{\rm M}$, $B_{\rm m} \preceq B \preceq B_{\rm M}$ and $C_{\rm m} \preceq C \preceq C_{\rm M}$, are all Metzler and Hurwitz.

If there is no uncertainty in the system parameters, on the basis of Propositions 1 and 2, one can readily obtain the following corollary for **Problem NCTNS**:

Corollary 1: **Problem NCTNS** can be solved if the following conditions are satisfied: 1) $BKC \succeq 0$, 2) $A - \beta BKC$ is Metzler, 3) $\alpha(A - \lambda_1 BKC) \prec -\sigma$.

In order to overcome the difficulties of solving **Problem NCTNS** and **Problem RNCTNS**, a **System Augmentation Approach** needs to be developed. For the following closed-loop state-space model: For $i \in \mathscr{I}$:

$$\dot{\xi}_i(t) = (A - \lambda_i BKC)\xi_i(t) \Longleftarrow \begin{cases} \dot{\xi}_i(t) = A\xi_i(t) + \lambda_i B\tilde{u}_i(t) \\ \tilde{y}_i(t) = C\xi_i(t) \\ \tilde{u}_i(t) = -K\tilde{y}_i(t) \end{cases} \tag{10}$$

one can introduce a new state component $\tilde{u}_i(t)$ and define the state variable as $\bar{x}_i(t) = \begin{bmatrix} \xi_i^{\rm T}(t) & \tilde{u}_i^{\rm T}(t) \end{bmatrix}^{\rm T}$, then an equivalent augmented system of (10) is obtained as follows:

$$\mathbf{E}\dot{\bar{x}}_i(t) = \mathbf{A}_i \bar{x}_i(t) \tag{11}$$

where

$$\mathbf{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_i = \begin{bmatrix} A & \lambda_i B \\ -KC & -I \end{bmatrix} \quad \forall i \in \mathscr{I}.$$

The system augmentation approach for SOF consensus is derived based on the descriptor system in (11). The basic idea is to construct a Lyapunov function for the stability of the augmented system in (11). With this idea, some results based on Propositions 1 and 2 are obtained in the following theorems.

Theorem 1: **Problem NCTNS** can be solved if and only if there are matrices $P_i > 0$ such that the following conditions hold: 1) $BKC \succeq 0$, 2) $A - l_{\max}BKC$ is Metzler, 3)

$$\mathbf{A}_i^{\rm T}\mathbf{P}_i + \mathbf{P}_i^{\rm T}\mathbf{A}_i + 2\sigma \mathbf{E}^{\rm T}\mathbf{P}_i < 0 \tag{12}$$

where $\mathbf{A}_i$ are defined in (11)

$$\mathbf{P}_i = \begin{bmatrix} P_i & 0 \\ \frac{1}{2}KC & \frac{1}{2}I \end{bmatrix}, \quad i \in \mathscr{I}.$$

Proof: Notice that conditions 1) and 2) are as same as those in Proposition 1. What needs to be proven as well is that condition 3) is the counterpart of the corresponding condition 3) in Proposition 1.

Sufficiency of 3): Define a nonsingular matrix as

$$T = \begin{bmatrix} I & 0 \\ -KC & I \end{bmatrix}.$$

By performing a congruent transformation on (12) with $T^{\rm T}$ and T, one can get

$$\begin{aligned} \Psi_i &:= T^{\rm T}\big(\mathbf{A}_i^{\rm T}\mathbf{P}_i + \mathbf{P}_i^{\rm T}\mathbf{A}_i + 2\sigma \mathbf{E}^{\rm T}\mathbf{P}_i\big)T \\ &= \begin{bmatrix} A_i^{\rm T}P_i + P_iA_i + 2\sigma P_i & \lambda_i P_i B \\ \lambda_i B^{\rm T}P_i & -I \end{bmatrix} < 0 \end{aligned} \tag{13}$$

from which the first leading principal signifies that matrices $\alpha(A_i) \prec -\sigma$, $i \in \mathscr{I}$, are Hurwitz.

Necessity of 3): If all matrices A_i are Hurwitz and $\alpha(A_i) \prec -\sigma$, then a set of matrices $Q_i > 0$, $i \in \mathscr{I}$ can be found to satisfy the following condition:

$$A_i^{\mathrm{T}} Q_i + Q_i A_i + 2\sigma Q_i < 0.$$

A set of scalars $c_i \succ 0$, $i \in \mathscr{I}$ can be found to satisfy that

$$-I - c_i \lambda_i^2 B^{\mathrm{T}} Q_i \left(A_i^{\mathrm{T}} Q_i + Q_i A_i + 2\sigma Q_i\right)^{-1} Q_i B < 0. \quad (14)$$

Notice that

$$\begin{aligned} &-I - c_i \lambda_i^2 B^{\mathrm{T}} Q_i \left(A_i^{\mathrm{T}} Q_i + Q_i A_i + 2\sigma Q_i\right)^{-1} Q_i B \\ &= -I - \lambda_i^2 B^{\mathrm{T}} c_i Q_i \left(A_i^{\mathrm{T}} c_i Q_i + c_i Q_i A_i + 2c_i\sigma Q_i\right)^{-1} c_i Q_i B. \end{aligned}$$

Let $P_i = c_i Q_i$, $i \in \mathscr{I}$, then (14) becomes

$$-I - \lambda_i^2 B^{\mathrm{T}} P_i \left(A_i^{\mathrm{T}} P_i + P_i A_i + 2\sigma P_i\right)^{-1} P_i B < 0. \quad (15)$$

By Schur complement equivalence and appropriate matrix manipulations, it follows that:

$$\mathbf{A}_i^{\mathrm{T}} \mathbf{P}_i + \mathbf{P}_i^{\mathrm{T}} \mathbf{A}_i + 2\sigma \mathbf{E}^{\mathrm{T}} \mathbf{P}_i = T^{-\mathrm{T}} \Psi_i T^{-1} < 0$$

where Ψ_i is defined as (13). The whole proof is completed. □

Theorem 2: **Problem RNCTNS** can be solved if there is a diagonal matrix $P > 0$ such that the following conditions can be satisfied: 1) $K \succeq 0$, 2) $A_{\mathrm{m}} - \beta B_{\mathrm{M}} K C_{\mathrm{M}}$ is Metzler, 3)

$$\mathbf{A}^{\mathrm{T}} \mathbf{P} + \mathbf{P}^{\mathrm{T}} \mathbf{A} + 2\sigma \mathbf{E}^{\mathrm{T}} \mathbf{P} < 0 \quad (16)$$

where

$$\mathbf{A} = \begin{bmatrix} A_{\mathrm{M}} & \lambda_1 B_{\mathrm{m}} \\ -KC_{\mathrm{m}} & -I \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P & 0 \\ \frac{1}{2} K C_{\mathrm{m}} & \frac{1}{2} I \end{bmatrix}. \quad (17)$$

Sketch of Proof: Conditions 1) and 2) are as same as those in Proposition 2. Hence, condition 3) is to be proven. By defining $\bar{T}$ as

$$\bar{T} = \begin{bmatrix} I & 0 \\ -KC_{\mathrm{m}} & I \end{bmatrix}$$

then it follows from Proposition 2, Lemma 2 as well as the proof of Theorem 1 that (16) (that is, condition 3) in Theorem 2 is comparable to condition 3) in Proposition 2. The proof is completed. □

B. Consensus Design With Convergence Rate Optimization

In the last subsection, the consensusability with convergence rate analysis of networked positive systems is discussed. In this section, the consensus design conditions via semidefinite programming are derived and the algorithms are also developed.

1) Consensus Design via Semidefinite Programming: First, for **Problem NCTNS**, the consensus design condition corresponding to Theorem 1 is developed in the following.

Theorem 3: **Problem NCTNS** can be solved under the necessary and sufficient conditions that there are matrices $P_i > 0$, $i \in \mathscr{I}$, K and M which satisfy:

1) $BKC \succeq 0$,
2) $A - l_{\max} BKC$ is Metzler,
3)

$$\Phi_i(M) = \begin{bmatrix} A^{\mathrm{T}} P_i + P_i A + 2\sigma P_i + \Gamma & \lambda_i P_i B - C^{\mathrm{T}} K^{\mathrm{T}} \\ \lambda_i B^{\mathrm{T}} P_i - KC & -I \end{bmatrix} < 0 \quad (18)$$

where $\Gamma = -C^{\mathrm{T}} K^{\mathrm{T}} M - M^{\mathrm{T}} K C + M^{\mathrm{T}} M$. Once the conditions hold, a controller matrix K is obtained.

Proof: Since conditions 1) and 2) remain the same as those in Theorem 1, we will prove condition 3).

Sufficiency of 3): It is worth noticing that, for any matrix M, $(KC - M)^{\mathrm{T}}(KC - M) \geq 0$ gives rise to

$$-C^{\mathrm{T}} K^{\mathrm{T}} K C \leq -C^{\mathrm{T}} K^{\mathrm{T}} M - M^{\mathrm{T}} K C + M^{\mathrm{T}} M.$$

Then it follows from (18) that (12) holds.

Necessity of 3): If **Problem NCTNS** can be solved, then one can always find matrices $P_i > 0$, $i \in \mathscr{I}$, such that the condition in (13) holds. By setting $M = KC$, it follows that:

$$-C^{\mathrm{T}} K^{\mathrm{T}} K C = -C^{\mathrm{T}} K^{\mathrm{T}} M - M^{\mathrm{T}} K C + M^{\mathrm{T}} M.$$

Substituting this into (12) gives rise to that (18) holds. On the basis of Theorem 1, **Problem NCTNS** can be solved if and only if conditions 1) to 3) are ensured. So far the proof process is finished. □

For **Problem RNCTNS**, performing a similar analysis as the proof in Theorem 3, the consensus design condition on the basis of Theorem 2 is obtained as follows.

Theorem 4: **Problem RNCTNS** can be solved if there are a diagonal matrix $P > 0$, matrices K and M, which enable the conditions below establish:

1) $K \succeq 0$,
2) $A_{\mathrm{m}} - \beta B_{\mathrm{M}} K C_{\mathrm{M}}$ is Metzler,
3)

$$\Phi_{\mathrm{R}}(M) = \begin{bmatrix} A_{\mathrm{M}}^{\mathrm{T}} P + P A_{\mathrm{M}} + 2\sigma P + \bar{\Gamma} & \lambda_1 P B_{\mathrm{m}} - C_{\mathrm{m}}^{\mathrm{T}} K^{\mathrm{T}} \\ \lambda_1 B_{\mathrm{m}}^{\mathrm{T}} P - K C_{\mathrm{m}} & -I \end{bmatrix} < 0 \quad (19)$$

where $\bar{\Gamma} = -C_{\mathrm{m}}^{\mathrm{T}} K^{\mathrm{T}} M - M^{\mathrm{T}} K C_{\mathrm{m}} + M^{\mathrm{T}} M$. Once the conditions hold, a controller matrix K is obtained.

Remark 3: It is well known that the positive SOF control problem of positive systems will result in bilinear inequality conditions. Although several well-known methods have been proposed for solving the positive SOF control problem of a single positive system [12], [14], [27], [31], however, those methods are not applicable to the problem of multiple positive systems.

Remark 4: Notice that the conditions from (18) to (19) belong to nonlinear/nonconvex matrix inequalities which are not easy to solve. However, if M is known, they will all become linear matrix inequalities (LMIs) which are convex. Furthermore, if one defines the LMI as diag($\Phi_1(M)$, $\Phi_2(M), \ldots, \Phi_n(M)$) $< \gamma I$ where γ is a scalar variable, from the proof in Theorem 3, the value of γ will reach its minimum value when $M = KC$. Therefore, if M is defined with suitable values, **Problem NCTNS** (and **Problem RNCTNS**) can be further solved by an iterative algorithm. The proof in Theorem 3 implies that the initialization of M would also determine whether the iterative algorithm leads to a solution.

By virtue of the theoretical results in Theorems 3 and 4, we develop an algorithm (given in Algorithm 1) to solve **Problem NCTNS** and **Problem RNCTNS**.

Algorithm 1 NCTNS1

Step 1: Initialize the indexes $k = 1$ and $\gamma^{(0)} = 0$. Considering (2) and (3), compute the initial matrix $M^{(k)}$: $A - \lambda_i BM^{(k)}$, $i \in \mathscr{I}$, are Hurwitz.

Step 2: Set $M = M^{(k)}$ and minimize $\gamma^{(k)}$

$$\text{Case 1 : s.t.} \begin{cases} BKC \succeq 0 \\ A - l_{\max} BKC \in \mathbb{M}^r \\ \operatorname{diag}(\Phi_1(M), \Phi_2(M), \ldots, \Phi_n(M)) < \gamma^{(k)} I \end{cases}$$

with respect to $P_i > 0$ and K

$$\text{Case 2 : s.t.} \begin{cases} K \succeq 0 \\ A_{\rm m} - \beta B_{\rm M} KC_{\rm M} \in \mathbb{M}^r \\ \Phi_{\rm R}(M) < \gamma^{(k)} I \end{cases}$$

with respect to $P > 0$ (diagonal) and K.

Step 3: If $0 \succeq \gamma^{(k)}$, K is thus confirmed. This algorithm is completed here. Otherwise, move to Step 4.

Step 4: If $|\gamma^{(k)} - \gamma^{(k-1)}|/\gamma^{(k)} \prec \eta$, where η is a pregiven tolerance, it is unable to get the required solution via this algorithm. End the algorithm. Otherwise, assign matrix $M^{(k+1)} = KC$ (respectively, $M^{(k+1)} = KC_{\rm m}$ for Case 2), update $k = k + 1$, then go to Step 2.

2) Convergence Rate Optimization via Semidefinite Programming: The robust case is considered and we assume that **Problem RNCTNS** is solvable with the conditions in Proposition 2 satisfied. According to Proposition 2, one has $K \succeq 0$, $A_{\rm m} - \beta B_{\rm M} KC_{\rm M}$ is Metzler and $\alpha(A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m}) \prec -\sigma$, which indicates that $A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m} \succeq A - \lambda_1 BKC \succeq A - \lambda_2 BKC \succeq \cdots \succeq A - \lambda_n BKC \succeq A_{\rm m} - \beta B_{\rm M} KC_{\rm M}$; moreover, $A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m}$ and $A - \lambda_i BKC$, $i \in \mathscr{I}$, are Metzler (and Hurwitz). According to Lemma 4, one has $\alpha(A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m}) \succeq \alpha(A - \lambda_1 BKC) \succeq \alpha(A - \lambda_2 BKC) \succeq \cdots \succeq \alpha(A - \lambda_n BKC)$. Hence, from (9), one has $\alpha(\mathbb{A}) = \alpha(A - \lambda_1 BKC) \preceq \alpha(A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m})$. That means one can improve the convergence rate by minimizing $\alpha(A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m})$. Now, the question is how to minimize $\alpha(A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m})$. Notice that $K \succeq 0$ and $B_{\rm m} KC_{\rm m} \succeq 0$. According to Lemma 4, one can minimize $\alpha(A_{\rm M} - \lambda_1 B_{\rm m} KC_{\rm m})$ by *increasing the values of the elements of matrix K*. To construct an optimization algorithm, one can define an objective function as $\|K\|_1$ over the conditions in Theorem 4 and try to maximize it. Therefore, based on the discussions above, and the result of Theorem 4, one can develop an iterative algorithm for convergence rate optimization, which is given in Algorithm 2.

Remark 5: Notice that the condition $\alpha(\mathbb{A}) = \alpha(A - \lambda_1 BKC)$ does not always hold in Proposition 1. For ***Problem NCTNS***, one cannot draw a similar conclusion for convergence rate optimization as ***Problem RNCTNS*** unless the conditions in Corollary 1 are satisfied.

Remark 6: Algorithm 1 is developed to solve ***Problem RNCTNS*** and ***Problem NCTNS***. In Algorithm 2, only ***Problem RNCTNS*** is considered on the basis of Theorem 4 for improving the convergence rate.

Algorithm 2 NCTNS2

Step 1: Set $k = 1$. For the multiagent system in (2) and (3), compute a feasible solution matrix $K^{(1)}$ by Algorithm 1. Then set $s^{(0)} \triangleq \|K^{(1)}\|_1$ and an initial matrix $M^{(1)}$ is obtained as $M^{(1)} = KC_{\rm m}$.

Step 2: Fix $M = M^{(k)}$

$$\text{Case 2 : Max } s^{(k)} \triangleq \|K\|_1 \quad \text{s.t.} \begin{cases} K \succeq K^{(k)} \\ A_{\rm m} - \beta B_{\rm M} KC_{\rm M} \in \mathbb{M}^r \\ \Phi_{\rm R}(M) < 0. \end{cases}$$

Step 3: If $|s^{(k)} - s^{(k-1)}|/s^{(k)} \prec \eta$, where η is a prescribed tolerance, then an optimized K is obtained. End the algorithm. Otherwise, assign $M^{(k+1)} = KC_{\rm m}$ and $K^{(k+1)} = K$, set $k = k + 1$ and go back to Step 2.

IV. SIMO Agents via Linear Programming

Due to the positivity of agents, in what follows, the consensus design with convergence rate optimization for **Problem RNCTNS** is to be discussed via the copositive Lyapunov function approach. An important lemma regarding SOF control for SIMO positive linear systems which will be used in proving the theorem is presented at first.

Lemma 5: Suppose that (1) is an SIMO positive linear system and there exists a row vector K such that matrix $A - BKC$ is Metzler. Then the following statements are equivalent:

1) Matrix $A - BKC$ is Hurwitz,

2) There exist a column vector $p \succ 0$ and a row vector K such that $p^{\rm T} A - KC \prec 0$ and $p^{\rm T} B = 1$ hold.

Proof: 2) $\Rightarrow$ 1): It follows from $p^{\rm T} A - KC \prec 0$ and $p^{\rm T} B = 1$ where $p \succ 0$ is a column vector that $p^{\rm T}(A - BKC) \prec 0$ holds. According to Lemma 3, $A - BKC$ is Hurwitz (and Metzler).

1) $\Rightarrow$ 2): From Lemma 3, it follows that if condition 1) holds, there must exist a column vector $q \succ 0$ such that $q^{\rm T}(A - BKC) \prec 0$ holds. Notice that $q^{\rm T} B \succ 0$ is a scalar. Selecting $p^{\rm T} = q^{\rm T}/(q^{\rm T} B)$, we have that $p^{\rm T} A - KC = (q^{\rm T}/(q^{\rm T} B))A - KC = (q^{\rm T}/(q^{\rm T} B))A - ((q^{\rm T} B)/(q^{\rm T} B))KC = (q^{\rm T}/(q^{\rm T} B))A - (q^{\rm T}/(q^{\rm T} B))BKC = (1/(q^{\rm T} B))q^{\rm T}(A - BKC) \prec 0$. Also, $p^{\rm T} B = (q^{\rm T}/(q^{\rm T} B))B = ((q^{\rm T} B)/(q^{\rm T} B)) = 1$. Therefore, we can conclude that a column vector $p \succ 0$ and a row vector K can be found to satisfy $p^{\rm T} A - KC \prec 0$ and $p^{\rm T} B = 1$ if condition 1) holds. □

Using the above lemma, the theorem for **Problem RNCTNS** is obtained. It should be pointing that the result is presented under the framework of linear programming and thus can be solved very efficiently.

Theorem 5: For SIMO agents, **Problem RNCTNS** can be solved if a column vector $p \succ 0$ and a row vector K exist such that the following conditions hold: 1) $K \succeq 0$, 2) $A_{\rm m} - \beta B_{\rm M} KC_{\rm M} \in \mathbb{M}^r$, 3) $p^{\rm T}(A_{\rm M} + \sigma I) - \lambda_1 KC_{\rm m} \prec 0$, 4) $p^{\rm T} B_{\rm m} = 1$.

The proof can be done by applying Lemma 5 to the statement (iii) of Proposition 2.

In Section III-B, we have discussed the consensus convergence rate and come up with the idea to improve the convergence rate by increasing the values of the elements of matrix K. Based on such an idea as well as the results in Theorem 5, one can develop a linear-programming-based algorithm for **Problem RNCTNS**.

Algorithm 3 NCTNS3

Maximize $\|K\|_1$ subject to

$$\text{SIMO}:\begin{cases} K \succeq 0 \\ A_{\mathrm{m}} - \beta B_{\mathrm{M}} K C_{\mathrm{M}} \in \mathbb{M}^r \\ p^{\mathrm{T}} A_{\mathrm{M}} - \lambda_1 K C_{\mathrm{m}} \prec 0 \\ p^{\mathrm{T}} B_{\mathrm{m}} = 1 \end{cases}$$

with respect to variables: vectors $p \succ 0$ and K.

Remark 7: In Algorithm 3, since the linear-programming-based problem is convex, the objective function $\|K\|_1$ obtained is globally optimal subject to the condition in Theorem 5. However, in Algorithm 2, the obtained $\|K\|_1$ is a locally maximal solution. Hence, for SIMO agents, one should use Algorithm 3 to solve **Problem RNCTNS** to obtain a satisfactory consensus convergence rate rather than those in Section III-B.

V. Case Studies

To validate the effectiveness of the proposed criteria, a numerical example and a positive tunnel diode circuit system are considered in this section. Implementing Algorithms 1 to 3 with the MATLAB solver SeDuMi, the results obtained under different cases are carried out to illustrate the effectiveness of the results.

A. Numerical MIMO Case

In this section, we validate the proposed LMI approaches for solving **Problem NCTNS** and **Problem RNCTNS**, using a simulated MIMO example. Consider a networked positive system in the form of (2) and (3) containing one leader and four followers. For each agent, the system matrices are

$$A = \begin{bmatrix} -10 & 6 & 4 \\ 4 & -8 & 4 \\ 7 & 5 & -12 \end{bmatrix},\quad B = \begin{bmatrix} 3 & 0 \\ 1 & 0 \\ 2 & 2 \end{bmatrix},\quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}.$$

Suppose that

$$A_{\mathrm{m}} = \begin{bmatrix} -10.1 & 5.8 & 4 \\ 3.9 & -8.2 & 4 \\ 7 & 4.9 & -12 \end{bmatrix},\quad B_{\mathrm{m}} = \begin{bmatrix} 2.9 & 0 \\ 0.9 & 0 \\ 1.9 & 1.9 \end{bmatrix}$$

$$C_{\mathrm{m}} = \begin{bmatrix} 1.9 & 0 & 0 \\ 0 & 1.9 & 2.9 \end{bmatrix},\quad A_{\mathrm{M}} = \begin{bmatrix} -9.9 & 6.2 & 4 \\ 4.1 & -7.8 & 4 \\ 7 & 5.1 & -12 \end{bmatrix}$$

$$B_{\mathrm{M}} = \begin{bmatrix} 3.1 & 0 \\ 1.1 & 0 \\ 2.1 & 2.1 \end{bmatrix},\quad C_{\mathrm{M}} = \begin{bmatrix} 2.1 & 0 & 0 \\ 0 & 2.1 & 3.1 \end{bmatrix}.$$

Moreover, suppose the graph in Fig. 1 is used for modeling the communication among agents, where the information of leader can be sensed only by Followers 1 and 2. It is assumed that the Laplacian matrix of follower systems is as follows:

TABLE I

Spectral Abscissae $\alpha(\mathbb{A})$ Corresponding to (20)–(22)

Theorem/Algorithm	Theorem 3	Theorem 4	Algorithm 3
K	(20)	(21)	(22)
α	−0.71244	−0.40154	−0.51816

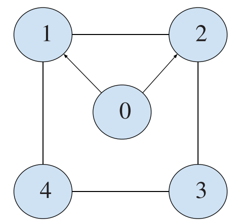

Fig. 1. Communication graph.

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

and the diagonal matrix representing the interconnection from the leader to followers is $G = \mathrm{diag}(1, 1, 0, 0)$. In this case, the eigenvalues of $L + G$ are $\lambda_1 = 0.3820$, $\lambda_2 = 2.3820$, $\lambda_3 = 2.6180$, $\lambda_4 = 4.6180$. Also, $l_{\max} = 3$. We have implemented Algorithms 1 and 2, and found that the conditions in Theorem 3 with $\sigma = 0.6$ are feasible with

$$K = \begin{bmatrix} 0.27866 & 0.093495 \\ 0.093495 & -0.070142 \end{bmatrix} \tag{20}$$

and the conditions in Theorem 4 with $\sigma = 0.3$ are feasible with

$$K = \begin{bmatrix} 0.26492 & 0.078795 \\ 0.078795 & 0.023437 \end{bmatrix}. \tag{21}$$

It can be observed from (20) that the requirement of nonnegative controller gain which is necessary in the single-input single-output (SISO) case becomes unnecessary now. With solution (21), we implemented Algorithm 3 by using Yalmip, and found the following optimized solution:

$$K = \begin{bmatrix} 0.26492 & 0.078798 \\ 0.078798 & 0.16181 \end{bmatrix}. \tag{22}$$

The spectral abscissae $\alpha(\mathbb{A})$ corresponding to the solutions from (20)–(22) are listed in Table I.

We can find that the spectral abscissae $\alpha(\mathbb{A})$ with (20) and (21) are $-0.71244 \prec -0.6$ and $-0.40154 \prec -0.3$, respectively. By comparing the spectral abscissae $\alpha(\mathbb{A})$ with (21) and (22), we can see that the consensus convergence rate has been improved significantly.

B. Positive Tunnel Diode Circuit

It has been shown in many research works that positive systems theory does play an important role in modeling, analysis, and control design for positive electric circuit systems [15]–[17]. To verify the effectiveness of the linear-programming-based approaches for **Problem RNCTNS**, we consider a positive circuit networked system with identical

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

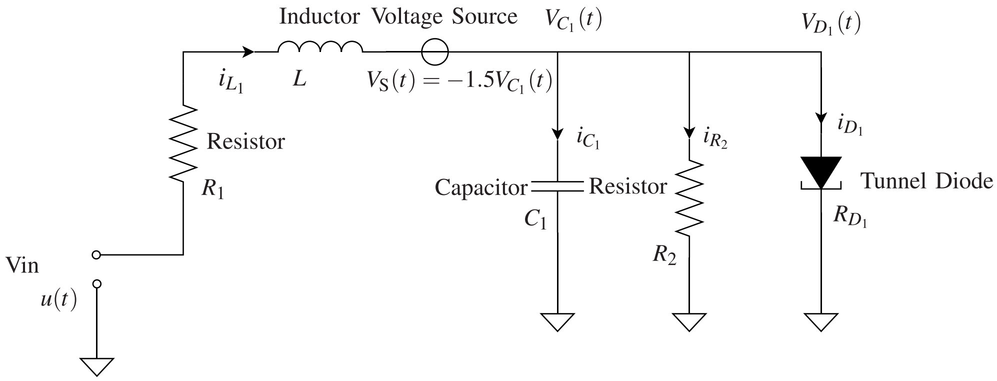

Fig. 2. Positive tunnel diode circuit.

TABLE II
SPECTRAL ABSCISSAE $\alpha(\mathbb{A})$ CORRESPONDING TO (24) AND (25)

Theorem/Algorithm	Theorem 5	Algorithm 4
K	(24)	(25)
α	-0.19646	-0.86716

positive tunnel diode circuit systems as agents. The positive tunnel diode circuit shown in Fig. 2 consists of two resistors, one capacitor, one inductor, one tunnel diode and one voltage source. Using Kirchhoff's Laws, the dynamic system is characterized by

$$\begin{cases} i_{L_1}(t) = C\dot{V}_{C_1}(t) + \dfrac{V_{C_1}(t)}{R_2} + \dfrac{V_{C_1}(t)}{R_D} \\ 1.5V_{C_1}(t) + u(t) = L_1\dot{i}_{L_1}(t) + V_{C_1}(t) + R_1 i_{L_1}(t). \end{cases} \tag{23}$$

Let the voltage of capacitor $V_{C_1}(t)$ and the current of Inductor $i_{L_1}(t)$ be state/output variables, (23) can be expressed as the positive system in (2). The parameters of the circuit in Fig. 2 are set as $R_1 = 0.5\ \Omega$, $C_1 = 1\ F$, $L_1 = 1\ H$, and $R_2 = 2\ \Omega$. R_{D_1} is modeled as an uncertain parameter [33] and set to be $2\ \Omega \preceq R_{D_1} \preceq 4\ \Omega$. One of the advantages of Algorithm 3 is that one can add some linear constraints on the controller gain such as $K \preceq K_{\mathrm{M}}$ which is essential from a practical point of view. In this case, we set $K \preceq [10, 10]$. Suppose that this SIMO case has the same number of agents and the communication graph as in Example 1.

By using the linear program of Theorem 5, a feasible solution is found as

$$K = \begin{bmatrix} 0.10798 & 0.7 \end{bmatrix}. \tag{24}$$

To show the effectiveness of Algorithm 4, we implemented it with Yalmip and obtained the optimized solution for **Problem RNCTNS** as follows:

$$K = \begin{bmatrix} 0.10827 & 10 \end{bmatrix}. \tag{25}$$

The spectral abscissae $\alpha(\mathbb{A})$ corresponding to the solutions from (24) and (25) are shown in Table II. From Table II, we can see that the spectral abscissa obtained via Algorithm 4, i.e., (25), is much greater, which has shown the effectiveness of Algorithm 4. The consensus tracking performance with the obtained controllers is illustrated using the circuit with the above parameters and $R_{D_1} = 2\ \Omega$. The evolution of voltage $V_{C_1}(t)$ (solid line) and current $i_{L_1}(t)$ (dotted line) is shown in Figs. 3–6 from which we can see that controller (25) has demonstrated faster consensus tracking performance compared with controller (24) (the initial states of Agents 0 to 4 are $[1, 2]^{\mathrm{T}}$, $[0.3, 0.5]^{\mathrm{T}}$, $[0.6, 1]^{\mathrm{T}}$, $[1.2, 1.5]^{\mathrm{T}}$ and $[2, 3]^{\mathrm{T}}$, respectively).

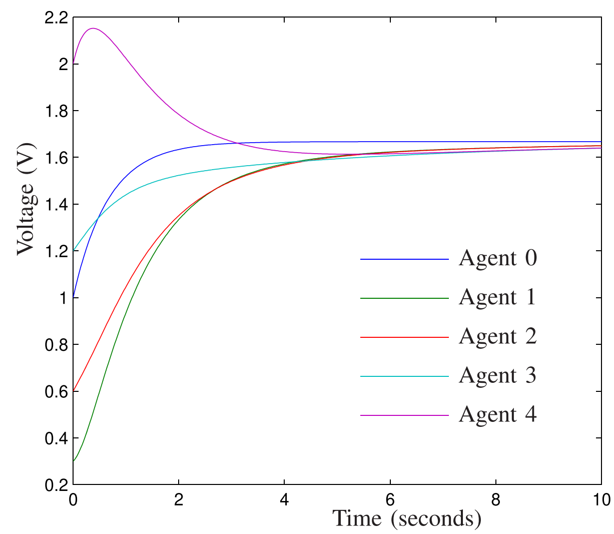

Fig. 3. Evolution of $V_{C_1}(t)$ of the circuits with controller (24).

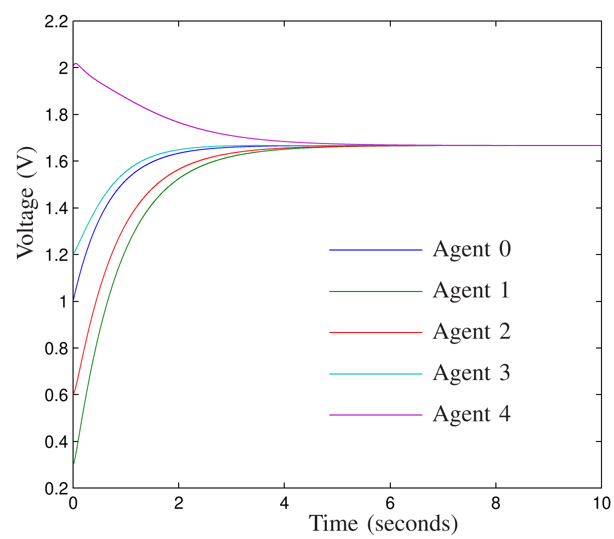

Fig. 4. Evolution of $V_{C_1}(t)$ of the circuits with controller (25).

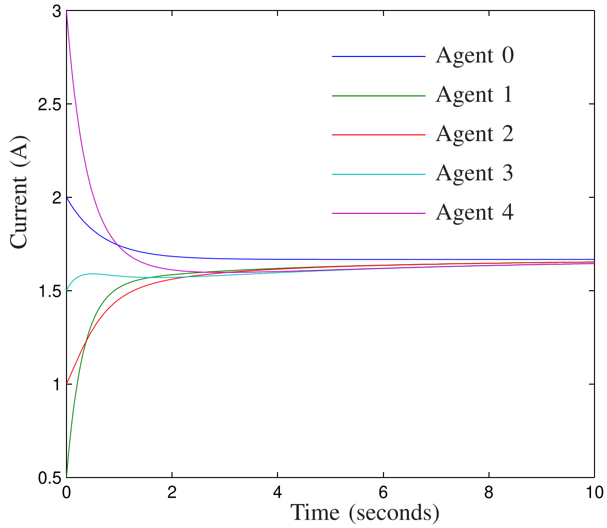

Fig. 5. Evolution of $i_{L_1}(t)$ of the circuits with controller (24).

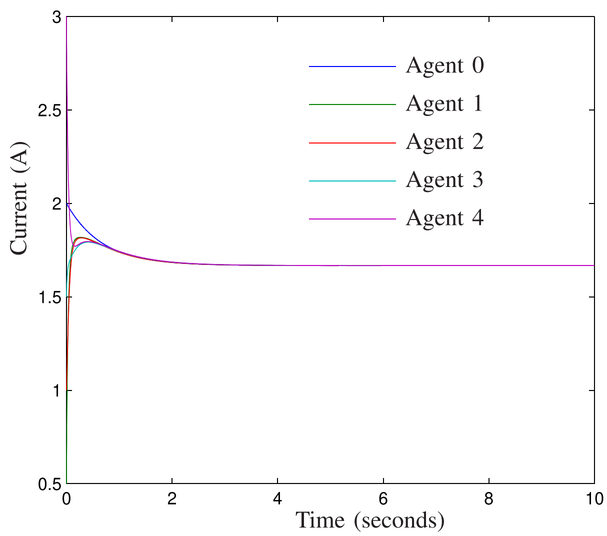

Fig. 6. Evolution of $i_{L_1}(t)$ of the circuits with controller (25).

The well-known approaches [12], [14], [27], [31] can solve the SOF control problem for a single positive system. For instance, by using the approach in [31] for this example, a controller is obtained in following:

$$K = \begin{bmatrix} 0.5827 & -0.5234 \end{bmatrix} \tag{26}$$

which can only guarantee partially conditions of Proposition 1. Controller (26) can only guarantee that

$$A_1 = A - \lambda_1 BKC = \begin{bmatrix} -1 & 1 \\ 0.27743 & -0.30008 \end{bmatrix} \tag{27}$$

is Metzler and Hurwitz as it has eigenvalues -1.2824 and -0.017663. However

$$A_2 = A - \lambda_2 BKC = \begin{bmatrix} -1 & 1 \\ -0.88799 & 0.74674 \end{bmatrix}$$

$$A_3 = A - \lambda_3 BKC = \begin{bmatrix} -1 & 1 \\ -1.0255 & 0.87026 \end{bmatrix}$$

$$A_4 = A - \lambda_4 BKC = \begin{bmatrix} -1 & 1 \\ -2.1909 & 1.9171 \end{bmatrix}$$

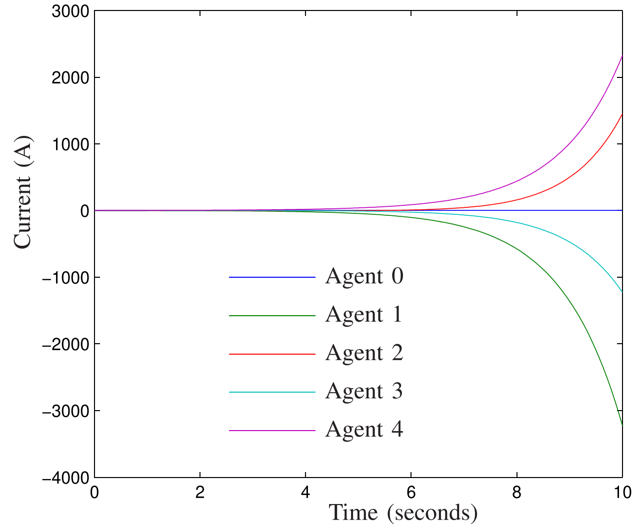

Fig. 7. Evolution of $V_{C_1}(t)$ of the circuits with controller (26).

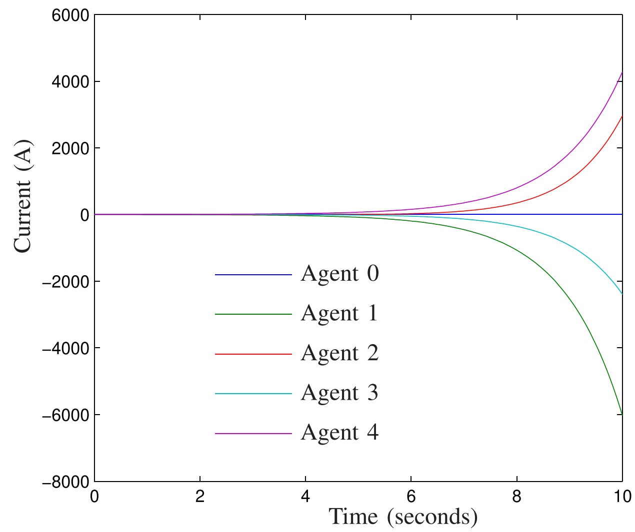

Fig. 8. Evolution of $i_{L_1}(t)$ of the circuits with controller (26).

are not Metzler and

$$BKC = \begin{bmatrix} 0 & 0 \\ 0.5827 & -0.5234 \end{bmatrix}$$

is not positive. Using this controller, the evolution of voltage $V_{C_1}(t)$ (solid line) and current $i_{L_1}(t)$ (dotted line) is shown in Figs. 7 and 8 from which we can see that neither consensus tracking nor positivity has been achieved.

C. Platoon Control of Holonomic Vehicles

The platooning of connected and automated vehicles has recently attracted extensive research interests due to its potential benefits to road traffic, e.g., enhancing highway safety, improving traffic capacity, and smoothness, and reducing fuel consumption [20]. Platoon control aims to ensure that all the vehicles in a group move at the same speed while maintaining the desired space between adjacent vehicles. In this example, the algorithm is applied to the platoon control of holonomic vehicles moving on a plane [13]. It is assumed that each holonomic vehicle in the group is assigned with a fixed and known relative position within the target formation. Both damping

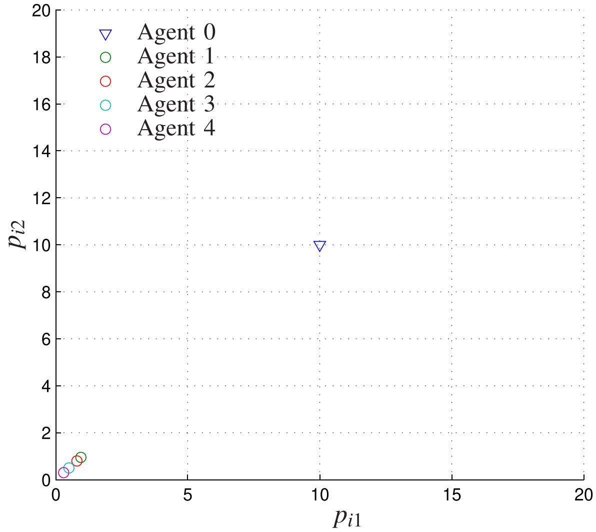

Fig. 9. Platooning of vehicles with controller (30) ($t = 0$ s).

and actuator dynamics are considered in these vehicles. The n vehicles are modeled as, for $i \in \mathscr{I}$

$$\begin{cases} \dot{p}_i(t) = q_i(t) \\ \dot{q}_i(t) = -\alpha_1 q_i(t) + \upsilon(t) \\ \dot{\upsilon}_i(t) = -\alpha_2 \upsilon_i(t) + l(t) \end{cases} \tag{28}$$

where $p_i(t) := [p_{i1}(t), p_{i1}(t)]^{\mathrm{T}} \in \mathbb{R}^2$, and $q_i(t) := [q_{i1}(t), q_{i1}(t)]^{\mathrm{T}} \in \mathbb{R}^2$ are the position and the velocity of the ith vehicle, respectively, and $\upsilon_i(t) := [\upsilon_{i1}(t), \upsilon_{i1}(t)]^{\mathrm{T}} \in \mathbb{R}^2$ is an actuator state, and $l_i(t) := [l_{i1}(t), l_{i1}(t)]^{\mathrm{T}} \in \mathbb{R}^2$ is a control input. The parameters $\alpha_1 \succeq 0$ and $\alpha_2 \succ 0$ characterize the damping and the actuator dynamics. The leader vehicle is represented by

$$\begin{cases} \dot{p}_0(t) = q_0(t) \\ \dot{q}_0(t) = -\alpha_1 q_0(t) + \upsilon(t) \\ \dot{\upsilon}_0(t) = -\alpha_2 \upsilon_0(t). \end{cases} \tag{29}$$

In addition, it is assumed that the measured output is

$$y_i(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_i(t) \\ q_i(t) \\ \upsilon_i(t) \end{bmatrix}.$$

It is obvious that the vehicles represented in (28) and (29) are positive systems. The platoon control of vehicles is said to be achieved if their velocity vectors converge to the same value and the positions maintain a prescribed separation, that is, $p_i(t) - d_i \rightarrow p_j(t) - d_j$ and $\upsilon_i(t) \rightarrow \upsilon_j(t)$ as $t \rightarrow \infty$ for any $i, j \in \mathscr{I}$, where $d_i = [d_{i1}, d_{i2}]^{\mathrm{T}} \in \mathbb{R}^2$ is the constant target position of the ith vehicle relative to the center of the formulation. Since each vehicle is assumed to be holonomic, and they evolve independently in the two planar directions, we can solve the consensus problem in each direction separately. Suppose here we have the same number of agents and the communication graph as in Example 1. The system parameters are chosen as $\alpha_1 = 0$, and $\alpha_2 = 1$. In this case, the conditions $BKC \succeq 0$ and $A - l_{\max} BKC \in \mathbb{M}^r$ are ignored to obtain a feasible solution. Then Algorithm 1 gives

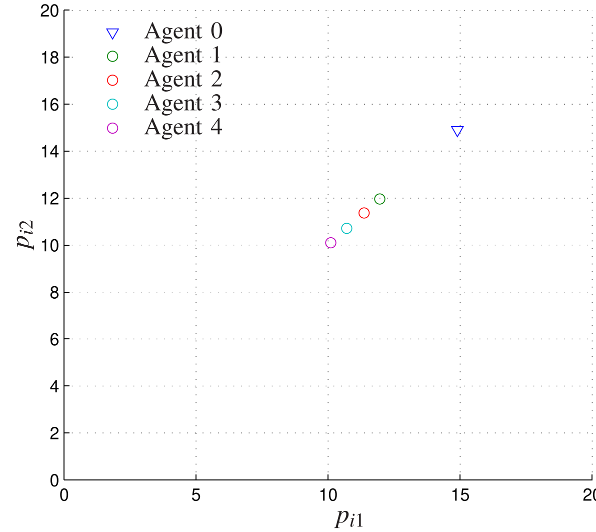

Fig. 10. Platooning of vehicles with controller (30) ($t = 5$ s).

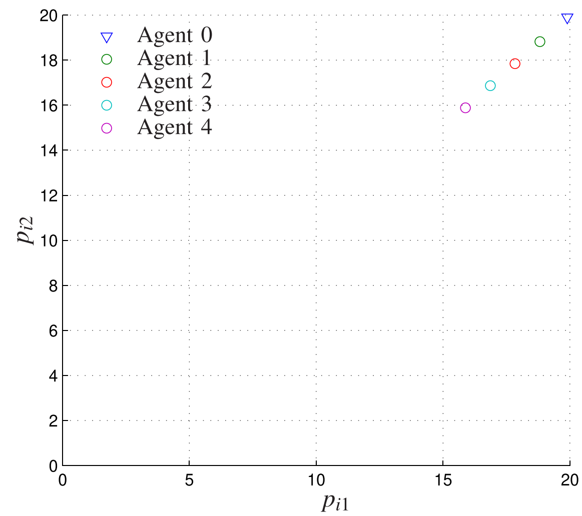

Fig. 11. Platooning of vehicles with controller (30) ($t = 10$ s).

a feasible solution as follows:

$$K = \begin{bmatrix} 73.555 & 248.65 \end{bmatrix}. \tag{30}$$

For simulation, it is assumed that

$$d_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \; d_3 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \; d_4 = \begin{bmatrix} -4 \\ -4 \end{bmatrix}.$$

Using (30), the platooning of vehicles is shown in Figs. 9–11.

VI. Conclusion

In this article, the consensus tracking problem has been solved for positive networked systems with SOF control. Both the nominal and uncertain cases have been considered. All the agents are positive state-space models, and the communication graph among followers is undirected and connected. Three consensus analysis conditions for nominal and uncertain networked positive systems have been obtained applying the positive systems theory. Then by introducing the system augmentation approach, two consensus design conditions for positive MIMO agents have been obtained, and several semidefinite programming algorithms were developed for the solution. In addition, due to the positivity of systems, a linear-programming-based approach has been proposed for the consensus design of positive SIMO agents. Case studies have been carried out to show that the theoretical results and the proposed approaches are effective for the solution.

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