Decentralized H_2 Control for Discrete-Time Networked Systems With Positivity Constraint

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***Abstract*— In this brief, we study the decentralized H_2 state-feedback control problem for networked discrete-time systems with positivity constraint. This problem (for a single positive system), raised recently in the area of positive systems theory, is known to be challenging due to its inherent nonconvexity. In contrast to most works, which only provide sufficient synthesis conditions for a single positive system, we study this problem within a *primal–dual scheme*, in which *necessary and sufficient* synthesis conditions are proposed for networked positive systems. Based on the equivalent conditions, we develop a primal–dual iterative algorithm for solution, which helps prevent from converging to a local minimum. In the simulation, two illustrative examples are employed for verification of our proposed results.**

***Index Terms*— Discrete-time systems, H_2 state-feedback control, networked systems, positive systems.**

I. Introduction

Among various classes of dynamic systems, there is a special type of systems named *positive dynamic systems*. The first systematic introduction for such kinds of systems can be traced back to a book on fundamental systems theory published by Luenberger [24]. Generally speaking, a positive system can be regarded as a dynamic system whose states and outputs are constrained to be nonnegative given that its inputs and initial states are nonnegative [10]. During the past two decades, there has been a large quantity of research devoted to the investigation of positive systems from a variety of engineering and scientific communities, due to its broad applications in systems biology, pharmacokinetics, and electric circuits [5], [17], [18], [19], [20]. A strong motivation behind the development of positive systems theory is that, in the physical world, many descriptor variables are usually constrained to be nonnegative due to their intrinsic characteristics or physical laws, such as the material flows in a compartmental network [2]. Meanwhile, positive systems theory also finds its way in modeling stochastic or probabilistic processes, since probabilities are intrinsically nonnegative quantities, such as Markov chains [27].

In recent years, a considerable amount of effort has been devoted to addressing the analysis and synthesis issues of positive systems with different performance indices (see [11], [16], [25], [29], [33], [35], [36] and references therein). Taking advantage of the positivity property of systems, one can discover some nice features that are not usually present in the analysis of nonpositive systems. Exploring such nice features is exactly the major object in the study of positive systems theory. As we know, synthesizing a closed-loop system with positivity constraint is, in general, not an easy task by using the fundamental theory of nonpositive systems simply. However, this challenge can be surprisingly circumvented, since a stable positive system admits a diagonal Lyapunov matrix in the stability condition. Because of such a nice property, the state-feedback controller design problem for a closed-loop system with positivity constraint becomes much easier. In addition to the systems' stability, it is found that such a diagonal feature also exists in the KYP-type linear matrix inequality characterization for H_∞ performance of positive systems [26], [32]. Therefore, the positivity-preserving and/or structured H_∞ state-feedback control problem of positive systems can be straightforwardly formulated as a semidefinite programming problem that is convex. Disappointingly, recent studies have shown that, such a diagonal feature does not exist in the H_2 performance characterization for positive systems anymore. The main reason is that, the Lyapunov matrix appearing in the H_2 performance characterization is exactly the controllability (or observability) Gramian of the closed-loop system, which is usually not a diagonal matrix [4], [30]. Thus, the positivity-preserving state-feedback control of positive systems under H_2 performance becomes fairly challenging due to nonconvexity. To tackle such a challenging issue for a single positive system, continuous-time systems were considered, and a useful algorithm for the solvability of suboptimal gains has been developed. In particular, Ebihara et al. [6], [7], [8] have made significant contributions to address the issues of both continuous-time and discrete-time positive dynamic systems and proposed a couple of conditions that are represented as linear matrix inequalities for the computation of suboptimal gains.

More recently, the study of networked positive systems has become a new trend, as a large-scale system that consists of multiple interconnected positive subsystems may exhibit complicated and peculiar characteristics [22], [23], [31]. As such, it is worth investigating networked positive systems in the area of positive systems theory [15], [25], [34]. In this brief, we are going to address the H_2 state-feedback control problem for discrete-time networked systems with positivity constraint. The main contributions of this work in comparison with [8] are summarized as follows: 1) the H_2 state-feedback control problem for a single positive system is extended to that for networked positive systems; 2) novel and effective synthesis characterizations for the positive H_2 control are derived in terms of matrix inequalities; and 3) two tractable optimization algorithms are developed by using the proposed theoretical results.

Notations: We use $\mathbb{R}$ to represent the sets of real numbers. For symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, the notation $A > B$ (respectively, $A \geq B$) means that $A - B$ is positive definite (respectively, positive semidefinite). For matrices $A, B \in \mathbb{R}^{m\times n}$, the notation $A \succ B$ (respectively, $A \succeq B$) means that $A - B$ is positive (respectively, nonnegative). The symbol $\mathrm{tr}(P)$ denotes the trace of matrix $P \in \mathbb{R}^{n\times n}$. The symbol $\mathrm{diag}(A_1, A_2, \ldots, A_N)$ denotes the block diagonal matrix in which the matrices $A_1, A_2, \ldots, A_N$ are diagonal elements. The symbol $*$ represents the off-diagonal element of a symmetric

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matrix. Metzler matrix A is denoted by $A \in \mathbb{M}$. Matrices are assumed in compatible dimensions.

II. Preliminaries and Problem Formulation

A. Preliminaries

In this section, we present some preliminary results to pave the way for further analysis on H_2 state-feedback control of networked discrete-time systems with positivity constraint.

First, the following discrete-time linear system is considered:

$$\mathcal{G}: \begin{cases} x(k+1) = \mathcal{A}x(k) + \mathcal{E}w(k) \\ z(k) = \mathcal{C}x(k) \end{cases} \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{E} \in \mathbb{R}^{n\times m}$, and $\mathcal{C} \in \mathbb{R}^{r\times n}$ are system matrices, and $x(k)$, $w(k)$, and $z(k)$ are system state, disturbance input, and performance output, respectively. The basic definition and a useful result for the positivity of system (1) are provided as follows.

Definition 1 [10]: We call system (1) a positive system, if for any $x(0) \succeq 0$ and input $w(k) \succeq 0$, the system state variable $x(k) \succeq 0$ and the system output $z(k) \succeq 0$ for $t \succeq 0$.

Lemma 1 [10]: System (1) is positive if and only if its system matrices are all nonnegative.

In what follows, we assume that system (1) is always a positive system. The H_2 norm of system (1) (denoted by $\|\mathcal{G}\|_2$) can be characterized by the following lemma.

Lemma 2 [4], [30]: For a given $\gamma > 0$, the system in (1) is asymptotically stable with H_2 norm $\|\mathcal{G}\|_2 < \gamma$, if and only if one of the following two equivalent conditions holds.

1) *Primal:* $\exists\ P > 0$ and $Z > 0$, such that $\mathrm{tr}(Z) < \gamma^2$

$$\begin{bmatrix} P & \mathcal{A}P & \mathcal{E} \\ * & P & 0 \\ * & * & I \end{bmatrix} > 0 \tag{2}$$

$$\begin{bmatrix} Z & \mathcal{C}P \\ * & P \end{bmatrix} > 0. \tag{3}$$

2) *Dual:* $\exists\ Q > 0$ and $W > 0$, such that $\mathrm{tr}(W) < \gamma^2$

$$\begin{bmatrix} Q & \mathcal{A}^{\mathrm{T}}Q & \mathcal{C}^{\mathrm{T}} \\ * & Q & 0 \\ * & * & I \end{bmatrix} > 0 \tag{4}$$

$$\begin{bmatrix} W & \mathcal{E}^{\mathrm{T}}Q \\ * & Q \end{bmatrix} > 0. \tag{5}$$

The linear matrix inequality conditions of this lemma are very fundamental conclusions in the Gramian-based H_2 performance characterization [4], [28], [30].

B. Problem Formulation

Consider a discrete-time networked system constituting N agents, where the ith agent is described by

$$\begin{aligned} x_i(k+1) &= A_{ii}x_i(k) + \sum_{i\neq j}^{N} A_{ij}x_j(k) + B_iu_i(k) + E_iw_i(k) \\ z_i(k) &= C_ix(k) + D_iu_i(k), \quad i = 1, 2, \ldots, N \end{aligned} \tag{6}$$

where $A_{ii} \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{r\times n}$, $D_i \in \mathbb{R}^{c\times p}$, and $E_i \in \mathbb{R}^{n\times p}$ are the ith agent's system matrices, and $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^m$, $w_i(k) \in \mathbb{R}^p$, and $z_i(k) \in \mathbb{R}^r$ are the ith agent's local system state, local control input, disturbance input, and performance output, respectively. The interconnections of N agents are characterized by $\sum_{i\neq j}^{N} A_{ij}x_j(k)$ $(j = 1, 2, \ldots, N)$. Similar to the problem setting in [8], we assume that E_i and A_{ij} $(j = 1, 2, \ldots, N)$ are nonnegative. System (6) can be expressed in a compact form

$$G: \begin{cases} x(k+1) = Ax(k) + Bu(k) + Ew(k) \\ z(k) = Cx(k) + Du(k) \end{cases} \tag{7}$$

where

$$\begin{aligned} x^{\mathrm{T}}(k) &:= \left[x_1^{\mathrm{T}}(k), x_2^{\mathrm{T}}(k), \ldots, x_N^{\mathrm{T}}(k)\right]^{\mathrm{T}} \\ u^{\mathrm{T}}(k) &:= \left[u_1^{\mathrm{T}}(k), u_2^{\mathrm{T}}(k), \ldots, u_N^{\mathrm{T}}(k)\right]^{\mathrm{T}} \\ w^{\mathrm{T}}(k) &:= \left[w_1^{\mathrm{T}}(k), w_2^{\mathrm{T}}(k), \ldots, w_N^{\mathrm{T}}(k)\right]^{\mathrm{T}} \\ z^{\mathrm{T}}(k) &:= \left[z_1^{\mathrm{T}}(k), z_2^{\mathrm{T}}(k), \ldots, z_N^{\mathrm{T}}(k)\right]^{\mathrm{T}} \\ A &:= \left[A_{ij}\right]_{N\times N} \\ B &:= \mathrm{diag}(B_1, B_2, \ldots, B_N) \\ E &:= \mathrm{diag}(E_1, E_2, \ldots, E_N) \\ C &:= \mathrm{diag}(C_1, C_2, \ldots, C_N) \\ D &:= \mathrm{diag}(D_1, D_2, \ldots, D_N). \end{aligned}$$

The following local state-feedback law:

$$u_i(k) = K_ix_i(k), \quad i = 1, 2, \ldots, N. \tag{8}$$

With the state-feedback gain $K_i \in \mathbb{R}^{m\times n}$ is applied to the system in (6), and then, a closed-loop form is obtained as follows:

$$G_{\mathrm{F}}: \begin{cases} x(k+1) = (A+BK)x(k) + Ew(t) \\ z(k) = (C+DK)x(t) \end{cases} \tag{9}$$

where $K := \mathrm{diag}(K_1, K_2, \ldots, K_N)$.

In this brief, we investigate the decentralized H_2 state-feedback control problem for discrete-time networked linear systems with positivity constraint. Based on the above model settings, the definition of the problem that we are going to solve is given as follows.

Problem PH2SC: Considering the positive discrete-time linear system in (6) with the local state-feedback control law in (8), given any $\gamma > 0$ and initial state $x(0) > 0$, determine a set of gain matrices K_i $(i = 1, 2, \ldots, N)$, such that the closed-loop system in (9) is asymptotically stable with H_2 performance $\|G_{\mathrm{F}}\|_2 < 0$, and the state trajectories of agents remain nonnegative.

III. Main Results

The analysis and synthesis conditions of **Problem PH2SC** are derived in this section, by virtue of positive systems theory and H_2 performance analysis.

A. Positive Decentralized H_2 Control Analysis and Synthesis

It is well known that, even for positive systems, the Lyapunov matrix in the H_2 performance characterization is the controllability (or observability) Gramian of the closed-loop system, which is usually not diagonal [4], [30]. The following analysis result for the positive decentralized H_2 control of system (9) can be readily obtained by Lemmas 1 and 2.

Proposition 1: For $\gamma \succ 0$, **Problem PH2SC** is solved by gain matrix K, if and only if the following two conditions hold simultaneously.

1) Matrices $A + BK \succeq 0$, $C + DK \succeq 0$.
2) *Primal:* $\exists\ P > 0$ and $Z > 0$, such that $\mathrm{tr}(Z) < \gamma^2$, and

$$\begin{bmatrix} P & (A+BK)P & E \\ * & P & 0 \\ * & * & I \end{bmatrix} > 0 \tag{10}$$

$$\begin{bmatrix} Z & (C+DK)P \\ * & P \end{bmatrix} > 0 \tag{11}$$

or Dual: $\exists\ Q > 0$ and $W > 0$ such that $\mathrm{tr}(W) < \gamma^2$

$$\begin{bmatrix} Q & (A+BK)^{\mathrm{T}}Q & (C+DK)^{\mathrm{T}} \\ * & Q & 0 \\ * & * & I \end{bmatrix} > 0 \tag{12}$$

$$\begin{bmatrix} W & E^{\mathrm{T}}Q \\ * & Q \end{bmatrix} > 0. \tag{13}$$

In the following theorem, a novel primal–dual characterization that is equivalent to the condition we summarized in Proposition 1 is derived for the analysis of **Problem PH2SC**.

Theorem 1: For $\gamma > 0$, **Problem PH2SC** is solved by gain matrix K, if and only if one of the following two conditions holds.

1) *Primal:* $\exists\ P > 0$, $Z > 0$, and a scalar $\alpha > 0$, such that $\mathrm{tr}(Z) < \gamma^2$, and

$$\begin{bmatrix} P+\alpha BKK^{\mathrm{T}}B^{\mathrm{T}} & AP & E & -\alpha BK \\ * & P & 0 & P \\ * & * & I & 0 \\ * & * & * & \alpha I \end{bmatrix} > 0 \tag{14}$$

$$\begin{bmatrix} Z+\alpha DKK^{\mathrm{T}}D^{\mathrm{T}} & CP & -\alpha DK \\ * & P & P \\ * & * & \alpha I \end{bmatrix} > 0 \tag{15}$$

and matrices $A+BK \succeq 0$, $C+DK \succeq 0$.

2) *Dual:* $\exists\ Q > 0$, $W > 0$, and a scalar $\beta > 0$, such that $\mathrm{tr}(W) < \gamma^2$

$$\begin{bmatrix} Q+\beta K^{\mathrm{T}}K & A^{\mathrm{T}}Q & C^{\mathrm{T}} & -\beta K^{\mathrm{T}} \\ * & Q & 0 & QB \\ * & * & I & D \\ * & * & * & \beta I \end{bmatrix} > 0 \tag{16}$$

$$\begin{bmatrix} W & E^{\mathrm{T}}Q \\ * & Q \end{bmatrix} > 0 \tag{17}$$

and matrices $A+BK \succeq 0$, $C+DK \succeq 0$.

Proof: Since the nonnegative constraints on matrices $A + BK \succeq 0$ and $C + DK \succeq 0$ are common in conditions 1) and 2), one only needs to show that inequalities (14) and (15) are equivalent to inequalities (10) and (11). Define two nonsingular matrices as follows:

$$T_1 = \begin{bmatrix} I & 0 & 0 & BK \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & 0 & DK \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Performing a congruent manipulation to inequality (14) by T_1 and T_1^{T} yields

$$\begin{bmatrix} P & (A+BK)P & E & 0 \\ * & P & 0 & P \\ * & * & I & 0 \\ * & * & * & \alpha I \end{bmatrix} > 0 \tag{18}$$

which indicates that inequality (10) holds. Performing a similar matrix manipulation to inequality (15) by T_2 and T_2^{T}, we have

$$\begin{bmatrix} Z & (C+DK)P & 0 \\ * & P & P \\ * & * & \alpha I \end{bmatrix} > 0 \tag{19}$$

which also indicates that inequality (11) holds. This completes the sufficiency part.

On the other hand, assuming that inequalities (10) and (11) hold, there always exist two sufficiently large scalars $\alpha_1 > 0$ and $\alpha_2 > 0$, such that

$$\begin{bmatrix} P & (A+BK)P & E \\ * & P & 0 \\ * & * & I \end{bmatrix} > \begin{bmatrix} 0 & 0 & 0 \\ * & (1/\alpha_1)PP^{\mathrm{T}} & 0 \\ * & * & 0 \end{bmatrix} \geq 0 \tag{20}$$

$$\begin{bmatrix} Z & (C+DK)P \\ * & P \end{bmatrix} > \begin{bmatrix} 0 & 0 \\ * & (1/\alpha_2)PP^{\mathrm{T}} \end{bmatrix} \geq 0. \tag{21}$$

Taking $\alpha = \max\{\alpha_1, \alpha_2\}$, and by Schur complement equivalence, we have inequalities (18) and (19) hold. Performing a congruent manipulation to inequality (18) by T_1^{-1} and $T_1^{-\mathrm{T}}$ yields inequality (14). Similarly, pre- and post-multiplying inequality (19) by T_2^{-1} and $T_2^{-\mathrm{T}}$ yields inequality (15). This completes the necessity part. Based on the above discussion, we have that inequalities (14) and (15) are equivalent to inequalities (10) and (11).

To show the equivalence of (16) and (12), we can similarly define

$$T_3 = \begin{bmatrix} I & 0 & 0 & -K^{\mathrm{T}} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Along the same lines, we can readily show that inequality (16) is equivalent to inequality (12). The proof is completed. □

Remark 1: A novel primal–dual characterization is established for the H_2 state-feedback control of positive discrete-time networked systems. The most prominent characteristic of Theorem 1 is that, it gives a *necessary and sufficient* condition in which the Lyapunov matrix P (or Q) and the controller gain K are completely separated. This can be observed from the inequalities in (10)–(17). The property allows us to parametrize the controller gains without imposing any specific structures on the Lyapunov matrix. In other words, we can impose some particular constraints on the controller K, such as sparsity, positiveness, and negativenes, along with a free parameter P in the computation, which is going to be presented in the sequel.

By employing the useful results that we have initially concluded in Theorem 1, the corresponding *necessary and sufficient conditions* on the synthesis of networked positive systems are derived in the following theorem for **Problem PH2SC**.

Theorem 2: For given $\gamma \succ 0$, **Problem PH2SC** is solved by gain matrix $K = (1/\alpha)F$ (or $K = (1/\beta)F$), if and only if one of the following two conditions holds.

$$\begin{bmatrix} P+\alpha BKK^{\mathrm{T}}B^{\mathrm{T}} & AP & E & \alpha BK \\ * & P & 0 & P \\ * & * & I & 0 \\ * & * & * & \alpha I \end{bmatrix} > \begin{bmatrix} \alpha B(K-M)(K-M)^{\mathrm{T}}B^{\mathrm{T}} & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix} \geq 0 \tag{22}$$

$$\begin{bmatrix} Z+\alpha DKK^{\mathrm{T}}D^{\mathrm{T}} & CP & \alpha DK \\ * & P & P \\ * & * & \alpha I \end{bmatrix} > \begin{bmatrix} \alpha D(K-M)(K-M)^{\mathrm{T}}D^{\mathrm{T}} & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \geq 0 \tag{23}$$

1) *Primal:* $\exists\ P > 0$, $Z > 0$, F, M, and a scalar $\alpha > 0$, such that $\mathrm{tr}(Z) < \gamma^2$, and

$$\begin{bmatrix} \Phi & AP & E & -BF \\ * & P & 0 & P \\ * & * & I & 0 \\ * & * & * & \alpha I \end{bmatrix} > 0 \tag{24}$$

$$\begin{bmatrix} \Lambda & CP & -DF \\ * & P & P \\ * & * & \alpha I \end{bmatrix} > 0 \tag{25}$$

and matrices $A\alpha + BF \succeq 0$, $C\alpha + DF \succeq 0$ where $\Phi := P - \alpha BMM^{\mathrm{T}}B^{\mathrm{T}} + BFM^{\mathrm{T}}B^{\mathrm{T}} + BMF^{\mathrm{T}}B^{\mathrm{T}}$ and $\Lambda := Z - \alpha DMM^{\mathrm{T}}D^{\mathrm{T}} + DFM^{\mathrm{T}}D^{\mathrm{T}} + DMF^{\mathrm{T}}D^{\mathrm{T}}$.

2) *Dual:* $\exists\ Q > 0$, $W > 0$, L, M, and a scalar $\beta > 0$, such that $\mathrm{tr}(W) < \gamma^2$

$$\begin{bmatrix} Q-\beta M^{\mathrm{T}}M+L^{\mathrm{T}}M+M^{\mathrm{T}}L & A^{\mathrm{T}}Q & C^{\mathrm{T}} & -L^{\mathrm{T}} \\ * & Q & 0 & QB \\ * & * & I & D \\ * & * & * & \beta I \end{bmatrix} > 0 \tag{26}$$

$$\begin{bmatrix} W & E^{\mathrm{T}}Q \\ * & Q \end{bmatrix} > 0 \tag{27}$$

and matrices $A\beta + BL \succeq 0$, $C\beta + DL \succeq 0$.

Proof: Taking $F = \alpha K$ and substituting it into inequalities as in (22) and (23), shown at the bottom of the previous page, we have

$$\begin{bmatrix} \Phi & AP & E & \alpha BK \\ * & P & 0 & P \\ * & * & I & 0 \\ * & * & * & \alpha I \end{bmatrix} > 0 \tag{28}$$

$$\begin{bmatrix} \Lambda & CP & \alpha DK \\ * & P & P \\ * & * & \alpha I \end{bmatrix} > 0 \tag{29}$$

which can also be represented as (22) and (23), indicating that inequalities (14) and (15) hold. This completes the sufficiency part.

Assume that inequalities (14) and (15) hold. Taking $M = K$, then inequalities (14) and (15) lead to (28) and (29), since $\alpha BKK^{\mathrm{T}}B^{\mathrm{T}} = \alpha BKK^{\mathrm{T}}B^{\mathrm{T}} + \alpha B(K-M)(K-M)^{\mathrm{T}}B^{\mathrm{T}} = -\alpha BMM^{\mathrm{T}}B^{\mathrm{T}} - \alpha BKM^{\mathrm{T}}B^{\mathrm{T}} - \alpha BMK^{\mathrm{T}}B^{\mathrm{T}}$ and $\alpha DKK^{\mathrm{T}}D^{\mathrm{T}} = \alpha DKK^{\mathrm{T}}D^{\mathrm{T}} + \alpha D(K-M)(K-M)^{\mathrm{T}}D^{\mathrm{T}} = -\alpha DMM^{\mathrm{T}}D^{\mathrm{T}} - \alpha DKM^{\mathrm{T}}D^{\mathrm{T}} - \alpha DMK^{\mathrm{T}}D^{\mathrm{T}}$. Moreover, $A\alpha + BF = A\alpha + \alpha BK \succeq 0$ and $C\alpha + DF = C\alpha + \alpha DK \succeq 0$. Therefore, the condition in 1) of Theorem 2 is equivalent to the condition in 1) of Theorem 1.

Regarding condition 2), by taking $L = \beta K$ and following a similar line as above, we can readily show that condition 2) in Theorem 2 is equivalent to condition 2) in Theorem 1. The whole proof is completed. □

Remark 2: By introducing two additional variables, that is, α (or β) and F, another novel equivalent condition is derived for the synthesis of **Problem PH2SC** in Theorem 2 where the actual gain matrix is solved implicitly by $K = (1/\alpha)F$ (or $K = (1/\beta)F$). Note that there is no conservatism in the synthesis result, although we have introduced new variables to the condition.

B. Optimization Algorithms for Solution

In this section, two semidefinite programming algorithms are developed to solve **Problem PH2SC**, among which the first one is associated with the following results.

Proposition 2: For given $\gamma \succ 0$, **Problem PH2SC** is solved by a gain matrix $K = SH^{-1}$, if $\exists\ P > 0$, $Z > 0$, $S := \mathrm{diag}(S_1, S_2, \ldots, S_N)$, and $H := \mathrm{diag}(H_1, H_2, \ldots, H_N)$, such that the following two conditions hold.

1)

$$AH + BS \succeq 0, \quad CH + DS \succeq 0. \tag{30}$$

2) $\mathrm{tr}(Z) < \gamma^2$

$$\begin{bmatrix} P & AH+BS & E \\ * & H+H^{\mathrm{T}}-P & 0 \\ * & * & I \end{bmatrix} > 0 \tag{31}$$

$$\begin{bmatrix} Z & CH+DS \\ * & H+H^{\mathrm{T}}-P \end{bmatrix} > 0 \tag{32}$$

where H_i $(i = 1, 2, \ldots, N)$ is an M-matrix (that is, H_i is a matrix of which the eigenvalues have nonnegative real parts, and the off-diagonal elements are nonpositive [3]).

Proof: It follows from (31) and (32) that $H + H^{\mathrm{T}} > P > 0$, which further indicates that the eigenvalues of H locate at the open right half-plane. Since the off-diagonal elements of H_i are nonpositive, the matrix H_i (or H) is an M-matrix. Therefore, we have $H^{-1} \succeq 0$. For condition 1), we have matrices $(AH + BS)H^{-1} = A + BK \succeq 0$ and $(CH + DS)H^{-1} = C + DK \succeq 0$, which guarantees the positivity. Using [[8], Lemma 1], and a change of variables readily yields condition 2). □

As the constraints we derived in Proposition 2 are represented as linear matrix inequalities, a convex optimization algorithm for minimizing $\rho := \gamma^2$ can be developed as follows.

Algorithm H2SC1:

$$\begin{aligned} &\text{Minimize } \rho \\ &\quad \text{s.t.} \quad \{\mathrm{tr}(Z) < \rho, (30), (31), (32)\} \\ &\quad \text{w.r.t.} \quad \{P > 0, Z > 0, S, H \in \mathbb{M}\}. \end{aligned}$$

Although **Algorithm H2SC1** remains conservative, it can provide us with good starting points for the primal–dual iterative algorithm to be developed. By using the results concluded in Theorem 2, we can develop a primal–dual iterative algorithm for the solvability of **Problem PH2SC** as follows.

Algorithm H2SC2:

Step 1. Set $i = 1$, $\rho^{(0)} = 0$, $\delta = 0$, $\tilde{\gamma} = \tilde{\gamma}_0 \succ 0$. Compute an $M^{(1)}$ using **Algorithm H2SC1**.

Step 2. Fix $M = M^{(i)}$, minimize $\rho^{(i)} = \rho$

$$\text{s.t.} \begin{cases} \mathrm{tr}(Z) < \rho, \\ (24), \\ (25), \qquad \text{w.r.t.}\{P > 0, Z > 0, \alpha > 0, F\}. \\ A\alpha + BF \succeq 0, \\ C\alpha + DF \succeq 0, \end{cases}$$

If $\rho^{(i)} \preceq \tilde{\gamma}^2$, a feasible $K = (1/\alpha)F$ is found. **STOP**. Otherwise, go to next step.

Step 3. If $|\rho^{(i)} - \delta|/\rho^{(i)} \prec \theta$ (a prescribed positive tolerance), **STOP**. Otherwise, go to next step.

Step 4. If $|\rho^{(i)} - \rho^{(i-1)}|/\rho^{(i)} \prec \theta$, set $\delta = \rho^{(i)}$, $i = i+1$, update $M^{(i)} = (1/\alpha)F$, go to **Step 5**. Otherwise, set $i = i+1$, update $M^{(i)} = (1/\alpha)F$, then go to **Step 2**.

Step 5. Fix $M = M^{(i)}$, minimize $\rho^{(i)} = \rho$

$$\text{s.t.} \begin{cases} \mathrm{tr}(W) < \rho, \\ (26), \\ (28), \qquad \text{w.r.t.}\{Q > 0, W > 0, \beta > 0, L\}. \\ A\beta + BL \succeq 0, \\ C\beta + DL \succeq 0, \end{cases}$$

If $\rho^{(i)} \preceq \tilde{\gamma}^2$, a feasible $K = (1/\beta)L$ is found. **STOP**. Otherwise, go to next step.

TABLE I

H_2 PERFORMANCE OF THE CLOSED-LOOP SYSTEM WITH CONTROLLERS IN EXAMPLE 1

Controllers	(33) and (34)	(35) and (36)
H_2 Performance	4.5663	4.2074

Step 6. If $|\rho^{(i)} - \delta|/\rho^{(i)} \prec \theta$, **STOP**. Otherwise, go to next step.

Step 7. If $|\rho^{(i)} - \rho^{(i-1)}|/\rho^{(i)} \prec \theta$, set $\delta = \rho^{(i)}$, $i = i+1$, update $M^{(i)} = (1/\beta)L$, go to **Step 2**. Otherwise, set $i = i+1$, update $M^{(i)} = (1/\beta)L$, then go to **Step 5**.

Remark 3: In Step 1, we compute a starting point $M^{(1)}$ using **Algorithm H2SC1**. We denote the H_2 performance of the closed-loop system with $M^{(1)}$ by γ_0. **Algorithm H2SC2** contains two different inner loops (from Steps 2 to 4 or from Steps 5 to 7) and a single outer loop (from Steps 2 to 7). The first inner loop with the starting point $M^{(1)}$ can always find a set of new points K guaranteeing that $\rho^{(i+1)} \preceq \rho^{(i)} \preceq \gamma_0^2$, that is, $\gamma^{(i+1)} \preceq \gamma^{(i)} \preceq \gamma_0$ for $i \succeq 1$. While the first inner loop gets stuck at a local minimum $K = M^{(i)}$, it will jump to the second inner loop, which possesses the same convergence property as the first one. Therefore, throughout the overall iterations of **Algorithm H2SC2**, we have that $\gamma^{(i+1)} \preceq \gamma^{(i)} \preceq \gamma_0$ for $i \succeq 1$.

IV. ILLUSTRATIVE EXAMPLES

In this section, two illustrative examples are employed to verify the proposed results and algorithms in Section III. The algorithms are implemented with Yalmip and SeDuMi in MATLAB 2014a.

A. Example 1

The positive discrete-time linear system in (6) is considered, in which

$$A_{11} = \begin{bmatrix} 0 & 0.06 & 0 \\ 0.39 & 0.45 & 0.36 \\ 0.37 & 0 & 0.39 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.18 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0.11 & 0 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 0 & 0.4 & 0 \\ 0 & 0 & 0.1 \\ 0.08 & 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.9 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1.5845 \\ 0 \\ 1.2905 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 1.8621 \\ 1.6147 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.2 \\ 1.3 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1.5 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.20 & 0.22 \\ 0.17 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.29 & 0 \\ 0 & 0.17 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0.8 & 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.4 & 0.8 & 0 \end{bmatrix}.$$

Using **Algorithm H2SC1** to the networked positive system, two controller gains were obtained as follows:

$$K_1 = \begin{bmatrix} 0.1052 & 0.0531 & 0.0167 \\ -0.0296 & -0.3166 & -0.2758 \end{bmatrix} \tag{33}$$

$$K_2 = \begin{bmatrix} -0.3114 & -0.0632 & 0.1936 \\ 0.0177 & 0.0027 & -0.4752 \end{bmatrix}. \tag{34}$$

Using **Algorithm H2SC2** with the starting point (33) and (34), two improved controller gains were obtained after five iterations

$$K_1 = \begin{bmatrix} 0.0047 & 0 & 0 \\ -0.0043 & -0.375 & -0.3 \end{bmatrix} \tag{35}$$

$$K_2 = \begin{bmatrix} -0.3333 & -0.0667 & 0.1724 \\ 0 & 0 & -0.5 \end{bmatrix}. \tag{36}$$

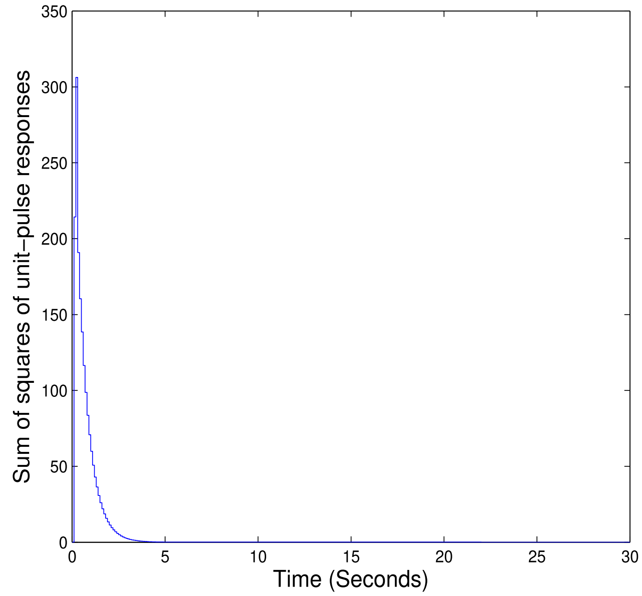

Fig. 1. Sum of squares of system unit-pulse responses with controllers (35) and (36).

The H_2 performance of the closed-loop system with controllers (33)–(36) is summarized in Table I. From the table, we can see that **Algorithm H2SC1** provided an acceptable solution, and **Algorithm H2SC2** provided an improved solution. Notice that in the time domain, the H_2 norm of system has the following characterization: $\|G_F\|_2^2 = \Sigma_{k=0}^{\infty} \text{tr}\{\tilde{G}_F(k)\tilde{G}_F^T(k)\}$, where $\tilde{G}_F(k)$ is the unit-pulse response matrix [30]. For instance, the matrix $\tilde{G}_F(k)$ in this example is a 4×4 matrix of which the element in the first row and second column represents the unit-pulse response of system channel Input 2 to Output 1. Therefore, the sum of squares of system unit-pulse responses (sampling period is 0.1 s) with controller gains (35) and (36) is shown in Fig. 1 from which we can see that the suggested control strategy appears to be effective to control the networked positive system in this example. Besides, we can further verify the positivity of the closed-loop system with controller gains (35) and (36) by Lemma 1.

B. Example 2

The positive discrete-time linear system in (6) is considered, in which the system matrices are

$$A_{11} = \begin{bmatrix} 0.7 & 0.2 \\ 0.1 & 0.5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.3 & 0.9 \\ 0 & 0.59 \end{bmatrix}$$

$$A_{33} = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.5 \end{bmatrix}, \quad A_{12} = A_{21}^T \begin{bmatrix} 0.27 & 0 \\ 0.23 & 0.31 \end{bmatrix}$$

$$A_{13} = A_{31}^T \begin{bmatrix} 0.24 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{23} = A_{32}^T \begin{bmatrix} 0.04 & 0 \\ 0.3 & 0.2 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.46 \\ 0.65 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.29 \\ 0.75 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0.55 \\ 0.42 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.11 \\ 0.14 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.01 \\ 0.96 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0.97 \\ 0.12 \end{bmatrix}$$

$$D_1 = 0.5, \quad D_2 = 0.1, \quad D_3 = 0.19, \quad C_1 = \begin{bmatrix} 0.26 & 0.75 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.89 & 0.72 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.4 & 0.93 \end{bmatrix}.$$

It can be easily verified that this open-loop system is unstable. Using **Algorithm H2SC1** to the networked positive system, three controller gains were obtained as follows:

$$K_1 = \begin{bmatrix} -0.7050 & -1.7991 \end{bmatrix} \tag{37}$$

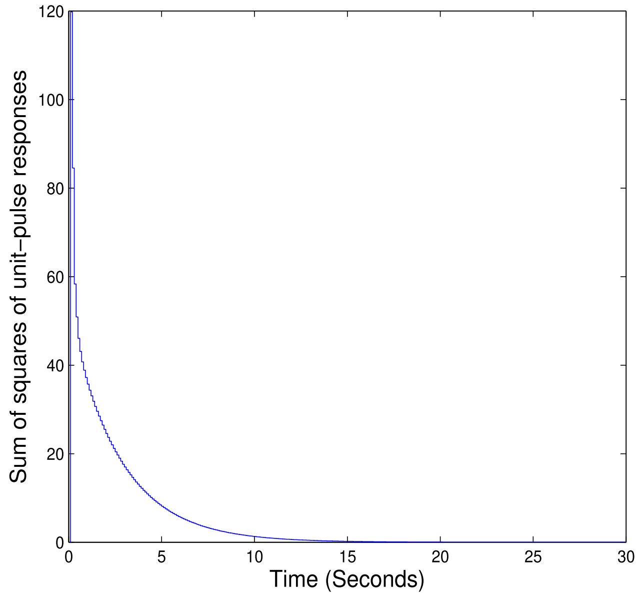

Fig. 2. Sum of squares of system unit-pulse responses with controllers (40) to (42).

TABLE II
H_2 PERFORMANCE OF THE CLOSED-LOOP SYSTEM WITH CONTROLLERS IN EXAMPLE 2

Controllers	(38) to (40)	(41) to (43)
H_2 Performance	3.8715	3.5468

$$K_2 = \begin{bmatrix} 0.0017 & -0.6106 \end{bmatrix} \quad (38)$$
$$K_3 = \begin{bmatrix} -0.2002 & 0.0091 \end{bmatrix}. \quad (39)$$

Using **Algorithm H2SC2** with the starting point of (37)–(39), three improved controller gains were obtained after seven iterations

$$K_1 = \begin{bmatrix} -0.7143 & -1.8182 \end{bmatrix} \quad (40)$$
$$K_2 = \begin{bmatrix} 0 & -0.6146 \end{bmatrix} \quad (41)$$
$$K_3 = \begin{bmatrix} -0.2062 & 0 \end{bmatrix}. \quad (42)$$

The H_2 performance of the closed-loop system with controllers (37)–(42) is summarized in Table II. From the table, we can see that **Algorithm H2SC1** provided an acceptable solution as the starting point, and **Algorithm H2SC2** provided an improved solution. The system unit-pulse response (sampling period is 0.1 s) with controller gains (40)–(42) under the expected H_2 performance is shown in Fig. 2. The suggested control strategy appears to be very effective to control the networked positive system. Besides, we can further verify the positivity of the closed-loop system with controller gains (40)–(42) by Lemma 1.

V. CONCLUSION

This brief studied the H_2 state-feedback control problem for networked discrete-time systems with positivity constraint. This challenging problem for a single positive system was raised in the field of positive systems theory recently. In contrast to most works, which only provide sufficient synthesis conditions, the problem was studied within a *primal–dual framework*, and the *necessary and sufficient* synthesis conditions were proposed. Based on the derived conditions, a primal–dual iterative algorithm was developed for solution. Two illustrative examples were employed for the verification of our proposed results and algorithms. In the future, we will extend our approach to address the cooperative H_2 control issues of interconnected positive systems or stochastic systems that may exhibit positive property [9], [21], [37] as well as the observed-based consensus problem of positive agents [1], [12], [13], [14].

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