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Positive Consensus of Fractional-Order Multiagent Systems Over Directed Graphs

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***Abstract*— This article investigates the positive consensus problem of a special kind of interconnected positive systems over directed graphs. They are composed of multiple fractional-order continuous-time positive linear systems. Unlike most existing works in the literature, we study this problem for the first time, in which the communication topology of agents is described by a directed graph containing a spanning tree. This is a more general and new scenario due to the interplay between the eigenvalues of the Laplacian matrix and the controller gains, which renders the positivity analysis fairly challenging. Based on the existing results in spectral graph theory, fractional-order systems (FOSs) theory, and positive systems theory, we derive several necessary and/or sufficient conditions on the positive consensus of fractional-order multiagent systems (PCFMAS). It is shown that the protocol, which is designed for a specific graph, can solve the positive consensus problem of agents over an additional set of directed graphs. Finally, a comprehensive comparison study of different approaches is carried out, which shows that the proposed approaches have advantages over the existing ones.**

***Index Terms*— Directed graphs, fractional-order multiagent systems, positive consensus, positive fractional-order systems (FOSs), positive systems.**

I. Introduction

A. Background and Motivation

Among various classes of dynamic systems, there is a special type of systems named positive dynamic systems [1], which can be traced back to a book [2] on fundamental systems theory published by David Luenberger in 1979. Generally, a positive system can be regarded as a dynamic system whose states and outputs are constrained to be nonnegative given that its inputs and initial states are nonnegative. Such kind of systems has attracted much attention in the field of control recently (see [3]–[7] and the references therein). An important motivation behind positive systems theory is that, in the physical world, many descriptor variables are usually constrained to be nonnegative due to their intrinsic characteristics or physical laws, such as the material flows in a compartmental network [8], and the probabilities of Markov chains in stochastic processes quantities [9].

Recently, the research of collective behaviors in an interconnected positive system, referred to as a positive multiagent system, has attracted much attention due to the research and development of networked systems, such as the dynamical buffer networks and the epidemic spreading processes. Moreover, even simple dynamic models such as fractional/integer order integrators and first-order lags with positive gains, as well as their series/parallel connections, are all positive, which often represent some typical systems of moving objects. Although each of their dynamics is simple, the global behavior of a group of them is complicated, and deserves investigation, especially in the area of multiagent systems [10], [11]. Indeed, examples of positive multiagent systems are also ubiquitous (see [12] and [13]).

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B. Related Works and Challenges

Analysis and synthesis of interconnected positive systems using positive systems theory have attracted much attention in recent years. For instance, Ebihara *et al.* [10] proposed a novel framework for the analysis and synthesis of interconnected systems constructed by heterogeneous positive systems. Sun *et al.* [14] investigated the consensus problem of multiagent systems using the property of positive switched linear systems. Ogura and Preciado [15] proposed an optimization framework for the design of a networked positive linear system whose structure switches over a Markov process. In particular, one hot issue is the positive consensus problem of multiagent systems. The objective of positive consensus is to design a controller such that the overall system can reach consensus and meanwhile the states of all the agents remain nonnegative throughout the evolutionary process. Despite the pioneering research on the positive consensus of multiagent systems over undirected graphs by Valcher and Zorzan [16] and Liu *et al.* [17], we have seen very limited progress in positive consensus of multiagent systems over directed graphs. This is because conventional positive systems theory, which is used to effectively analyze the positive consensus problem with undirected topologies, would unfortunately fail in the case of directed topologies. This challenge was further tackled by Wu and Su [18], Yang *et al.* [19], and Liu *et al.* [20], and sufficient conditions of positive consensus are given.

In contrast to the works on integer-order positive multiagent systems mentioned above, the study of positive fractional-order interconnected systems is a recent new trend since many fractional-order interconnected systems (and the individual agent) also contain nonnegative variables (see [21]–[23] and the references therein). Hien and Chu [24] proposed a decentralized control strategy for positive fractional-order interconnected systems with heterogeneous time-varying delays. Huong and Nahavandi *et al.* [25] designed the positive reduced-order distributed functional observers for positive fractional-order interconnected time-delay systems. More recently, Ye *et al.* [26] investigated the positive consensus problem of fractional-order multi-agent systems (PCFMAS) over undirected graphs. It should be noted that the consensus problem of fractional-order multiagent systems is difficult since they possess more complex and general dynamics (compared with the integer-order systems) [27]–[32]. Furthermore, this problem will become more challenging while agents interacting over a directed graph due to the interplay between the (complex) eigenvalues of the Laplacian matrix and the controller gains, which renders the positivity analysis infeasible.

C. Contributions

This note studies the PCFMAS over directed graphs. The main contributions of this work in comparison with [16]–[19], [26], [33] are summarized as follows: 1) this is the first attempt to address this

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problem for positive fractional-order multiagent systems over directed graphs, which includes the integer-order positive multiagent systems as a special case; 2) stability results are extended from real fractional-order systems (FOSs) to complex FOSs, and several necessary and/or sufficient conditions are proposed for analysis and synthesis of positive consensus; and 3) the protocol, which is designed for a given directed graph, can also solve the positive consensus problem of agents over an additional set of directed graphs.

Notation: The notation $\otimes$ means Kronecker product. The imaginary unit is denoted as j ($j^2 = -1$). The symbol A^* denotes the conjugate transpose of matrix A. For symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, the notation $A > B$ (respectively, $A \geq B$) means that $A - B$ is positive definite (respectively, positive semidefinite). For matrices $A, B \in \mathbb{R}^{m\times n}$, the notation $A \succ B$ (respectively, $A \succeq B$) means that $A - B$ is positive (respectively, nonnegative). For matrix A, $A \in \mathbb{M}$ means that A is Metzler, and $A \in \mathbb{H}$ means that A is Hurwitz. The spectral abscissa of matrix A is represented by $\xi(A)$. For a complex matrix $A \in \mathbb{C}^{p\times p}$, $\mathrm{Re}(A)$ denotes the real part and $\mathrm{Im}(A)$ denotes the imaginary part. The symbol $\mathbb{N}^+$ denotes the set of positive integers. The symbol I_p denotes a $p \times p$ identity matrix. Matrix A is an M-matrix if its off-diagonal elements are nonpositive, and its eigenvalues have nonnegative real parts. Matrices are assumed having compatible dimensions if not stated specifically.

II. Preliminaries and Problem Formulation

A. Preliminaries

The following results [1], [34], [35] pave the way for further analysis on the PCFMAS over directed graphs.

1) Fractional-Order Fundamentals: The Caputo fractional derivative and integral of order $\alpha \in (n-1, n), n \in \mathbb{N}^+$, for a continuous function $f(t)$ are, respectively, defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} f^{(n)}(\tau)\mathrm{d}\tau$$

and

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau$$

where $f^{(n)}(\cdot)$ denotes the nth-order derivative of $f(\cdot)$ and $L(\cdot)$ denotes the Gamma function

$$\Gamma(\alpha) = \int_0^t e^{-t} t^{\alpha-1}\mathrm{d}t. \tag{1}$$

Consider a fractional-order continuous-time linear system

$$D^\alpha x(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad \alpha \in (0, 2) \tag{2}$$

where $\mathcal{A} \in \mathbb{C}^{p\times p}$ and $\mathcal{B} \in \mathbb{C}^{r\times r}$ are system matrices with appropriate dimensions, and $x(t) \in \mathbb{C}^p$ and $u(t) \in \mathbb{C}^r$ are the system state and control input, respectively. Moreover, we have the following useful results.

Definition 1 [36, Definition 1]: When $\mathcal{A}$ and $\mathcal{B}$ are real matrices, the FOS in (2) is called positive if for any nonnegative initial value and input, its state always remains nonnegative, that is, $x(t) \geq 0$ for $t \geq 0$.

Lemma 1 [36, Theorem 2]: When $\mathcal{A}$ and $\mathcal{B}$ are real matrices, the FOS in (2) is positive if and only if $\mathcal{A}$ is Metzler and $\mathcal{B}$ is nonnegative.

Lemma 2 [35]: When $\mathcal{A}$ and $\mathcal{B}$ are real matrices, the positive FOS in (2) with order $\alpha \in (0, 1]$ and $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied.

1) Matrix $\mathcal{A}$ is Hurwitz.
2) There is a diagonal matrix $P > 0$ such that

$$P\mathcal{A} + \mathcal{A}^{\mathrm{T}}P < 0 \text{ or } \mathcal{A}P + P\mathcal{A}^{\mathrm{T}} < 0.$$

Lemma 3 [1]: When A_1 and A_2 are real matrices, $A_1, A_2 \in \mathbb{M}$, $A_1 \preceq A_2 \Rightarrow \xi(A_1) \leq \xi(A_2)$.

2) Key Lemmas:

Lemma 4: The FOS in (2) with $u(t) = 0$ is asymptotically stable if and only if $|\arg(\lambda_i(\mathcal{A}))| > \alpha\pi/2$ where $\lambda_i(\mathcal{A})$, $i = 1, 2, \ldots, p$, denotes the eigenvalues of $\mathcal{A}$ and $\arg(\cdot)$ denotes the argument of a complex number.

Proof: Denotes $x(t) := \mathrm{Re}(x(t)) + j\mathrm{Im}(x(t))$ where $\mathrm{Re}(x(t)) \in \mathbb{R}^p$ and $\mathrm{Im}(x(t)) \in \mathbb{R}^p$. The FOS in (2) with $u(t) = 0$ can be rewritten as $D^\alpha\mathrm{Re}(x(t)) + D^\alpha\mathrm{Im}(x(t))j = \mathrm{Re}(\mathcal{A})\mathrm{Re}(x(t)) - \mathrm{Im}(\mathcal{A})\mathrm{Im}(x(t)) + \mathrm{Im}(\mathcal{A})\mathrm{Re}(x(t))j + \mathrm{Re}(\mathcal{A})\mathrm{Im}(x(t))j$, $\alpha \in [1, 2)$. For clearer illustration, we represent it into the following real form:

$$\begin{bmatrix} D^\alpha\mathrm{Re}(x(t)) \\ D^\alpha\mathrm{Im}(x(t)) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(\mathcal{A}) & -\mathrm{Im}(\mathcal{A}) \\ \mathrm{Im}(\mathcal{A}) & \mathrm{Re}(\mathcal{A}) \end{bmatrix}\begin{bmatrix} \mathrm{Re}(x(t)) \\ \mathrm{Im}(x(t)) \end{bmatrix}. \tag{3}$$

Therefore, the complex FOS in (2) with $u(t) = 0$ can be equivalently characterized by the real system in (3). By [35, Th. 2], it is known that (3) is asymptotically stable if and only if $|\arg(\lambda_i(\tilde{\mathcal{A}}))| > \alpha\pi/2$ ($i = 1, 2, \ldots, p$), where

$$\tilde{\mathcal{A}} := \begin{bmatrix} \mathrm{Re}(\mathcal{A}) & -\mathrm{Im}(\mathcal{A}) \\ \mathrm{Im}(\mathcal{A}) & \mathrm{Re}(\mathcal{A}) \end{bmatrix}. \tag{4}$$

Define

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} -jI_p & -jI_p \\ I_p & -I_p \end{bmatrix}, \quad T^* = \frac{1}{\sqrt{2}j}\begin{bmatrix} -I_p & jI_p \\ -I_p & -jI_p \end{bmatrix}.$$

Via simple matrix manipulations, we have

$$T^*\tilde{\mathcal{A}}T = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \bar{\mathcal{A}} \end{bmatrix}$$

where $\bar{\mathcal{A}}$ is the conjugate of $\mathcal{A}$. Notice that we have $|\arg(\lambda_i(\mathcal{A}))| > \alpha\pi/2$ if and only if $|\arg(\lambda_i(\bar{\mathcal{A}}))| > \alpha\pi/2$. Therefore, we have $|\arg(\lambda_i(\mathcal{A}))| > \alpha\pi/2$ if and only if $|\arg(\lambda_i(\tilde{\mathcal{A}}))| > \alpha\pi/2$, which further indicates that the FOS in (2) [or (3)] with $u(t) = 0$ is asymptotically stable if and only if $|\arg(\lambda_i(\mathcal{A}))| > \alpha\pi/2$ for $i = 1, 2, \ldots, p$. □

By Lemma 4, we can readily obtain the following conclusion for $\alpha \in (0, 1)$.

Lemma 5 [35]: The FOS in (2) with order $\alpha \in (0, 1)$ and $u(t) = 0$ is asymptotically stable if matrix $\mathcal{A}$ is Hurwitz.

Lemma 6 [35]: The FOS in (2) with order $\alpha \in [1, 2)$ and $u(t) = 0$ is asymptotically stable if and only if the following matrix is Hurwitz:

$$\begin{bmatrix} \mathcal{A}\sin(\alpha\pi/2) & \mathcal{A}\cos(\alpha\pi/2) \\ -\mathcal{A}\cos(\alpha\pi/2) & \mathcal{A}\sin(\alpha\pi/2) \end{bmatrix}. \tag{5}$$

Proof: Notice that the matrix in (5) is Hurwitz is equivalent to that $(\sin(\alpha\pi/2) + \cos(\alpha\pi/2)j)\mathcal{A}$ is Hurwitz. Then along the line in the proof of Lemma 4 and using [35, Th. 3], it can be shown that, the FOS in (2) with order $\alpha \in [1, 2)$ and $u(t) = 0$, is asymptotically stable if and only if the matrix in (5) is Hurwitz. □

3) Graph Theory: The topology of a multiagent system can be properly described by its graph. A graph is called directed if all its edges are directed from one vertex to another, and thus undirected graphs can be regarded as a special type of directed graphs. In this note, we investigate a multiagent system with directed communication topology that can be described by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \ldots, N\}$ is the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. For any $k, l \in \mathcal{V}$, we assume that $(k, l) \in \mathcal{E}$ if and only if agent l is able to access the full state of agent k. A path of graph $\mathcal{G}$ is defined as a sequence $\{k, l, m, \ldots, o, p, q\}$ such that all the successive tuples $(k, l), (l, m), \ldots, (o, p), (p, q) \in \mathcal{E}$. Graph $\mathcal{G}$ is assumed to contain a spanning tree, that is, there is a root $k \in \mathcal{V}$ such that there exists a path from k to any other vertex $l \in \mathcal{V}$. The adjacency matrix of graph $\mathcal{G}$ is defined and denoted as an $N \times N$ matrix Λ where $[\Lambda]_{kl} = 1$ if $(l, k) \in \mathcal{E}$ and $[\Lambda]_{kl} = 0$ otherwise. It is assumed that graph $\mathcal{G}$

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contains no self-loops, that is, $[\Lambda]_{kk} = 0, k \in \mathcal{V}$. Define the neighbor set for any vertex $k \in \mathcal{V}$ as $\aleph_k := \{l \in \mathcal{V} : (l, k) \in \mathcal{E}\}$. The Laplacian matrix of graph $\mathcal{G}$ is defined and denoted as an $N \times N$ matrix L such that, $k, l \in \mathcal{V}$

$$[L]_{kl} = \begin{cases} \sum_{m=1}^{N} [\Lambda]_{km}, & \text{if } k = l \\ -[\Lambda]_{kl}, & \text{if } k \neq l. \end{cases} \tag{6}$$

By the well-known results in [34], the eigenvalues, λ_i, $i = 1, 2, \ldots, N$, of a Laplacian matrix L containing a spanning tree, are in general complex which can be ordered as $0 = \mathrm{Re}(\lambda_1(L)) < \mathrm{Re}(\lambda_2(L)) \leq \cdots \leq \mathrm{Re}(\lambda_N(L))$. Moreover, $\beta(L) := \max\{\sum_{m=1}^{N} [\Lambda]_{km} : k = 2, 3, \ldots, N\}$.

B. Problem Formulation

Consider a homogeneous multiagent system constituting of N identical agents in a directed communication topology, and the dynamics of each agent can be described by the following positive fractional-order linear system:

$$D^{\alpha} x_k(t) = A x_k(t) + B u_k(t), \quad k = 1, 2, \ldots, N \tag{7}$$

where $\alpha \in (0, 2)$, $x_k(t) := [x_{k1}, x_{k2}, \ldots, x_{kp}]^{\mathrm{T}} \in \mathbb{R}^p$ is the state of agent k, and $u_k(t) \in \mathbb{R}^r$ is the input of agent k. The pair (A, B) is assumed to be stabilizable. By Lemma 1, the system in (7) is positive if and only if $A \in \mathbb{R}^{p \times p}$ is Metzler and $B \in \mathbb{R}^{p \times r}$ is nonnegative.

The following distributed state-feedback protocol is utilized:

$$u_k(t) = K \sum_{l=1}^{N} [\Lambda]_{kl} (x_l - x_k), \quad k = 1, 2, \ldots, N \tag{8}$$

where K is the controller gain matrix to be determined. For convenience of expression, define the global state $x(t) := [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pn}$. Then the overall multiagent system in (7) can be described as

$$D^{\alpha} x(t) = \Omega x(t) \tag{9}$$

where $\Omega = I_N \otimes A - L \otimes BK$.

Based on the above model settings, the problem to be solved is formulated and defined as follows.

Problem PCFMAS: Considering the positive fractional-order multiagent system in (7) with the state-feedback protocol in (8), given any nonnegative initial states $x_k(0) \succeq 0$, $k = 1, 2, \ldots, N$, determine gain matrix K such that the consensus of the multiagent system in (7) is achievable, that is, $\lim_{t \to \infty} (x_l(t) - x_k(t)) = 0, \ \forall l, k = 1, \ldots, N$, and the state trajectory of each agent remains nonnegative, that is, $x_k(t) \succeq 0, k = 1, 2, \ldots, N$ for $t \succeq 0$.

III. Main Results

In this section, several necessary and sufficient conditions and analyses on the solvability of problem PCFMAS are derived using graph theory and positive systems theory. Equation (10), as shown at the bottom of the page.

Theorem 1: Problem PCFMAS with $\alpha \in (0, 2)$ is solved by gain matrix K if and only if all the following conditions are satisfied.

1) $BK \succeq 0$.
2) $A - \beta(L) BK \in \mathbb{M}$.
3) $|\arg(\lambda_i(A - \lambda_k(L) BK))| \succ \alpha\pi/2$ where $i = 1, 2, \ldots, p$, $k = 2, 3, \ldots, N$.

Proof: Notice that Ω, as shown at the bottom of the page. By Lemma 1, the positivity of the dynamics of all the agents, that is, the positivity of system (9) is preserved if and only if system matrix Ω is Metzler. By definition, $\Omega \in \mathbb{M}$ if and only if BK is nonnegative and $A - \sum_{m=1}^{N} [\Lambda]_{km} BK, k = 2, 3, \ldots, N$, are Metzler. As $\beta(L) = \max\{\sum_{m=1}^{N} [\Lambda]_{km} : k = 2, 3, \ldots, N\}$, it is easy to see that, $A - \sum_{m=1}^{N} [\Lambda]_{km} BK, k = 2, 3, \ldots, N$, are Metzler if and only if $A - \beta(L) BK$ is Metzler. So, the positivity of the multiagent system in (9) is preserved if and only if condition 1) $BK \succeq 0$ and condition 2) $A - \beta(L) BK \in \mathbb{M}$.

Define $e_k(t) = \sum_{l=1}^{N} [\Lambda]_{kl} (x_l - x_k)$, $k = 1, 2, \ldots, N$, and $e(t) = [e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pn}$, we have

$$e(t) = -(L \otimes I_p) x(t) \tag{11}$$

and

$$D^{\alpha} e(t) = (I_N \otimes A - L \otimes BK) e(t). \tag{12}$$

Since graph $\mathcal{G}$ contains a spanning tree, there always exists a coordinate transformation for $e(t)$ [37]–[40] such that the overall dynamic system in (12) becomes

$$D^{\alpha} \epsilon_k(t) = A_k \epsilon_k(t), \quad k = 2, \ldots, N \tag{13}$$

where $A_k := A - \lambda_k(L) BK$. Note that $\lim_{t \to \infty} e_k(t) = 0$, $\forall k = 1, \ldots, N$ is equivalent to $\lim_{t \to \infty} \epsilon_k(t) = 0, \ \forall k = 2, \ldots, N$. Therefore, the consensus problem of system (12) is equivalently formulated as the stabilization problem of the $N - 1$ systems in (13). By Lemma 4, one can obtain the condition 3) $|\arg(\lambda_i(A_k))| \succ \alpha\pi/2$ where $i = 1, 2, \ldots, p$, $k = 1, 2, \ldots, n$ for consensus. The whole proof is completed. □

Remark 1: A necessary and sufficient analysis condition of fractional-order multiagent systems over directed graphs is established in Theorem 1, in which the eigenvalues λ_k, $k = 2, \ldots, N$, of graph $\mathcal{G}$ are generally complex numbers. Due to the interplay between complex eigenvalues and controller gain, conventional positive systems theory cannot be applied to addressing the positive consensus design problem. Hence, the positive consensus design

$$\begin{bmatrix} (WA^{\mathrm{T}} + AW)\sin(\alpha\pi/2) - 2\theta BQB^{\mathrm{T}} & (AW - WA^{\mathrm{T}})\cos(\alpha\pi/2) \\ (AW - WA^{\mathrm{T}})\cos(\alpha\pi/2) & (WA^{\mathrm{T}} + AW)\sin(\alpha\pi/2) - 2\theta BQB^{\mathrm{T}} \end{bmatrix} < 0 \tag{10}$$

$$\Omega = I_N \otimes A - L \otimes BK = \begin{bmatrix} A - \sum_{m=1}^{N} [\Lambda]_{1m} BK & [\Lambda]_{12} BK & \cdots & \cdots & [\Lambda]_{1(N-1)} BK & [\Lambda]_{1N} BK \\ [\Lambda]_{21} BK & A - \sum_{m=1}^{N} [\Lambda]_{2m} BK & \cdots & \cdots & [\Lambda]_{2(N-1)} BK & [\Lambda]_{2N} BK \\ \vdots & \vdots & \ddots & \cdots & \ddots & \vdots \\ [\Lambda]_{N1} BK & [\Lambda]_{N2} BK & \cdots & \cdots & [\Lambda]_{N(N-1)} BK & A - \sum_{m=1}^{N} [\Lambda]_{Nm} BK \end{bmatrix}.$$

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problem becomes much more complicated than that with undirected connected graphs [26].

Theorem 2: Problem PCFMAS with $\alpha \in (0, 1)$ is solved by gain matrix $K = QB^{\mathrm{T}}W^{-1}$ if there exist a positive-definite M-matrix $W > 0$, a matrix $Q > 0$ and a sufficiently large scalar $\mu \succ 0$ such that all the following conditions are satisfied.

1) $BQB^{\mathrm{T}} \succeq 0$.
2) $AW - \beta(L)BQB^{\mathrm{T}} + \mu W \succeq 0$.
3) $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(L))BQB^{\mathrm{T}} < 0$.

Proof: Since W is a positive definite M-matrix [41], we have $W^{-1} \succeq 0$ and $W^{-1} > 0$. Taking $K = QB^{\mathrm{T}}W^{-1}$, then condition 1) $BQB^{\mathrm{T}} \succeq 0$ and $W^{-1} \succeq 0$ lead to $BK = BQB^{\mathrm{T}}W^{-1} \succeq 0$. Moreover, if condition 2) $AW - \beta(L)BQB^{\mathrm{T}} + \mu W = (A - \beta(L)BQB^{\mathrm{T}}W^{-1} + \mu I)W \succeq 0$ holds, postmultiplying it by matrix $W^{-1} \succeq 0$ yields $A - \beta(L)BQB^{\mathrm{T}}W^{-1} + \mu I = A - \beta(L)BK + \mu I \succeq 0$ where $\mu \succ 0$ is a sufficiently large scalar. Therefore, we have $BK \succeq 0$ and $A - \beta(L)BK \in \mathbb{M}$.

For $W^{-1} > 0$, it is known that condition 3) $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(L))BQB^{\mathrm{T}} < 0$ is equivalent to $A^{\mathrm{T}}W^{-1} + W^{-1}A - 2\mathrm{Re}(\lambda_2(L))W^{-1}BQB^{\mathrm{T}}W^{-1} < 0$. Taking $K = QB^{\mathrm{T}}W^{-1}$, we define the Lyapunov equation for $A_k = A - \lambda_k(L)BK$ as $\Phi_k(W, \lambda_k(L)) = (A - \lambda_k(L)BK)^*W + W(A - \lambda_k(L)BK) = A^{\mathrm{T}}W + WA - 2\mathrm{Re}(\lambda_k(L))WBQB^{\mathrm{T}}W$, $k = 2, \ldots, N$. Since $0 = \mathrm{Re}(\lambda_1(L)) < \mathrm{Re}(\lambda_2(L)) \leq \cdots \leq \mathrm{Re}(\lambda_N(L))$ and $A^{\mathrm{T}}W + WA - 2\mathrm{Re}(\lambda_2(L))WBQB^{\mathrm{T}}W < 0$, we have $\Phi_N(W, \lambda_N(L)) \leq \Phi_{N-1}(W, \lambda_{N-1}(L)) \leq \cdots \leq \Phi_3(W, \lambda_3(L)) \leq \Phi_2(W, \lambda_2(L)) = A^{\mathrm{T}}W + WA - 2\mathrm{Re}(\lambda_2(L))WBQB^{\mathrm{T}}W < 0$. It follows from the Lyapunov stability theory [2] that system matrices $A_k = A - \lambda_k(L)BK$, $k = 2, \ldots, N$, are Hurwitz stable, and thus the FOSs in (13) with $\alpha \in (0, 1)$ are asymptotically stable by Lemma 5. The whole proof is completed. □

Corollary 1: Suppose the system matrices $A \in \mathbb{M}$ and $B \succeq 0$ are unknown but fixed, and they have known upper and lower bounds such that $A \in [\check{A}, \hat{A}]$, $B \in [\check{B}, \hat{B}]$, and $\check{A} \in \mathbb{M}$ with $\check{B} \succeq 0$, then problem PCFMAS $\alpha \in (0, 1)$ is solved by gain matrix $K = QB^{\mathrm{T}}W^{-1}$ if there exist a diagonal matrix $W > 0$ and $Q > 0$ such that all the following conditions hold.

1) $Q \succeq 0$.
2) $\check{A}W - \beta(L)\hat{B}Q\hat{B}^{\mathrm{T}} \in \mathbb{M}$.
3) $W\check{A}^{\mathrm{T}} + \check{A}W - 2\mathrm{Re}(\lambda_2(L))\hat{B}Q\hat{B}^{\mathrm{T}} \in \mathbb{M}$.
4) $W\hat{A}^{\mathrm{T}} + \hat{A}W - 2\mathrm{Re}(\lambda_2(L))\check{B}Q\check{B}^{\mathrm{T}} < 0$.

Proof: As $B \succeq 0$, conditions 1), 2), and 3) imply that $\forall A \in [\check{A}, \hat{A}]$, $\forall B \in [\check{B}, \hat{B}]$, $BQB^{\mathrm{T}} \succeq 0$, $AW - \beta(L)BQB^{\mathrm{T}} \in \mathbb{M}$, and $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(L))BQB^{\mathrm{T}} \in \mathbb{M}$, respectively. It is easy to see $W\hat{A}^{\mathrm{T}} + \hat{A}W - 2\mathrm{Re}(\lambda_2(L))\check{B}Q\check{B}^{\mathrm{T}} \succeq WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(L))BQB^{\mathrm{T}}$, then by Lemmas 2 and 3, and the proof in Theorem 2, we have $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2)BQB^{\mathrm{T}} < 0$ when condition 4) $W\hat{A}^{\mathrm{T}} + \hat{A}W - 2\mathrm{Re}(\lambda_2(L))\check{B}Q\check{B}^{\mathrm{T}} < 0$ holds. Therefore, the FOSs in (13) subject to interval uncertainties are asymptotically stable for $\alpha \in (0, 1)$. The whole proof is completed. □

Theorem 2 and Corollary 1 have provided the synthesis conditions on the PCFMAS with $\alpha \in (0, 1)$. Next, we will go even further to investigate the problem with $\alpha \in (1, 2)$ and propose the conditions for positive consensus. For clearer illustration, we define $\theta_k = \mathrm{Re}(\lambda_k(L))\sin(\alpha\pi/2) - \mathrm{Im}(\lambda_k(L))\cos(\alpha\pi/2)$ and $\gamma_k = \mathrm{Im}(\lambda_k(L))\sin(\alpha\pi/2) + \mathrm{Re}(\lambda_k(L))\cos(\alpha\pi/2)$ for $k = 2, 3, \ldots, N$.

Theorem 3: Problem PCFMAS with $\alpha \in [1, 2)$ is solved by gain matrix $K = QB^{\mathrm{T}}W^{-1}$ if there exist an M-matrix $W > 0$, a matrix $Q > 0$ and a sufficiently large scalar $\mu \succ 0$ such that all the following conditions are satisfied.

1) $BQB^{\mathrm{T}} \succeq 0$.
2) $AW - \beta(L)BQB^{\mathrm{T}} + \mu W \succeq 0$.
3) $\theta := \min\{\theta_k\} \succ 0$.
4) The inequality in (10) holds for $k = 2, 3, \ldots, N$.

Proof: As the conditions in 1) and 2) are derived regarding the positivity of the system, the proof of them is similar to that in Theorem 2, and thus omitted here. Next, we will give the proof for the consensus condition of agents. Define

$$\check{A} = \begin{bmatrix} A\sin(\alpha\pi/2) & -A\cos(\alpha\pi/2) \\ A\cos(\alpha\pi/2) & A\sin(\alpha\pi/2) \end{bmatrix}$$

$$\check{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} > 0, \quad \Theta_k = \begin{bmatrix} \theta_k I_p & \gamma_k I_p \\ -\gamma_k I_p & \theta_k I_p \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \bar{W} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} > 0, \quad \check{K} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$$

where $1 \succeq \sin(\alpha\pi/2) \succ 0$ and $-1 \prec \cos(\alpha\pi/2) \preceq 0$. Taking $K = QB^{\mathrm{T}}W^{-1}$, then condition 4) can be represented as $\check{W}\check{A}^{\mathrm{T}} + \check{A}\check{W} - 2\theta\check{B}\check{Q}\check{B}^{\mathrm{T}} < 0$. By following the line in the proof of Theorem 2 and condition 3) $\theta := \min\{\theta_k\} \succ 0$, we can show that $\check{W}\check{A}^{\mathrm{T}} + \check{A}\check{W} - 2\theta_k\check{B}\check{Q}\check{B}^{\mathrm{T}} \leq \check{W}\check{A}^{\mathrm{T}} + \check{A}\check{W} - 2\theta\check{B}\check{Q}\check{B}^{\mathrm{T}} < 0$ for $k = 2, 3, \ldots, N$. The inequalities $\check{W}\check{A}^{\mathrm{T}} + \check{A}\check{W} - 2\theta_k\check{B}\check{Q}\check{B}^{\mathrm{T}}$ for $k = 2, 3, \ldots, N$, can be rewritten as

$$(\check{A} - \Theta_k\check{B}\check{K})\check{W} + \check{W}(\check{A} - \Theta_k\check{B}\check{K})^{\mathrm{T}} < 0$$

which further indicates that $U_k := \check{A} - \Theta_k\check{B}\check{K}$ is Hurwitz. Define

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} -jI_p & -jI_p \\ I_p & -I_p \end{bmatrix}, \quad S^* = \frac{1}{\sqrt{2}j}\begin{bmatrix} -I_p & jI_p \\ -I_p & -jI_p \end{bmatrix}.$$

It follows from (14), as shown at the bottom of the page, that $(\sin(\alpha\pi/2) + \cos(\alpha\pi/2)j)A_k$ is also Hurwitz. Regarding the A_k as a unit and using the similarity transformation to (14) again, we show that (14) is similar to

$$\check{A} - \lambda_k(L)\check{B}\check{K} := \begin{bmatrix} A_k\sin(\alpha\pi/2) & A_k\cos(\alpha\pi/2) \\ -A_k\cos(\alpha\pi/2) & A_k\sin(\alpha\pi/2) \end{bmatrix}$$

which means that $\check{A} - \lambda_k(L)\check{B}\check{K}$ for $k = 2, 3, \ldots, N$, are Hurwitz. By Lemma 6, it follows that the FOSs in (13) with $\alpha \in [1, 2)$ are asymptotically stable. The whole proof is completed. □

IV. Numerical Simulation

In this section, a comprehensive comparison study of solving problem PCFMAS is made between our proposed approaches and those in [16]–[19] and [26].

A. Fractional-Order $\alpha \in (0, 1)$

This example borrowed from [26] aims to compare the approach of Theorem 2 with that proposed in [26]. Consider a four-agent system in (7) where the system matrices (see [26, Example 1]) are

$$A = \begin{bmatrix} -16 & 1 & 1 & 1 \\ 2 & -4 & 2 & 3 \\ 1 & 4 & -7 & 3 \\ 2 & 2 & 3 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 0.25 \\ 1.25 \\ 0.25 \\ 0.25 \end{bmatrix}.$$

Note that, according to Lemma 4, this FOS with $\alpha \in (0, 1)$ is unstable since its eigenvalues are $\{8.5321, -16.2367, -3.5033, -8.7921\}$.

$$S^*U_kS := \begin{bmatrix} (\sin(\alpha\pi/2) + \cos(\alpha\pi/2)j)(A - \lambda_k BK) & 0 \\ 0 & \overline{(\sin(\alpha\pi/2) + \cos(\alpha\pi/2)j)(A - \lambda_k BK)} \end{bmatrix} \in \mathbb{H} \tag{14}$$

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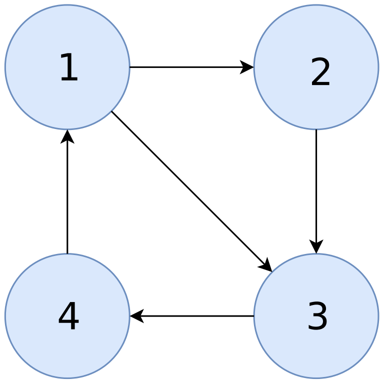

Fig. 1. Directed communication graph $\mathcal{G}$.

The communication topology is described by a directed graph $\mathcal{G}$ (see Fig. 1) whose Laplacian matrix is

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{15}$$

By the above model settings, it is easy to see that $\beta(L) = 2$. The Laplacian eigenvalues are, respectively, $\lambda_1(L) = 0$, $\lambda_2(L) = 1.5 + 0.866j$, $\lambda_3(L) = 1.5 - 0.866j$ and $\lambda_4(L) = 2$. Since the approach proposed in [26] requires that the communication graphs be undirected and connected, it cannot address such a positive consensus problem in this case. In the simulation, we set $\mu = 10\,000$. Solving the linear matrix constraints in Theorem 2, we can obtain that

$$K = \begin{bmatrix} 0.4908 & 1.5120 & 0.4273 & 0.4791 \end{bmatrix}. \tag{16}$$

It can be verified that

$$BK = \begin{bmatrix} 0.1227 & 0.3780 & 0.1068 & 0.1198 \\ 0.6135 & 1.8900 & 0.5342 & 0.5989 \\ 0.1227 & 0.3780 & 0.1068 & 0.1198 \\ 0.1227 & 0.3780 & 0.1068 & 0.1198 \end{bmatrix} \succ 0$$

$$A - \beta(L)BK = \begin{bmatrix} -16.2454 & 0.2440 & 0.7863 & 0.7604 \\ 0.7731 & -7.7800 & 0.9317 & 1.8022 \\ 0.7546 & 3.2440 & -7.2137 & 2.7604 \\ 1.7546 & 1.2440 & 2.7863 & -7.2396 \end{bmatrix} \in \mathbb{M}.$$

Also, the eigenvalues of A_2, A_3, and A_4 are

$$\begin{aligned} &A_2 : \{-2.0840 - 1.0933j, -8.8463 - 0.8576j, \\ &\qquad -9.6350 + 0.3068j, -16.3627 - 0.0492j\} \\ &A_3 : \{-2.0840 + 1.0933j, -8.8463 + 0.8576j, \\ &\qquad -9.6350 - 0.3068j, -16.3627 + 0.0492j\} \\ &A_4 : \{-2.9934, -16.4035, -9.5409 \pm 0.7270j\} \end{aligned}$$

respectively. Therefore, the conditions in Theorem 1 are all satisfied for $\alpha \in (0, 1)$.

To show the robustness of the controller (16), we consider another directed graph $\mathcal{G}_1$ (see Fig. 2) whose Laplacian matrix is

$$L_1 := \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 2 \end{bmatrix}. \tag{17}$$

It is easy to see that $\beta(L_1) = 2$ and the Laplacian eigenvalues are, respectively, $\lambda_1(L_1) = 0$, $\lambda_2(L_1) = 2 + 1j$, $\lambda_3(L_1) = 2 - 1j$ and $\lambda_4(L_1) = 3$. Therefore, this graph belongs to the set: $\{(\mathcal{G},\ L) \mid 1.5 \preceq \mathrm{Re}(\lambda_2(L)),\ 2 \succeq \beta(L)\}$. Also, the eigenvalues of A_2, A_3, and A_4 are

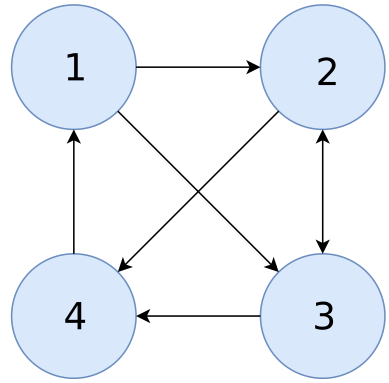

Fig. 2. Directed communication graph $\mathcal{G}_1$.

$$\begin{aligned} &A_2 : \{-3.1935 - 1.2452j, -9.1213 - 1.3387j, \\ &\qquad -9.7752 + 0.4132j, -16.3886 - 0.0687j\} \\ &A_3 : \{-3.1935 + 1.2452j, -9.1213 + 1.3387j, \\ &\qquad -9.7752 - 0.4132j, -16.3886 + 0.0687j\} \\ &A_4 : \{-4.0351, -16.4948, -10.0940 \pm 0.8788j\} \end{aligned}$$

respectively. All the conditions in Theorem 1 are satisfied for $\alpha \in (0, 1)$ as well.

B. Fractional Order $\alpha \in [1, 2)$

1) Integer-Order $\alpha = 1$: This example borrowed from [19] is employed for the comparison of our proposed approach in Theorem 3 and those in [16]–[19] and [26]. Consider a four-agent system in (7) over a directed graph, where the system matrices (see [19, Example 1]) are

$$A = \begin{bmatrix} -3 & 2 & 3 \\ 1 & -4 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}.$$

Note that this system with $\alpha = 1$ is unstable since its eigenvalues are $\{4.2891, -3.3546, -4.9346\}$. The communication topology is described by the graph $\mathcal{G}$ of the previous example and its Laplacian matrix is given in (15). As graph $\mathcal{G}$ has a directed spanning tree, the approaches in [17] (undirected graphs) and [18] (strongly-connected balanced directed graphs) cannot address the positive consensus problem in this case. Moreover, using the approach proposed in [19] to solve the positive consensus problem in this example, unfortunately, no solutions are obtained. This is because the solvability of the algorithm developed in [19] heavily relies on the initial value, and no solution is guaranteed. Solving the linear matrix constraints in Theorem 3, we can obtain that

$$K = \begin{bmatrix} 0.2984 & 0.1371 & 0.1903 \\ -0.0876 & -0.0361 & 5.7993 \end{bmatrix}. \tag{18}$$

Substituting (18) into the conditions of Theorem 1, we have $BK \succeq 0$ and $A - \beta(L)BK \in \mathbb{M}$. Also, the eigenvalues of A_2, A_3, and A_4 are

$$\begin{aligned} &A_2 : \{-3.3203 - 0.9459j, -5.0992 - 0.1216j \\ &\qquad -21.0977 - 10.2005j\} \\ &A_3 : \{-3.3203 + 0.9459j, -5.0992 + 0.1216j \\ &\qquad -21.0977 + 10.2005j\} \\ &A_4 : \{-3.6614, -5.2696, -27.0919\} \end{aligned}$$

respectively. Therefore, the conditions in Theorem 1 are all satisfied for $\alpha = 1$.

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2) Fractional-Order $\alpha = 1.3$: Consider a four-agent system in (7) where the system matrices are

$$A = \begin{bmatrix} -16 & 10 & 2 & 1 \\ 4 & -4 & 5 & 5 \\ 1 & 10 & -7 & 3 \\ 2 & 10 & 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2.5 \\ 12.5 \\ 2.5 \\ 2.5 \end{bmatrix}.$$

This system with fractional order $\alpha = 1.3$ is unstable since its eigenvalues are $\{9.2487, -19.0312, -11.7861, -6.4314\}$. The communication topology of agents is described by the graph $\mathcal{G}$ in the previous example, with its Laplacian matrix given in (15). $\theta = 0.9434$. Solving the matrix constraints in Theorem 3, we have

$$K = \begin{bmatrix} 0.1594 & 1.8668 & 0.1831 & 0.0999 \end{bmatrix}. \tag{19}$$

It can be verified that

$$BK = \begin{bmatrix} 0.3986 & 4.6670 & 0.4578 & 0.2497 \\ 1.9928 & 23.3350 & 2.2891 & 1.2485 \\ 0.3986 & 4.6670 & 0.4578 & 0.2497 \\ 0.3986 & 4.6670 & 0.4578 & 0.2497 \end{bmatrix} \succ 0$$

$$A - \beta(L)BK = \begin{bmatrix} -16.7971 & 0.6660 & 1.0844 & 0.5006 \\ 0.0144 & -50.6700 & 0.4218 & 2.5030 \\ 0.2029 & 0.6660 & -7.9156 & 2.5006 \\ 1.2029 & 0.6660 & 2.0844 & -1.4994 \end{bmatrix} \in \mathbb{M}.$$

Also, the eigenvalues of A_2, A_3, and A_4 are

$$\begin{aligned} A_2 &: \{-38.4416 - 19.4452j, -16.8833 - 0.3879j \\ &\quad -0.6741 - 1.0584j - 8.6626 - 0.2745j\} \\ A_3 &: \{-38.4416 + 19.4452j, -16.8833 + 0.3879j \\ &\quad -0.6741 + 1.0584j, -8.6626 + 0.2745j\} \\ A_4 &: \{-50.7076, -16.8403, -0.6635, -8.6707\} \end{aligned}$$

respectively. Therefore, the conditions in Theorem 1 are all satisfied for $\alpha = 1.3$.

To show the robustness of controller (19), we considered another graph topology $\mathcal{G}_2$ (see Fig. 3) for a seven-agent system whose Laplacian matrix is

$$L_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

It is easy to see that $\beta(L_2) = 2$ and the Laplacian eigenvalues are, respectively, $\lambda_1(L_2) = 0$, $\lambda_2(L_2) = 1.5 + 0.866j$, $\lambda_3(L_2) = 1.5 - 0.866j$, $\lambda_4(L_2) = \lambda_5(L_2) = 2$, and $\lambda_6(L_2) = \lambda_7(L_2) = 3$. Therefore, this graph belongs to the set: $\{(\mathcal{G}, L) \mid \mathrm{Re}(\lambda_k(L)) \sin(\alpha\pi/2) - \mathrm{Im}(\lambda_k(L)) \cos(\alpha\pi/2) \succeq 0.9434,\ 2 \succeq \beta(L)\}$. The eigenvalues of A_2, A_3, A_4 (A_5) and A_6 (A_7) are

$$\begin{aligned} A_2 &: \{-38.4416 - 19.4452j, -16.8833 - 0.3879j \\ &\quad -0.6741 - 1.0584j, -8.6626 - 0.2745j\} \\ A_3 &: \{-38.4416 + 19.4452j, -16.8833 + 0.3879j \\ &\quad -0.6741 + 1.0584j, -8.6626 + 0.2745j\} \\ A_4\ (A_5) &: \{-50.7076, -16.8403, -0.6635, -8.6707\} \\ A_6\ (A_7) &: \{-74.1860, -17.0503, -1.2571, -8.8298\} \end{aligned}$$

respectively. Therefore, the conditions in Theorem 1 are all satisfied for $\alpha = 1.3$ as well.

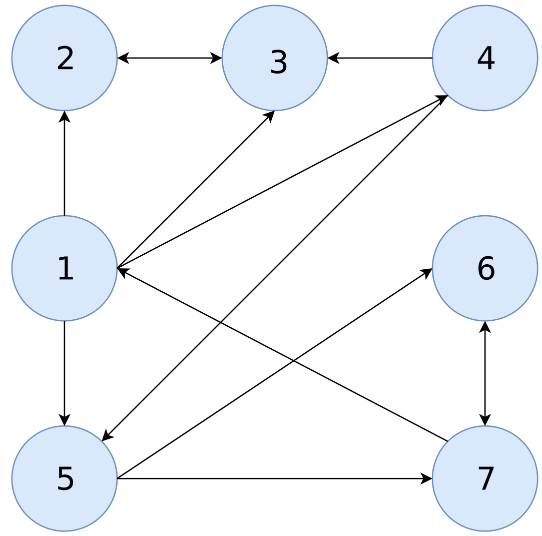

Fig. 3. Directed communication graph $\mathcal{G}_2$.

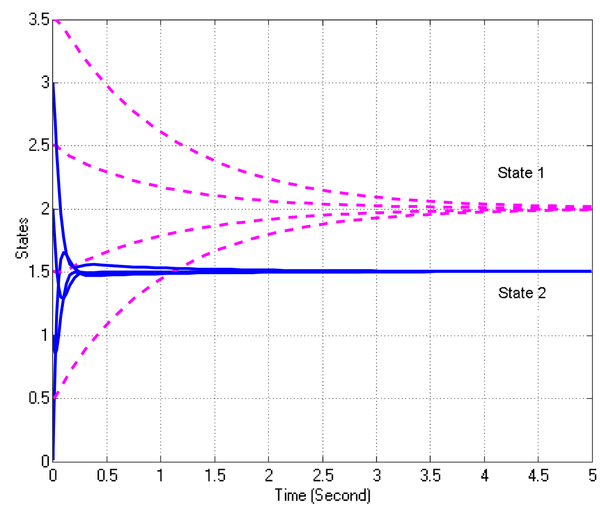

Fig. 4. Consensus result of agents with controller (21).

C. Fractional-Order Linear Electric Circuit

Consider a fractional-order linear electric network consisting of multiple positive electric circuits [22] as shown in Fig. 2 (seven agents). By Kirchhoff's voltage law, we have

$$\begin{cases} e(t) = L_1 D^\alpha i_1(t) + R(i_1(t) - i_2(t)) \\ R(i_1(t) - i_2(t)) = L_2 D^\alpha i_2(t). \end{cases} \tag{20}$$

Choosing $i_1(t)$ and $i_2(t)$ as the two state variables and $e(t)$ as the control input, leads to the system in (2) with the system matrices

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{R_1}{L_1} \\ \frac{R_1}{L_2} & -\frac{R_1}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix}.$$

The values of the parameters are chosen as $R_1 = 1\ \Omega$ and $L_1 = L_2 = 1$ H. Letting $\alpha = 1.1$, and assuming the communication topology of agents is described by the graph $\mathcal{G}$ in Example A with its Laplacian matrix (15), a feasible solution is found as

$$K = \begin{bmatrix} 15.7041 & 0.0568 \end{bmatrix}. \tag{21}$$

Using the obtained controller in (21), the consensus result of agents is shown in Fig. 4 where the initial values of Agents 1–4 are,

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respectively, $[0 \quad 0.5]^{\mathrm{T}}$, $[1 \quad 1.5]^{\mathrm{T}}$, $[2 \quad 2.5]^{\mathrm{T}}$, $[3 \quad 3.5]^{\mathrm{T}}$. We can see from Fig. 4 that the positivity of agents is preserved while achieving consensus.

V. Conclusion

This article has investigated the consensus issue for positive fractional-order interconnected systems over directed graphs. The objective of positive consensus is to design a controller such that the overall system can reach consensus and meanwhile the states of all the agents remain nonnegative throughout the evolutionary process. Using the spectral graph theory, FOSs theory, and positive systems theory, several necessary and/or sufficient conditions on the PCFMAS have been derived. A comprehensive comparison study of different approaches has been conducted and shown that the proposed approaches outperformed the recent published works. In the future, we will explore the positive consensus issue of nonlinear fractional-order linear dynamics with order $\alpha \in (0, 2)$.

References

[1] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, vol. 50. Hoboken, NJ, USA: Wiley, 2011.

[2] D. G. Luenberger, *Introduction to Dynamical Systems*. Hoboken, NJ, USA: Wiley, 1979.

[3] X. Zhao, Y. Yin, L. Liu, and X. Sun, "Stability analysis and delay control for switched positive linear systems," *IEEE Trans. Autom. Control*, vol. 63, no. 7, pp. 2184–2190, Jul. 2018.

[4] M. A. Rami and D. Napp, "Characterization and stability of autonomous positive descriptor systems," *IEEE Trans. Autom. Control*, vol. 57, no. 10, pp. 2668–2673, Oct. 2012.

[5] H. M. Trinh, P. T. Nam, and P. N. Pathirana, "Linear functional state bounding for positive systems with disturbances varying within a bounded set," *Automatica*, vol. 111, Jan. 2020, Art. no. 108644. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109819305059

[6] S. Zhu, Q.-L. Han, and C. Zhang, "L_1-stochastic stability and L_1-gain performance of positive Markov jump linear systems with time-delays: Necessary and sufficient conditions," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3634–3639, Feb. 2017.

[7] L. V. Hien, H. M. Trinh, and P. N. Pathirana, "On ℓ_1-gain control of 2-D positive Roesser systems with directional delays: Necessary and sufficient conditions," *Automatica*, vol. 112, Feb. 2020, Art. no. 108720. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109819305837

[8] G. Bastin and V. Guffens, "Congestion control in compartmental network systems," *Syst. Control Lett.*, vol. 55, no. 8, pp. 689–696, Aug. 2006.

[9] E. Seneta, *Non-negative Matrices and Markov Chains*. New York, NY, USA: Springer, 2006.

[10] Y. Ebihara, D. Peaucelle, and D. Arzelier, "Analysis and synthesis of interconnected positive systems," *IEEE Trans. Autom. Control*, vol. 62, no. 2, pp. 652–667, Feb. 2016.

[11] A. Rantzer and M. E. Valcher, "A tutorial on positive systems and large scale control," in *Proc. IEEE Conf. Decis. Control (CDC)*, Dec. 2018, pp. 3686–3697.

[12] F. Knorn, M. J. Corless, and R. N. Shorten, "A result on implicit consensus with application to emissions control," in *Proc. 50th IEEE Conf. Decis. Control Eur. Control Conf.*, Dec. 2011, pp. 1299–1304.

[13] R. Pahuja, H. K. Verma, and M. Uddin, "A wireless sensor network for greenhouse climate control," *IEEE Pervasive Comput.*, vol. 12, no. 2, pp. 49–58, Apr. 2013.

[14] Y. Sun, Y. Tian, and X.-J. Xie, "Stabilization of positive switched linear systems and its application in consensus of multiagent systems," *IEEE Trans. Autom. Control*, vol. 62, no. 12, pp. 6608–6613, Dec. 2017.

[15] M. Ogura and V. M. Preciado, "Optimal design of switched networks of positive linear systems via geometric programming," *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 2, pp. 213–222, Jun. 2017.

[16] M. E. Valcher and I. Zorzan, "On the consensus of homogeneous multiagent systems with positivity constraints," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5096–5110, Oct. 2017.

[17] J. J. R. Liu, J. Lam, and Z. Shu, "Positivity-preserving consensus of homogeneous multiagent systems," *IEEE Trans. Autom. Control*, vol. 65, no. 6, pp. 2724–2729, Jun. 2020.

[18] H. Wu and H. Su, "Observer-based consensus for positive multiagent systems with directed topology and nonlinear control input," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 49, no. 7, pp. 1459–1469, Jul. 2019.

[19] N. Yang, Y. Yin, and J. Liu, "Positive consensus of directed multi-agent systems using dynamic output-feedback control," in *Proc. 58th IEEE Conf. Decis. Control (CDC)*, Dec. 2019, pp. 897–902.

[20] J. J. R. Liu, K.-W. Kwok, Y. Cui, J. Shen, and J. Lam, "Consensus of positive networked systems on directed graphs," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Mar. 1, 2021, doi: 10.1109/TNNLS.2021.3058184.

[21] T. Kaczorek, "Positive linear systems consisting of n subsystems with different fractional orders," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 6, pp. 1203–1210, Feb. 2011.

[22] T. Kaczorek, *Selected Problems of Fractional Systems Theory*, vol. 411. Berlin, Germany: Springer, 2011.

[23] T. Kaczorek, "Stability of fractional positive nonlinear systems," *Arch. Control Sci.*, vol. 25, no. 4, pp. 491–496, Dec. 2015.

[24] H. V. Le and K. T. Chu, "Robust control of positive fractional-order interconnected systems with heterogeneous delays," *Asian J. Control*, vol. 21, no. 1, pp. 596–608, Jan. 2019.

[25] H. Trinh, D. C. Huong, and S. Nahavandi, "Observer design for positive fractional-order interconnected time-delay systems," *Trans. Inst. Meas. Control*, vol. 41, no. 2, pp. 378–391, Jan. 2019.

[26] Y. Ye, Y. Lu, Y. Wu, R. Lu, and M. Xing, "Positive consensus in fractional-order interval networked systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 68, no. 7, pp. 2538–2542, Jul. 2021.

[27] J. Shen, J. Cao, and J. Lu, "Consensus of fractional-order systems with non-uniform input and communication delays," *Proc. Inst. Mech. Eng., I, J. Syst. Control Eng.*, vol. 226, no. 2, pp. 271–283, Feb. 2012.

[28] H. Su, Y. Ye, X. Chen, and H. He, "Necessary and sufficient conditions for consensus in fractional-order multiagent systems via sampled data over directed graph," *IEEE Trans. Syst., Man, Cybern. Syst.*, vol. 51, no. 4, pp. 2501–2511, Apr. 2021.

[29] Y. Ye and H. Su, "Consensus of delayed fractional-order multiagent systems with intermittent sampled data," *IEEE Trans. Ind. Informat.*, vol. 16, no. 6, pp. 3828–3837, Jun. 2020.

[30] J. Chen, B. Chen, and Z. Zeng, "Synchronization and consensus in networks of linear fractional-order multi-agent systems via sampled-data control," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 8, pp. 2955–2964, Aug. 2020.

[31] X. Peng, H. Wu, and J. Cao, "Global nonfragile synchronization in finite time for fractional-order discontinuous neural networks with nonlinear growth activations," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 30, no. 7, pp. 2123–2137, Jul. 2019.

[32] H. Bao, J. H. Park, and J. Cao, "Adaptive synchronization of fractional-order output-coupling neural networks via quantized output control," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 7, pp. 3230–3239, Jul. 2021.

[33] J. J. R. Liu, N. Yang, K.-W. Kwok, and J. Lam, "Positive consensus of directed multi-agent systems," *IEEE Trans. Autom. Control*, early access, Aug. 30, 2021.

[34] F. R. Chung and F. C. Graham, *Spectral Graph Theory*, no. 92. Providence, RI, USA: American Mathematical Soc., 1997.

[35] B. Shafai and A. Oghbaee, "Positive observer design for fractional order systems," in *Proc. World Autom. Congr. (WAC)*, Aug. 2014, pp. 531–536.

[36] T. Kaczorek, "Fractional positive continuous-time linear systems and their reachability," *Int. J. Appl. Math. Comput. Sci.*, vol. 18, no. 2, pp. 223–228, Jun. 2008.

[37] X. Yang, X. Wan, C. Zunshui, J. Cao, Y. Liu, and L. Rutkowski, "Synchronization of switched discrete-time neural networks via quantized output control with actuator fault," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 32, no. 9, pp. 1–11, Sep. 2020.

[38] X. Wang, H. Su, M. Z. Q. Chen, and X. Wang, "Observer-based robust coordinated control of multiagent systems with input saturation," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 5, pp. 1933–1946, May 2018.

[39] Q. Ma and S. Xu, "Consensusability of first-order multiagent systems under distributed PID controller with time delay," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Jun. 4, 2021, doi: 10.1109/TNNLS.2021.3084366.

[40] J. J. R. Liu, J. Lam, B. Zhu, X. Wang, Z. Shu, and K.-W. Kwok, "Nonnegative consensus tracking of networked systems with convergence rate optimization," *IEEE Trans. Neural Netw. Learn. Syst.*, early access, Jun. 17, 2021, doi: 10.1109/TNNLS.2021.3085396.

[41] A. Berman and R. J. Plemmons, *Nonnegative Matrices in The Mathematical Sciences*. Philadelphia, PA, USA: SIAM, 1994.

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