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Proportional-Derivative Control of Discrete-Time Positive Systems: A State-Space Approach

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***Abstract*—The proportional-derivative (PD) control is investigated for discrete-time positive linear systems in this paper, which is a fundamental research problem in positive systems theory. Based on positive systems theory and Lyapunov theory, this paper proposes a systematic approach on the PD controller design for positive stabilization. The analyses and methods presented in this paper preserve both the necessity and sufficiency of the control problem, and the convergence of the algorithm is also analyzed. Finally, a numerical example on pest population control is employed in the simulation to verify the results.**

***Index Terms*—Discrete-time systems, PD control, Positive system**

I. Introduction

The research of positive linear systems has raised a lot of attention in both the scientific and engineering communities during the past decades. The systematic study of such kind of systems is provided by D. G. Luenberger in his book [1]. A positive linear system is a special dynamic system which preserves the positivity attributes of the system's variables [2]. The applications of positive systems broadly exist in industrial areas, such as epidemiology, economics and biology [3], [4]. A major reason is that, many physical quantities such as concentration, queue lengths and charge levels are intrinsically constrained to be nonnegative [5]. Besides, due to the nonnegative attribute of probabilities, many probabilistic models, such as Markov chains, can be described as positive systems [6].

Recently, many important results have been constantly proposed on positive systems [7]–[11]. We focus on the PD control of discrete-time positive linear systems in this paper. The design problem of PD controllers has been a crucial topic in many research areas, such as communication engineering [12], robotics [13], [14] and marine engineering [15], to name just a few. The reason is that it has some advantages, for instance, simple implementation, and shorter settling time. In addition, PD controllers can be used even when full-state feedback is not available, which can be advantageous in situations where some of the state variables are difficult or costly to measure. Nowadays, many industrial processes are modelled and represented by means of nonnegative physical quantities, and as such, serious destruction could arise once the positivity property is lost. In addition, the design of PID controllers for such a kind of systems is actually one of the fundamental research problems in positive systems theory. The contribution of this work includes providing a new answer to the problem, and giving novel insights to inspire future work.

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The contributions of our work are multi-fold. First, we propose a novel state-space formulation for the PD control design of discrete-time positive linear systems. Second, through utilizing the special structure of positive systems and PD controllers, several necessary and sufficient conditions are derived for the PD control of positive discrete systems, where the analyses and methods can be extended to study other PID control issues of positive systems in the future. Moreover, a convexification method is proposed for multi-input positive systems. Third, an algorithmic solution is developed through iterative linear matrix inequality (ILMI) method. The initialization and convergence of the algorithm are analyzed.

Notations: We use $\mathbb{R}$ to denote the set of real numbers. For symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, the notation $A \prec B$ (respectively, $A \preceq B$) means that $A - B$ is negative definite (respectively, negative semidefinite). The notation $A < B$ (respectively, $A \leq B$) means that $A - B$ is negative (respectively, nonpositive). We assume that, if not explicitly specified, the vectors and matrices in this paper have compatible dimensions.

II. Preliminaries

The discrete-time positive linear system is considered:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^q$ denote the system state, input and output, respectively.

Definition 1: [2] System (1) is a positive system if for any initial value $x(0) \geq 0$ and input $u(k) \geq 0$, $\forall k \geq 0$, the state $x(k) \geq 0$ and output $y(k) \geq 0$, $\forall k \geq 0$.

Lemma 1: [2] System (1) is a positive system if and only if A, B and C are nonnegative matrices.

Lemma 2: [2], [5], [16]–[18] For a nonnegative matrix A, we have the following equivalent statements:

1) System $x(k+1) = Ax(k)$ is asymptotically stable;
2) Matrix A is Schur stable, i.e., $\rho(A) < 1$;
3) Matrix $A - I$ is Hurwitz stable;

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4) $\exists$ diagonal $D \succ 0$ such that

$$A^{\mathrm{T}}DA \prec D \quad \text{or} \quad ADA^{\mathrm{T}} \prec D;$$

5) $\exists$ diagonal $D \succ 0$ such that

$$DA + A^{\mathrm{T}}D \prec 2D \quad \text{or} \quad AD + DA^{\mathrm{T}} \prec 2D.$$

The transfer function of a single-input single-output (SISO) discrete-time PD controller can be described as

$$U(z) = K_p Y(z) + \frac{K_d}{T_f + \Gamma(z)} Y(z) \tag{2}$$

where K_p and K_d are the proportional gain and the derivative gain, respectively. T_f is the derivative filter time constant. The symbols $U(z)$ and $Y(z)$ here are the z-transforms of input $u(t)$ and output $y(t)$ of system (1), respectively. The term $\Gamma(z)$ in (2) may take different forms if using different discretization methods. To preserve the system's stability [19], we use backward Euler method here and have

$$\Gamma(z) = \frac{T_s z}{z-1}$$

where $T_s > 0$ denotes the sampling period. Thus the multi-variable PD controller for a multi-input multi-output (MIMO) discrete system can be described as

$$U(z) = K_P Y(z) + K_D H_D(z) Y(z) \tag{3}$$

where

$$H_D(z) := \frac{1}{T_f + \Gamma(z)} I_q \tag{4}$$

is used to represent the transfer function matrix which is associated with the differentiator dynamics in discrete-time form [19]. With the above settings, we study the PD controller design of positive discrete systems in this paper.

Problem PDCPDS (PD Control of Positive Discrete Systems) Consider the discrete-time positive linear system in (1) with the PD controller in (3), determine the proportional and derivative gains K_P and K_D, respectively, so that the closed-loop system is asymptotically stable and its states and outputs always stay in the nonnegative orthant, i.e., $x(k) \geq 0$ and $y(k) \geq 0$ for $k \geq 0$.

Remark 1: Traditional PD controller design techniques [19], [20] are not applicable to positive systems since their positivity could be violated during the evolutionary process. This may cause severe issues when the variables are intrinsically nonnegative.

Remark 2: Due to the special dynamics of the differentiator (in the discrete-time domain) in (4) as well as the significant coupling between the gains K_P and K_D, **Problem PDCPDS**, which is to simultaneously reach the stability and positivity of the positive discrete system in (1), becomes fairly challenging.

III. Main Results

A state-space formulation is firstly proposed for the PD controller design of positive discrete systems in this section. Then we derive two necessary and sufficient conditions on positive stabilization for multi-input positive discrete systems. An LMI-based algorithmic solution is developed.

A. State-Space Formulation

We first transform the transfer function $H_D(z)$ in (4) as follows:

$$\begin{cases} \tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{B}y(k) \\ \tilde{y}(k) = \tilde{C}\tilde{x}(k) + \tilde{D}y(k) \end{cases} \tag{5}$$

where

$$\tilde{A} = \frac{T_f}{T_f + T_s} I_q, \quad \tilde{C} = -\frac{T_s}{T_f + T_s} I_q \tag{6}$$

and

$$\tilde{B} = \tilde{D} = \frac{1}{T_f + T_s} I_q. \tag{7}$$

This can be done through inverse z-transform and notice that T_f and T_s are positive constant parameters.

Hence the tuning of PD controller gains (3) can be formulated as an SOF control problem and

$$u(k) = K_P y(k) + K_D \tilde{y}(k). \tag{8}$$

Combining the dynamics of system (1) with the PD controller in (5) and (8), we have

$$\begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} = \begin{bmatrix} A + BK_P C + BK_D \tilde{D} C & BK_D \tilde{C} \\ \tilde{B} C & \tilde{A} \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}.$$

Furthermore, we define an augmented state $\hat{x}(k) := \begin{bmatrix} x(k)^{\mathrm{T}} & \tilde{x}(k)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and an augmented gain matrix $K := \begin{bmatrix} K_P & K_D \end{bmatrix}$, then the overall closed-loop system can be described as

$$\hat{x}(k+1) = (\hat{A} + \hat{B}K\hat{C})\hat{x}(k) \tag{9}$$

where

$$\hat{A} = \begin{bmatrix} A & 0 \\ \tilde{B}C & \tilde{A} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \\ \tilde{D}C & \tilde{C} \end{bmatrix}.$$

Therefore, we can address **Problem PDCPDS** by equivalently finding an SOF controller for the system in (9).

B. Stability-Positivity Analysis

We first give the following necessary and sufficient condition on the solvability of **Problem PDCPDS**.

Theorem 1: **Problem PDCPDS** is solvable if and only if matrices K_{P} and K_{D} fulfill the following conditions:

1) $BK_D \leq 0$,
2) $A + BK_P C + BK_D \tilde{D} C \geq 0$,
3) all the eigenvalues of matrix

$$\begin{bmatrix} A + BK_P C + BK_D \tilde{D} C & BK_D \tilde{C} \\ \tilde{B} C & \tilde{A} \end{bmatrix} \tag{10}$$

are within the unit circle over $\mathbb{C}$,
OR all the eigenvalues of matrix

$$\begin{bmatrix} A + BK_P C + BK_D \tilde{D} C - I & BK_D \tilde{C} \\ \tilde{B} C & \tilde{A} - I \end{bmatrix} \tag{11}$$

are in the open left half-plane over $\mathbb{C}$.

Proof. For stability, by *Lemma 2*, we have that system (9) is asymptotically stable if and only if matrix $\hat{A} + \hat{B}K\hat{C}$ is Schur or equivalently $\hat{A} + \hat{B}K\hat{C} - I$ is Hurwitz, which is equivalent to condition 3). For positivity, notice that $BK_D \leq 0$ if and only if $BK_D\tilde{C} \geq 0$ since $\tilde{C}$ is a negative diagonal matrix. As $\tilde{A}$ and $\tilde{B}$ are positive matrices, conditions 1) and 2) hold if and only if $\hat{A} + \hat{B}K\hat{C}$ is a nonnegative matrix, which is equivalent to the positivity of system (9) by *Lemma 1*. □

Then we propose a necessary and sufficient solvability condition for **Problem PDCPDS** for multi-input positive discrete systems as follows.

Theorem 2: **Problem PDCPDS** is solved by K_P and K_D if and only if scalar $\eta > 0$, and diagonal matrices $P_1 \succ 0$ and $P_2 \succ 0$ fulfill the following conditions:

1) $BK_D \leq 0$,
2) $A + BK_PC + BK_D\tilde{D}C \geq 0$,
3) the following matrix is negative definite:

$$\begin{bmatrix} \Gamma_1 & P_1C^{\mathrm{T}}\tilde{B}^{\mathrm{T}} & \eta BK_P + P_1C^{\mathrm{T}} & \eta BK_D + P_1C^{\mathrm{T}}\tilde{D}^{\mathrm{T}} \\ \star & \Gamma_2 & 0 & P_2\tilde{C}^{\mathrm{T}} \\ \star & \star & -\eta I & 0 \\ \star & \star & \star & -\eta I \end{bmatrix} \tag{12}$$

where

$$\Gamma_1 := AP_1 + P_1A^{\mathrm{T}} - 2P_1 - \eta B(K_PK_P^{\mathrm{T}} + K_DK_D^{\mathrm{T}})B^{\mathrm{T}},$$
$$\Gamma_2 := \tilde{A}P_2 + P_2\tilde{A}^{\mathrm{T}} - 2P_2.$$

Proof. We need to prove that the condition 3) in *Theorem 3* and the condition 3) in *Theorem 1* are equivalent. Notice that matrix (12) is congruent to the matrix

$$\begin{bmatrix} \tilde{\Gamma}_1 & BK_D\tilde{C}P_2 + P_1C^{\mathrm{T}}\tilde{B}^{\mathrm{T}} & P_1C^{\mathrm{T}} & P_1C^{\mathrm{T}}\tilde{D}^{\mathrm{T}} \\ \star & \Gamma_2 & 0 & P_2\tilde{C}^{\mathrm{T}} \\ \star & \star & -\eta I & 0 \\ \star & \star & \star & -\eta I \end{bmatrix} \tag{13}$$

with respect to a nonsingular matrix

$$U := \begin{bmatrix} I & 0 & -BK_P & -BK_D \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where $\tilde{\Gamma}_1 := P_1(A + BK_PC + BK_D\tilde{D}C)^{\mathrm{T}} + (A + BK_PC + BK_D\tilde{D}C)P_1 - 2P_1$. By *Lemma 2*, matrix (10) is Schur stable if and only if there exist diagonal $P_1, P_2 \succ 0$ such that

$$\begin{bmatrix} \tilde{\Gamma}_1 & BK_D\tilde{C}P_2 + P_1C^{\mathrm{T}}\tilde{B}^{\mathrm{T}} \\ \star & \Gamma_2 \end{bmatrix} \prec 0. \tag{14}$$

Using Schur complement equivalence, one can see that (14) holds if and only if there exists $\eta > 0$ such that matrix (13) is negative definite, which implies the results as desired. □

Remark 3: The above theorem gives a necessary and sufficient condition on the solvability of **Problem PDCPDS** for general positive discrete systems. However, notice that condition 3) in *Theorem 3* is a nonlinear matrix inequality that cannot be effectively solved. Thus convexification or linearization techniques are required to design the PD controller gains K_P and K_D for multi-input systems.

C. Stability-Positivity Design

Based on the previous analyses, we propose the PD controller design for positive discrete systems as follows.

Theorem 3: **Problem PDCPDS** is solved by K_P and K_D if and only if there exist matrices R_P, R_D, S_P, S_D, a scalar $\eta > 0$, and diagonal matrices $P_1 \succ 0$, $P_2 \succ 0$ such that

1) $BR_D \leq 0$,
2) $\eta A + BR_PC + BR_D\tilde{D}C \geq 0$,
3) the following matrix is negative definite

$$\begin{bmatrix} \Lambda_1 & P_1C^{\mathrm{T}}\tilde{B}^{\mathrm{T}} & BR_P + P_1C^{\mathrm{T}} & BR_D + P_1C^{\mathrm{T}}\tilde{D}^{\mathrm{T}} \\ \star & \Gamma_2 & 0 & P_2\tilde{C}^{\mathrm{T}} \\ \star & \star & -\eta I & 0 \\ \star & \star & \star & -\eta I \end{bmatrix} \tag{15}$$

where

$$\Lambda_1 := AP_1 + P_1A^{\mathrm{T}} - 2P_1 - BR_PS_P^{\mathrm{T}} - S_PR_P^{\mathrm{T}}B^{\mathrm{T}} + \eta S_PS_P^{\mathrm{T}} - BR_DS_D^{\mathrm{T}} - S_DR_D^{\mathrm{T}}B^{\mathrm{T}} + \eta S_DS_D^{\mathrm{T}}.$$

Under the above conditions, the PD controller gains are

$$K_P = \frac{1}{\eta}R_P \quad \text{and} \quad K_D = \frac{1}{\eta}R_D.$$

Proof. Taking $K_P = (1/\eta)R_P$ and $K_D = (1/\eta)R_D$, one can see that conditions 1) and 2) in *Theorem 3* are equivalent to conditions 1) and 2) in *Theorem 3*, respectively. We need to prove the equivalence between the condition 3) in *Theorem 3* and the condition 3) in *Theorem 3*.

Sufficiency: If matrix (12) is negative definite, notice that there exist matrices $S_P = BK_P$ and $S_D = BK_D$ with $R_P = \eta K_P$ and $R_D = \eta K_D$ such that $\Lambda_1 = AP_1 + P_1A^{\mathrm{T}} - 2P_1 - \eta B(K_PK_P^{\mathrm{T}} + K_DK_D^{\mathrm{T}})B^{\mathrm{T}} = \Gamma_1$, and matrix (15) is reduced to matrix (12) thus negative definite.

Necessity: If matrix (15) is negative definite, notice the fact that $\Lambda_1 - \Gamma_1 = \eta(BK_P - S_P)(BK_P - S_P)^{\mathrm{T}} + \eta(BK_D - S_D)(BK_D - S_D)^{\mathrm{T}} \succeq 0$. Thus, we have

$$\text{Matrix (12)} - \text{Matrix (15)} \preceq 0 \tag{16}$$

which implies that matrix (12) is negative definite. □

Remark 4: The above theorem provides a necessary and sufficient condition for PD controller design of positive discrete systems with general input channels, which also preserves the sufficiency and necessity of **Problem PDCPDS**. The nonlinear terms in *Theorem 3* are decoupled by constructing auxiliary structures and the equivalence is preserved, which paves the way for developing algorithmic solutions.

Based on *Theorem 3*, an ILMI algorithm is developed in *Algorithm 1* (see the next page) to solve **Problem PDCPDS**.

Proposition 1: In *Algorithm 1*, $\xi^{(h+1)} \leq \xi^{(h)}$ for $h \geq 1$, i.e., $\{\xi^{(h)}\}_{h\geq 1}$ is a non-increasing sequence.

Proof. Define an intermediate variable $\Gamma_1^{(h)} := AP_1 + P_1A^{\mathrm{T}} - 2P_1 - \eta B(K_P^{(h)}K_P^{(h)\mathrm{T}} + K_D^{(h)}K_D^{(h)\mathrm{T}})B^{\mathrm{T}}$. Notice the fact that $\Lambda_1^{(h)} - \Gamma_1^{(h+1)} = (BR_P^{(h+1)} - \eta S_P^{(h)})(BR_P^{(h+1)} - \eta S_P^{(h)})^{\mathrm{T}} + \Lambda_1^{(h)} - \Gamma_1^{(h+1)} = (BR_D^{(h+1)} - \eta S_D^{(h)})(BR_D^{(h+1)} - \eta S_D^{(h)})^{\mathrm{T}} \succeq 0$ thus $\Gamma_1^{(h+1)} \preceq \Lambda_1^{(h)}$. Now we consider the relation between $\Lambda_1^{(h+1)}$ and $\Gamma_1^{(h+1)}$. Notice that if we take

$R_P = \eta K_P^{(k+1)}$ and $R_D = \eta K_D^{(k+1)}$, meanwhile keep other variables unchanged, then $\Lambda_1^{(h+1)}$ will be reduced to $\Gamma_1^{(h+1)}$, which implies that $\Lambda_1^{(h+1)} \preceq \Lambda_1^{(h)}$. Therefore we can conclude that $\xi^{(h+1)} \leq \xi^{(h)}$ for all $h \geq 1$. □

Since $\{\xi^{(h)}\}_{h\geq 1}$ is nonincreasing, we can obtain a successively improved sequence of controller gains. When $\xi^{(h)}$ is finally decreased below 0, the solution to **Problem PDCPDS** is obtained. Moreover, the initialization of $S_P^{(0)}$ and $S_D^{(0)}$ is formulated as a positive observer design problem [21], which is summarized in the following proposition.

Algorithm 1 PDCPDS Solver

Initialize iterator: $h = 0$ and threshold: $\xi^{(0)} > 0$.
Initialize gains: $S_P^{(0)}$ and $S_D^{(0)}$ such that the following matrix is Schur table and nonnegative:

$$\begin{bmatrix} A + S_P^{(0)}C + S_D^{(0)}\tilde{D}C & S_D^{(0)}\tilde{C} \\ \tilde{B}C & \tilde{A} \end{bmatrix} \tag{17}$$

while $\xi^{(h)} \geq 0$ **do**

Minimize ξ with respect to variables:

$$\eta > 0,\ P_1 \succ 0,\ P_2 \succ 0,\ R_P \text{ and } R_D$$

subject to constraints:
1) $BR_D \leq 0$,
2) $\eta A + BR_PC + BR_D\tilde{D}C \geq 0$,
3)

$$\begin{bmatrix} \Lambda_1^{(h)} & P_1C^{\mathrm{T}}\tilde{B}^{\mathrm{T}} & BR_P + P_1C^{\mathrm{T}} & BR_D + P_1C^{\mathrm{T}}\tilde{D}^{\mathrm{T}} \\ \star & \Gamma_2 & 0 & P_2\tilde{C}^{\mathrm{T}} \\ \star & \star & -\eta I & 0 \\ \star & \star & \star & -\eta I \end{bmatrix} \preceq \xi I$$

where

$$\Lambda_1^{(h)} := AP_1 + P_1A^{\mathrm{T}} - 2P_1 - BR_PS_P^{(h)\mathrm{T}} - S_P^{(h)}R_P^{\mathrm{T}}B^{\mathrm{T}} + \eta S_P^{(h)}S_P^{(h)\mathrm{T}} - BR_DS_D^{(h)\mathrm{T}} - S_D^{(h)}R_D^{\mathrm{T}}B^{\mathrm{T}} + \eta S_D^{(h)}S_D^{(h)\mathrm{T}}.$$

Update $h = h + 1$ and $\xi^{(h)} = \xi$.
Update $K_P^{(h)} = (1/\eta)R_P$ and $K_D^{(h)} = (1/\eta)R_D$.
if $\|\xi^{(h)} - \xi^{(h-1)}\|/\xi^{(h)} < \epsilon$ **then**
STOP
end
Update $S_P^{(h)} = BK_P^{(h)}$ and $S_D^{(h)} = BK_D^{(h)}$.

end
return $K_P^{(h)}$ and $K_D^{(h)}$.

Proposition 2: Matrix (17) is Schur stable and nonnegative if and only if matrices L_P, L_D and diagonal matrices $P_1 \succ 0$, $P_2 \succ 0$ fulfill the following conditions:

1) $L_D \leq 0$,
2) $P_1A + L_PC + L_D\tilde{D}C \geq 0$,
3) the following inequality holds:

$$\begin{bmatrix} \Omega_1 & L_D\tilde{C} + C^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_2 \\ \star & \tilde{A}P_2 + P_2\tilde{A}^{\mathrm{T}} \end{bmatrix} \prec 2\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{18}$$

where

$$\Omega_1 := AP_1 + P_1A^{\mathrm{T}} + L_PC + L_D\tilde{D}C + C^{\mathrm{T}}L_P + C^{\mathrm{T}}\tilde{D}^{\mathrm{T}}L_D.$$

Under the above conditions, initial values are

$$S_P^{(0)} = P_1^{-1}L_P \quad \text{and} \quad S_D^{(0)} = P_1^{-1}L_D.$$

Proof. Taking $L_P = P_1S_P^{(0)}$ and $L_D = P_1S_D^{(0)}$, by *Lemma 2*, matrix (17) is nonnegative if and only if conditions 1) and 2) hold, which follows from the fact that P_1 and P_2 are positive diagonal matrices. By *Lemma 2*, nonnegative matrix (17) is Schur stable if and only if

$$\begin{bmatrix} A + S_P^{(0)}C + S_D^{(0)}\tilde{D}C & S_D^{(0)}\tilde{C} \\ \tilde{B}C & \tilde{A} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \times \begin{bmatrix} A + S_P^{(0)}C + S_D^{(0)}\tilde{D}C & S_D^{(0)}\tilde{C} \\ \tilde{B}C & \tilde{A} \end{bmatrix} \prec 2\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

which is equivalent to Eqn (18). □

Remark 5: The above proposition provides an equivalent approach for the initialization of $S_P^{(0)}$ and $S_D^{(0)}$. Further notice that all the linear conditions in *Proposition 2* can be effectively solved by numerical solvers.

IV. Illustrated Example

A numerical simulation is conducted in this section for verification of the proposed theoretical results of PD controller design and algorithm.

We utilize a real-world example, namely the pest population control problem, to verify the effectiveness of our proposed theoretical results. The structured population dynamics for a certain pest can be described by *Leslie matrix model*, which was developed by the P. H. Leslie in 1940s and broadly used in ecology [22]. Consider the following Leslie model:

$$\begin{cases} x(k+1) = \begin{bmatrix} f_1 & f_2 & f_3 \\ s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \\ 0 & 0 \end{bmatrix} u(k) \\ y(k+1) = \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & c_3 & c_4 \end{bmatrix} x(k+1) \end{cases}$$

where $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & x_3(k) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^3$ denote the populations of juvenile, immature and mature pests, respectively. In the Leslie matrix A, f_i denotes the birth rate for the i-th age class and s_i denotes the survival rate from the i-th age class to the next age class. For matrix B, its structure means that the external inputs can only influence the juvenile and immature pests. For matrix C, its structure means that the sum of juvenile and immature pests, and the sum of immature and mature pests are estimated.

Notice that the above Leslie system is a positive linear system since the populations of pests are intrinsically nonnegative quantities. To exterminate a certain insect pest, we utilize the PD controller in (8) for positive stabilization, where the derivative filter time constant is set as $T_f = 0.5$ and the sampling period is set as $T_s = 1.0$. Besides, we assume that the model parameters are estimated as $f_1 = 0.25$, $f_2 = 0.60$, $f_3 = 0.56$, $s_1 = 0.35$, $s_2 = 0.25$, $b_1 = 0.60$, $b_2 = 0.90$, $b_3 = 0.12$, $c_1 = 0.75$, $c_2 = 1.20$, $c_3 = 0.90$ and $c_4 = 1.00$.

By *Proposition 2*, matrices S_P and S_D are initialized as

$$S_P^{(0)} = \begin{bmatrix} 0.1316 & -0.2781 \\ 0.1432 & 0.3084 \\ 0.3224 & 0.3189 \end{bmatrix},$$

and

$$S_D^{(0)} = \begin{bmatrix} -0.2151 & -0.2228 \\ -0.2180 & -0.2092 \\ -0.2893 & -0.2646 \end{bmatrix}.$$

Implement *Algorithm 1* using SeDuMi in MATLAB 2020b, we have $\{\xi^{(0)}, \xi^{(1)}, \xi^{(2)}\} = \{0.6050, 0.0029, -0.6525\}$ with

$$K_P^{(2)} = \begin{bmatrix} -0.2842 & 0.2825 \\ 0.2825 & 0.2960 \end{bmatrix} \; K_D^{(2)} = \begin{bmatrix} -0.2051 & -0.4466 \\ -0.4466 & -0.2353 \end{bmatrix}.$$

Using *Theorem 1*, we first verify the system's positivity:

$$BK_D^{(2)} = \begin{bmatrix} -0.5250 & -0.4798 \\ -0.0536 & -0.028200 \end{bmatrix} \leq 0$$

and $A + BK_P^{(2)}C + BK_D^{(2)}\tilde{D}C =$

$$\begin{bmatrix} 0.0503 & 0.3849 & 0.6761 \\ 0.3486 & 0.0128 & 0.0167 \\ 0 & 0.2500 & 0 \end{bmatrix} \geq 0,$$

then the asymptotic stability is verified by matrix (10)

$$\begin{bmatrix} 0.0503 & 0.3849 & 0.6761 & 0.1750 & 0.1599 \\ 0.3486 & 0.0128 & 0.0167 & 0.0179 & 0.0094 \\ 0 & 0.2500 & 0 & 0 & 0 \\ 0.5000 & 0.8000 & 0 & 0.3333 & 0 \\ 0 & 0.6000 & 0.6667 & 0 & 0.3333 \end{bmatrix}.$$

Its eigenvalues $\{0.8080, 0.0822, 0.3289, -0.2446 \pm 0.2923i\}$ are all within the unit circle over $\mathbb{C}$. The responses of system (1) with the designed PD controller versus P controller are plotted by solid and dashed lines in Fig. 1.

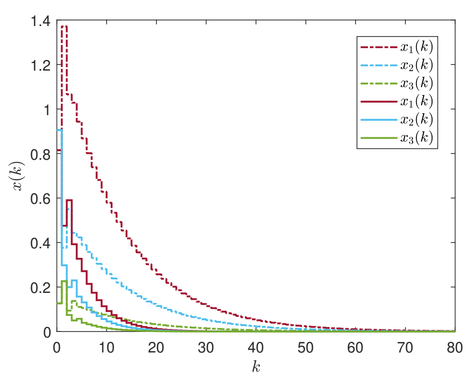

Fig. 1. Positive stabilization of Leslie system via PD control versus P control.

V. Conclusions

A fundamental research problem in positive systems theory, that is, the PD control of positive discrete systems, has been investigated. We have first proposed a state-space formulation, and derived equivalent conditions on the solvability of this problem. Moreover, a linearization-based technique has been employed for the multi-input systems. A systematic approach for the PD controller design has been provided, along with an LMI-based algorithm for computing the controller gain. The initialization and convergence of the algorithm have also been analyzed. Finally, a numerical example of pest population control has been used for verification of the theoretical results. Disturbance rejection of PD control for positive systems with interval uncertainty will be our future work.

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