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Stability and stabilization of periodic piecewise positive systems: A time segmentation approach

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Abstract
This paper is concerned with the stability analysis and stabilization of periodic piecewise positive systems. By constructing a time-scheduled copositive Lyapunov function with a time segmentation approach, an equivalent stability condition, determined via linear programming, for periodic piecewise positive systems is established. Based on the asymptotic stability condition, the spectral radius characterization of the state transition matrix is proposed. The relation between the spectral radius of the state transition matrix and the convergent rate of the system is also revealed. An iterative algorithm is developed to stabilize the system by decreasing the spectral radius of the state transition matrix. Finally, numerical examples are given to illustrate the results.

KEYWORDS
decay rate, periodic piecewise systems, positive systems, stability, stabilization

1 | INTRODUCTION

Positive systems, whose state is always in the nonnegative orthant, have drawn increasing attention in recent decades. Due to the positivity of the state, the systems feature a couple of advantages in theoretical research, including decrease of the complexity of stability conditions [1, 2], simplification of the characterization for some input–output gains, like L_1- and L-gains, which were first considered in Briat [3], and reduction of conservativeness of conditions for stability and input–output gain analysis for some kinds of positive systems [4, 5] and therefore have a wide range of applications in engineering fields, including disease transmission [6], population dynamics [7], networked fluid flow [8], and spacecraft rendezvous process [9].

Different kinds of systems with positivity have been investigated, including Markov jump systems [6, 10, 11], periodic systems [12, 13], singular systems [14–16], switched systems [17–19], and time delay systems [20, 21].

For linear continuous time-invariant positive systems, the stability, L_1-, and L-gain can be characterized by the linear inequality. This represents a significant reduction of the number of decision parameters for analyzing the stability of positive systems when compared with the linear matrix inequality (LMI) approach for that of general linear systems. Therefore, linear programming formulations, which are based on linear copositive Lyapunov functions, have been developed to study stability and input–output gain of different kinds of positive systems.

As a special kind of positive systems, periodic piecewise positive systems have numerous applications, including traffic systems [22] and medical treatments [23]. In previous work, the above applications were always modeled as positive switched systems, which can be found in Blanchini et al. [8] and Xiang et al. [24]. By ignoring the inherent periodicity and fixed time interval of each subsystem, the obtained results are more conservative for those practical applications. By using periodic piecewise positive systems for characterization, the obtained analytical results will be sharper. In recent years, increasing attention has been paid to periodic piecewise systems [25–27], which can be seen as a special kind of switched systems consisting of several time-invariant subsystems [28]. However, to our best knowledge, few results have been reported on the periodic piecewise positive systems due to the difficulties in characterizing the equivalent stability condition and incorporating the positivity constraint in stabilization. Therefore, in this paper, we are concerned with the stability and stabilization of the periodic piecewise positive systems.

In order to analyze the stability condition and stabilization of periodic piecewise positive systems, we should first review the previous results for periodic piecewise systems. For periodic piecewise systems, the existing results can be seen as the extension of the results for switched systems under dwell-time constraint [29, 30]. Since the switching order and the interval of each subsystem are fixed, the applied Lyapunov function changes from a subsystem-based one to a time-based one and the number of LMIs reduces significantly. Furthermore, due to the periodic property, for periodic piecewise systems with time delay, the initial states of the systems can be determined and the control synthesis can be achieved in forms of LMIs. Although extensive research efforts have been focused on stability condition and control synthesis of periodic piecewise systems, the conditions of the stability and stabilization are still subject to some defects, which are listed as follows:

- *There are some drawbacks in obtaining linear conditions for stability*. For the stability condition of periodic piecewise systems, a necessary and sufficient condition can be characterized through the spectral radius of the state transition matrix [26]. However, such a condition is nonlinear in the system matrix parameters. Hence, it is hard to be applied to obtaining conditions for stabilization and characterization of the input–output gain that are linear in the system matrix parameters. To overcome these difficulties, Zhu proposed a novel event-triggered feedback controller for nonlinear systems in Zhu et al. [31]; the authors in Li et al. [26] applied a discontinuous Lyapunov function to obtain a sufficient stability condition in terms of LMIs. In subsequent research [32], even though the authors proposed different kinds of Lyapunov functions to decrease the conservativeness of the stability conditions characterized by the system matrix, the necessity of the condition cannot be guaranteed.
- *It is hard to strike a balance between the complexity of the stabilization algorithm and the conservativeness of the stability condition*. When using a linear time-varying Lyapunov function to characterize the stability condition, the applied Lyapunov function can be continuous or discontinuous. For the discontinuous one, the stability condition is less conservative. However, the number of unknown parameters to be designed is large and coupling between those parameters exists. When fixing some unknown parameters of discontinuous Lyapunov functions or applying continuous Lyapunov functions to turn the stabilization problem into an LMI problem, the conservativeness of the stability conditions increases. Furthermore, for nonlinear Lyapunov function like the one with the matrix polynomial approach [32], a similar dilemma exists.

The above difficulties also exist in both periodic piecewise positive systems and positive switched systems under dwell-time constraint. In addition, as the positivity of the state should be guaranteed, it will be of ever-increasing difficulty to design controllers for the systems. Recently, some research on stability analysis and stabilization of linear continuous switched positive systems under dwell-time constraint can be found in earlier studies [17, 24, 33, 34]. By analyzing the stability via copositive or diagonal Lyapunov function, some sufficient stability conditions are provided. In the above-mentioned works, the positivity of the state is only applied to decreasing the number of unknown parameters in the condition, and the conservativeness of the condition cannot be reduced when the system is a positive system. In Xiang et al. [24], even though the stability condition becomes less conservative by dividing a copositive Lyapunov function into a number of pieces over a subsystem, the condition is still a sufficient stability condition and is difficult to be applied to stabilize the systems. Motivated by the challenging difficulties mentioned above, we endeavor to present new results of the stability and stabilization of the periodic piecewise positive systems.

In this paper, a time segmentation approach and a corresponding time-scheduled copositive Lyapunov function are proposed. Based on the Lyapunov function, an equivalent asymptotic stability condition is derived. Furthermore, based on the established equivalent asymptotic stability condition, the stabilization problem is solved by an iterative algorithm. The main contributions of this paper are given as follows:

1) **Stability:** We construct a novel interpolation function of the time-scheduled copositive Lyapunov function to analyze the stability. With the knowledge of the state transition matrix, we do further research to extend the result in Zheng and Wang [35]. The time segmentation approach and the interpolation function eliminate the conservativeness in the stability condition.
2) **Spectral radius characterization:** We show that the spectral radius of the state transition matrix of the periodic piecewise positive systems can be estimated by linear inequalities. With the increase of the number of inequalities, the estimated spectral radius decreases and finally converges to the true spectral radius. Instead of using the matrix polynomial approach in Li et al. [32], we give a new perspective to accurately estimate the convergent speed of the state by taking the advantage of the positivity property.
3) **Stabilization:** Our copositive Lyapunov function is continuous in each period. Compared with the discontinuous function in Zhu et al. [36], the number of designed parameters decreases, and the complexity of the control synthesis algorithm is reduced. Furthermore, by minimizing the spectral radius of the state transition matrix, a local minimum could be obtained by the proposed algorithm.

The rest of this paper is organized as follows. The definitions of positivity and asymptotic stability of a periodic piecewise positive system and some useful preliminaries are given in Section 2. The stability, spectral radius characterization, and stabilization issues of the periodic piecewise positive systems based on a time-scheduled copositive Lyapunov function are investigated in Section 3. Examples to illustrate the effectiveness of the obtained results are presented in Section 4, and Section 5 concludes the paper.

Notation: $\mathbb{R}^n$ denotes the set of all n-dimensional real vectors, $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices. A^{T} denotes the transpose of matrix A. T^- denotes the left-hand limit of T. $v_{[j]}$ denotes the j-th element of vector v. $A_{[ij]}$ denotes the i,j-th element of matrix A. $\mathbf{diag}(a_1,a_2,\ldots,a_n)$ denotes a diagonal matrix with elements $a_1,a_2,\ldots,a_n$. $\varpi_i(A)$ denotes the i-th eigenvalue of matrix A. $\rho(A)=\max_{i=1,2,\ldots,n}\{|\varpi_i(A)|\}$ denotes the spectral radius of matrix $A\in\mathbb{R}^{n\times n}$. $\prod_{j=j_1}^{j_n} M_j = M_{j_n}M_{j_{n-1}}\ldots M_{j_1}$ denotes the product of n matrices $M_{j_1},M_{j_2},\ldots,M_{j_n}$. 1_n denotes the n-dimensional column vector with each entry equals to 1. I_n denotes the $n\times n$-dimensional identity matrix. $\mathbb{N}=\{0,1,2,\ldots\}$, and $\mathbb{N}_+=\{1,2,\ldots\}$. $\mathbb{R}_+^n$ ($\mathbb{R}_{0,+}^n$) denotes the set of all n-dimensional real vectors whose entries are positive (nonnegative), $\mathbb{R}_+^{m\times n}$ ($\mathbb{R}_{0,+}^{m\times n}$) denotes the set of all $m\times n$ real matrices whose entries are positive (nonnegative). $\mathbb{M}^{n\times n}$ denotes the set of all $n\times n$ Metzler matrices whose off-diagonal entries are nonnegative. $v\succ(\geq)0$ means v is a positive (nonnegative) vector and satisfies $v\in\mathbb{R}_+^n\left(\mathbb{R}_{0,+}^n\right)$. $A\succ(\geq)0$ means A is a positive (nonnegative) matrix and satisfies $A\in\mathbb{R}_+^{m\times n}\left(\mathbb{R}_{0,+}^{m\times n}\right)$. For two vectors v_1 and v_2, $v_1\succ(\geq)v_2$ means v_1-v_2 is a positive (nonnegative) vector. For two matrices A and B, $A\succ(\geq)B$ means $A-B$ is a positive (nonnegative) matrix. For a matrix $A\in\mathbb{R}^{n\times m}$, $\mathcal{L}_R(A)=\min_{i=1,2,\ldots,n}\left\{(|A|1_m)_{[i]}\right\}$ $\left(\mathcal{L}_C(A)=\min_{i=1,2,\ldots,m}\left\{\left(1_n^{\mathrm{T}}|A|\right)\right\}\right)$, where $|A|=\left[|a_{[ij]}|\right]$ and $\mathcal{L}_R(A)=\mathcal{L}_C\left(A^{\mathrm{T}}\right)$. For a vector $v\in\mathbb{R}^n$, $\|v\|=\max_{i=1,2,\ldots,n}\left\{|v_{[i]}|\right\}$. For a matrix $A\in\mathbb{R}^{n\times m}$, $\|A\|=\sup_{\|v\|=1}\|Av\|=\||A|\times 1_m\|$. For a vector $v\in\mathbb{R}^n$, $\|v\|_1=\sum_{i=1}^n|v_{[i]}|$. For a matrix $A\in\mathbb{R}^{n\times m}$, $\|A\|_1=\sup_{\|v\|_1=1}\|Av\|_1=\left\|A^{\mathrm{T}}\right\|$.

2 | PROBLEM FORMULATION AND PRELIMINARIES

Consider a periodic piecewise system given as follows:

$$\dot{x}(t)=A(t)x(t)+B(t)u(t), \tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$ and $u(t)\in\mathbb{R}^{n_u}$ are the state vector and control input, respectively. $A(t)=A(t+T_p)$ and $B(t)=B(t+T_p)$ for all $t\geq 0$, and $T_p>0$ is the fundamental period. Furthermore, the time-varying matrices $A(t)$, $B(t)$ satisfy $A(t)=A_{\sigma(i)}$ and $B(t)=B_{\sigma(i)}$, when $t\in\left[t_{i-1,\sigma(i)-1},t_{i,\sigma(i)}\right)$ for any $i\in\{1,2,\ldots,m\}$, where $(\sigma(1),\sigma(2),\ldots,\sigma(m))$ is a cyclic permutation of $(1,2,\ldots,m)$ and $t_{0,\sigma(1)-1}=0$ and $t_{m,\sigma(m)}=T_p$. We also define the time interval $T_{\sigma(i)}=t_{i,\sigma(i)}-t_{i-1,\sigma(i)-1}$. According to Zhu et al. [36], when $u(t)=0$, some basic definitions and lemmas of system (1) on positivity and stability are recalled.

Definition 1 Positivity. A periodic piecewise system (1) is said to be positive if for any initial state $x(0)\geq 0$ and any cyclic permutation of $(\sigma(1), \sigma(2), \ldots, \sigma(m))$, its state $x(t)$ is in the nonnegative orthant for all $t \geq 0$.

Definition 2 Stability. A periodic piecewise system (1) is said to be asymptotically stable if system (1) is Lyapunov stable and for any nonnegative initial state, the state trajectory $x(t)$ asymptotically converges to zero.

Lemma 1 Positivity and stability conditions [36]. Consider a periodic piecewise positive system (1) with $u(t)=0$. The positivity and stability conditions are given below:

(i) System (1) is positive if and only if A_i is Metzler for all $i \in \{1, 2, \ldots, m\}$;
(ii) System (1) is asymptotically stable if and only if $\prod_{i=1}^{m} e^{A_i T_i}$ is a Schur matrix.

Furthermore, some properties of general matrices, nonnegative matrices, and Metzler matrices which will be used in the following are recalled.

Lemma 2 [37]. For a nonnegative matrix $Q \in \mathbb{R}_{0,+}^{n \times n}$, some properties are given as follows:

(i) Q is a Schur matrix if and only if there exists a vector $p \in \mathbb{R}_{+}^{n}$ such that $Qp \prec p$.
(ii) For a scalar $\gamma \in \mathbb{R}$, Q satisfies $\rho(Q) < \gamma$ if and only if there exists a vector $p \in \mathbb{R}_{+}^{n}$ such that $Qp \prec \gamma p$.

Lemma 3. Given a Metzler matrix $Q \in \mathbb{M}^{n \times n}$, when $\rho(Q) < 1$, $(I_n - Q)^{-1}$ exists and $(I_n - Q)^{-1}$ is a nonnegative matrix.

Lemma 3 can be directly derived from the property of Metzler Huwitz matrix in Berman and Plemmons [37]. Therefore, the proof is omitted here.

Lemma 4 [38]. Given a matrix $Q \in \mathbb{R}^{n \times n}$ and a scalar $M \in \mathbb{N}_{+}$, when $M \rightarrow$, $\left(I_n - \frac{1}{M} Q\right)^{-M} \rightarrow e^{Q}$.

In addition, some properties of -norm and function $\mathcal{L}_R(\cdot)$ $(\mathcal{L}_C(\cdot))$ are given as follows.

Lemma 5 [39]. For a matrix $Q \in \mathbb{R}^{n \times n}$ satisfying $\|Q\| < 1$, the inequality $\left(1 + \|Q\|\right)^{-1} \leq \left\|(I_n - Q)^{-1}\right\|_{\}\} \leq \left(1 - \|Q\|\right)^{-1}}$ holds.

Lemma 6. For two nonnegative matrices $Q \in \mathbb{R}_{0,+}^{n \times l}$ and $R \in \mathbb{R}_{0,+}^{l \times m}$, the following statements hold:

(i) $\mathcal{L}_R(QR) \geq \mathcal{L}_R(Q)\mathcal{L}_R(R)$;
(ii) $\mathcal{L}_C(QR) \geq \mathcal{L}_C(Q)\mathcal{L}_C(R)$.

Proof. Statement (i) is proved in the following:

$$
\begin{aligned}
\mathcal{L}_R(QR) &= \min_{i=1,2\ldots,n} \sum_{j=1}^{m} \sum_{k=1}^{l} q_{[ik]} r_{[kj]} \\
&= \min_{i=1,2,\ldots,n} \sum_{k=1}^{l} \sum_{j=1}^{m} q_{[ik]} r_{[kj]} \\
&\geq \left(\min_{i=1,2\ldots,n} \sum_{k=1}^{l} q_{[ik]} \right) \mathcal{L}_R(R) = \mathcal{L}_R(Q)\mathcal{L}_R(R).
\end{aligned}
\tag{2}
$$

According to inequality (2), we have

$$
\mathcal{L}_C(QR) = \mathcal{L}_R(R^{\mathrm{T}} Q^{\mathrm{T}}) \geq \mathcal{L}_R(R^{\mathrm{T}})\mathcal{L}_R(Q^{\mathrm{T}}) = \mathcal{L}_C(Q)\mathcal{L}_C(R)
$$

holds, and statement (ii) is proved.

Lemma 7. For a Metzler matrix $Q \in \mathbb{M}^{n \times n}$, when $\rho(Q) < 1$, the following statements hold:

(i) $\mathcal{L}_R\left[(I_n - Q)^{-1}\right] \geq \left(1 + \|Q\|\right)^{-1}$;
(ii) $\mathcal{L}_C\left[(I_n - Q)^{-1}\right] \geq \left(1 + \|Q\|_1\right)^{-1}$.

Proof. Assume $(I_n - Q)^{-1} 1_n = v_1$ and $\left(1 + \|Q\|\right)^{-1} 1_n = v_2$. According to Lemma 3, $v_1 \succ 0$ and $v_2 \succ 0$, when $\rho(Q) < 1$. Then the following two equations hold:

$$
(I_n - Q)v_1 = 1_n, \tag{3}
$$

$$
\left(1 + \|Q\|\right)v_2 = 1_n. \tag{4}
$$

Subtracting (3) from (4), we have $v_1 - Qv_1 - v_2 - \|Q\|v_2 = 0$, and hence $v_1 - v_2 - Qv_1 + Qv_2 - Qv_2 - \|Q\|v_2 = 0$, which gives

$$
(I_n - Q)(v_1 - v_2) = Qv_2 + \|Q\|v_2. \tag{5}
$$

Since $-Q1_n \preccurlyeq \|Q\|1_n$, Equation (5) gives $v_1 - v_2 = (I_n - Q)^{-1}\left(Q + \|Q\|I_n\right)1_n\left(1 + \|Q\|\right)^{-1} \geq 0$. It implies $\mathcal{L}_R\left[(I_n - Q)^{-1}\right] \geq \left(1 + \|Q\|\right)^{-1}$, which proves statement (i). The proof of statement (ii) is similar to statement (i), thus omitted here.

3 | MAIN RESULTS

3.1 | Stability analysis

Based on the transition matrix of system (1) and the properties of nonnegative matrices and Metzler matrices, an equivalent stability condition of system (1) in terms of state transition matrices is first discussed in this subsection. Theorem 1 below gives several equivalent stability conditions for system (1).

Theorem 1 Stability characterization via state transition matrices. *Consider periodic piecewise positive system (1) with $u(t)=0$, the following statements are equivalent:*

(i) *System (1) is asymptotically stable;*
(ii) *Matrix $\prod_{i=1}^{m} e^{A_iT_i}$ is a Schur matrix;*
(iii) *There exists a vector $p \in \mathbb{R}_+^{n_x}$ such that $\left(\prod_{i=1}^{m} e^{A_iT_i}\right)p \prec p$;*
(iv) *For any set of vectors $v_i \in \mathbb{R}_+^{n_x}$, there exist a scalar $k>0$ and a set of vector $p_i' \in \mathbb{R}_+^{n}$ such that*

$$e^{A_iT_i}p_i' + kv_i = p_{i+1}',\ i=1,2,\dots,m, \tag{6a}$$

$$p_{m+1}' \prec p_1'. \tag{6b}$$

Remark 1. Combining Lemma 1 and Lemma 2, one can find that conditions (i), (ii), and (iii) are equivalent. The equivalence of condition (iv) could be seen as an alternative way to revise the sufficient condition of Theorem 2.1 in Bougatef et al. [40] to a necessary and sufficient one, which has also been addressed in Remark 2.5 of Ait Rami and Napp [12]. By introducing a set of strictly positive vectors v_i, one can always guarantee the strictly positivity of the set of vectors p_i'.

One can see that the asymptotic stability conditions always contain matrix $e^{A_iT_i}$ in Theorem 1. It is hard to use these conditions to design a feedback controller of system (1) directly. By applying a time segmentation approach to each subsystem and constructing a time-scheduled copositive Lyapunov function, the asymptotic stability condition of system (1) can be solved via linear inequalities, and a sufficient condition is given in Proposition 1.

Proposition 1. Given a scalar $M \in \mathbb{N}_+$, periodic piecewise positive system (1) with $u(t)=0$ is asymptotically stable if there exist a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$, $i=1,2\dots,m$, $j=1,2,\dots,M$, such that

$$A_i^{\mathrm{T}}p_{i,j-1} - \frac{M}{T_i}p_{i,j-1} + \frac{M}{T_i}p_{i,j} \prec 0, \tag{7a}$$

$$A_i^{\mathrm{T}}p_{i,j} - \frac{M}{T_i}p_{i,j-1} + \frac{M}{T_i}p_{i,j} \prec 0, \tag{7b}$$

$$p_{i,M} = p_{i+1,0},\ i=1,2\dots,m-1, \tag{7c}$$

$$p_{m,M} \succ p_{1,0}. \tag{7d}$$

Proposition 1 can be seen as an extension of Theorem 2 in Zhu et al. [36] by applying time segmentation to the time interval of each subsystem into M parts. It can also be found as one computational approach in Briat [41]. Thus, the proof of Proposition 1 is omitted here. Based on the similar idea of applying the periodic time-scheduled copositive Lyapunov functions to analyze stability, one can also use linear inequalities to characterize the stability condition and input–output gains of continuous time positive periodic systems in previous studies [13, 42].

In an earlier study [24], Xiang et al. extended their previous results for general switched systems in Xiang et al. [43] to switched positive systems. It is concluded that, for a Hurwitz Metzler matrix $A \in \mathbb{M}^{n\times n}$, there exist a sufficiently large M and a set of vectors $p_j \in \mathbb{R}_+^{n}$ such that

$$A^{\mathrm{T}}p_{j-1} + \frac{M}{T}\left(p_j - p_{j-1}\right) \prec 0,\ j=1,2,\dots,M, \tag{8}$$

$$A^{\mathrm{T}}p_{j} + \frac{M}{T}\left(p_j - p_{j-1}\right) \prec 0,\ j=1,2,\dots,M, \tag{9}$$

hold, with $p_j = e^{A^{\mathrm{T}}(T-t_j)}p_M + (T-t_j)\phi$, where $t_j = jT/M$, $\phi \succ 0$ and $A^{\mathrm{T}}\phi \prec 0$. In the i-th subsystem, let $A_i \mapsto A$ and $p_{i,j} \mapsto p_j$; one can find that inequalities (7a) and (7b) are the same as (8) and (9). For periodic piecewise positive systems, the subsystem may be unstable and matrix A_i needs not be Hurwitz. By relaxing the conditions in Xiang et al. [24], Lemma 8 will show that, for any Metzler matrix A, one can find a sufficiently large M such that conditions (8) and (9) hold and p_0 and p_M satisfy $p_0 = e^{A^{\mathrm{T}}T}p_M + T\phi$, where $\phi \succ 0$.

Lemma 8. Given a Metzler matrix $A \in \mathbb{M}^{n\times n}$ and a scalar $T>0$. For any vector $p \in \mathbb{R}_+^n$ and scalar $k>0$, there exist a set of vectors $p_j \in \mathbb{R}_+^n$, a vector $v \in \mathbb{R}_+^n$ and sufficiently large scalar $M \in \mathbb{N}_+$ such that the following conditions hold:

$$A^{\mathrm{T}}p_{j-1} - \frac{M}{T}p_{j-1} + \frac{M}{T}p_j \prec 0, j=1,2,\ldots,M, \tag{10a}$$

$$A^{\mathrm{T}}p_{j} - \frac{M}{T}p_{j-1} + \frac{M}{T}p_j \prec 0, j=1,2,\ldots,M, \tag{10b}$$

$$p_0 = e^{A^{\mathrm{T}}T}p + kv, \tag{10c}$$

$$p_M = p. \tag{10d}$$

Proof. First, a set of vectors p_j is defined as follows:

$$p_j = p_{j-1} + \frac{T}{M}\phi_j, j=1,2,\ldots,M, \tag{11}$$

$$p_M = p, \tag{12}$$

where $\phi_j = (-\tilde{A}_M)^{M+1-j}(kq - A^{\mathrm{T}}p)$, $\tilde{A}_M = (\frac{T}{M}A^{\mathrm{T}} - I_n)^{-1}$ and $q \prec 0$ is an arbitrary vector. We let M satisfy $M > T\rho(A)$. According to Lemma 3, $\tilde{A}_M$ exists and satisfies $\tilde{A}_M \preccurlyeq 0$. Since $-\tilde{A}_M$ is a full rank nonnegative matrix, $\left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q \prec 0$, when $q \prec 0$. Define $p_0 = e^{A^{\mathrm{T}}T}p + kv$. By substituting p_0 into (11) and (12), we have

$$\begin{aligned} v &= \frac{1}{k}\left(p_0 - e^{A^{\mathrm{T}}T}p\right) \\ &= \frac{1}{k}\left(p - \frac{T}{M}\sum_{l=1}^{M}\phi_l - e^{A^{\mathrm{T}}T}p\right) \\ &= \frac{1}{k}\left\{p + \sum_{l=1}^{M}(-\tilde{A}_M)^{l-1}[(-\tilde{A}_M)p - p]\right\} \\ &\quad - \left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q - \frac{1}{k}e^{A^{\mathrm{T}}T}p \\ &= \frac{1}{k}\left[(-\tilde{A}_M)^M p - e^{A^{\mathrm{T}}T}p\right] - \left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q. \end{aligned} \tag{13}$$

Based on the property of matrix norm (Page 290 of [39]), vector $-\left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q$ in (13) satisfies the following inequality:

$$\begin{aligned} -\left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q &\preccurlyeq \left\|-\left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q\right\|1_n \\ &\preccurlyeq \frac{T}{M}\sum_{l=1}^{M}\|(-\tilde{A}_M)\|^l\|-q\|1_n \\ &= \frac{T}{M}\sum_{l=1}^{M}\left\|\left(I_n - \frac{T}{M}A^{\mathrm{T}}\right)^{-1}\right\|^l \\ &\quad \times\|-q\|1_n. \end{aligned} \tag{14}$$

According to Lemma 5, when $M > T\|A^{\mathrm{T}}\| \geq T\rho(A)$, the right-hand side of inequality (14) gives

$$\begin{aligned} &\frac{T}{M}\sum_{l=1}^{M}\left\|\left(I_n - \frac{T}{M}A^{\mathrm{T}}\right)^{-1}\right\|^l\|-q\|1_n \\ &\preccurlyeq \frac{T}{M}\sum_{l=1}^{M}\left(\frac{1}{1-\frac{T}{M}\|A^{\mathrm{T}}\|}\right)^l\|-q\|1_n \\ &\preccurlyeq \frac{T}{M}\sum_{l=1}^{M}\left(\frac{1}{1-\frac{T}{M}\|A^{\mathrm{T}}\|}\right)^M\|-q\|1_n \\ &= T\left(\frac{1}{1-\frac{T}{M}\|A^{\mathrm{T}}\|}\right)^M\|-q\|1_n. \end{aligned} \tag{15}$$

Function $\left(1-\frac{T}{M}\|A^{\mathrm{T}}\|\right)^{-M}$ monotonically decreases for $M > T\|A^{\mathrm{T}}\|$ as M increases. Choose M^* such that $M^* \in \mathbb{N}_+$ and $M^* > T\|A^{\mathrm{T}}\|$. When $M \geq M^*$, (15) satisfies the following inequality:

$$\frac{T}{M}\sum_{l=1}^{M}\left(\frac{1}{1-\frac{T}{M}\|A^{\mathrm{T}}\|}\right)^M\|-q\|1_n \preccurlyeq T\bar{\delta}_{M^*}\|-q\|1_n, \tag{16}$$

where $\bar{\delta}_{M^*} = \left(1-\frac{T}{M^*}\|A^{\mathrm{T}}\|\right)^{-M^*}$. Inequality (16) gives an upper bound of vector $-\left[\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l\right]q$; a lower bound of the vector is given in what follows. According to the definition of $\mathcal{L}_R(\cdot)$ and Lemma 6, one has

$$\begin{aligned} \frac{-T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^l q &\geq \frac{T}{M}\sum_{l=1}^{M}\mathcal{L}_R\left[(-\tilde{A}_M)^l\right]\mathcal{L}_R(-q)1_n \\ &\geq \frac{T}{M}\sum_{l=1}^{M}[\mathcal{L}_R(-\tilde{A}_M)]^l\mathcal{L}_R(-q)1_n \\ &= \frac{T}{M}\sum_{l=1}^{M}\left\{\mathcal{L}_R\left[\left(I_n - \frac{T}{M}A^{\mathrm{T}}\right)^{-1}\right]\right\}^l \\ &\quad \mathcal{L}_R(-q)1_n. \end{aligned} \tag{17}$$

According to Lemma 7, the right-hand side of (17) gives

$$
\begin{aligned}
&\frac{T}{M}\sum_{l=1}^{M}\left\{\mathcal{L}_R\left[\left(I_n-\frac{T}{M}A^{\mathrm{T}}\right)^{-1}\right]\right\}^{l}\mathcal{L}_R(-q)1_n\\
&\geq\frac{T}{M}\sum_{l=1}^{M}\left[\frac{1}{1+\frac{T}{M}\left\|A^{\mathrm{T}}\right\|}\right]^{l}\mathcal{L}_R(-q)1_n\\
&\geq T\left[\frac{1}{1+\frac{T}{M}\left\|A^{\mathrm{T}}\right\|}\right]^{M}\mathcal{L}_R(-q)1_n.
\end{aligned}
\tag{18}
$$

Function $\left(1+\frac{T}{M}\left\|A^{\mathrm{T}}\right\|\right)^{-M}$ monotonically decreases for $M\geq M^*$ as M increases. The right-hand side of inequality (18) gives

$$
\begin{aligned}
&T\left[\frac{1}{1+\frac{T}{M}\left\|A^{\mathrm{T}}\right\|}\right]^{M}\mathcal{L}_R(-q)1_n\\
&\geq T\lim_{M\to}\left\{\left[\frac{1}{1+\frac{T}{M}\left\|A^{\mathrm{T}}\right\|}\right]^{M}\right\}\mathcal{L}_R(-q)1_n\\
&=T\underline{\delta}\mathcal{L}_R(-q)1_n,
\end{aligned}
$$

where $\underline{\delta}=e^{-T\left\|A^{\mathrm{T}}\right\|}$. According to (16) and (18), the sum of $-\frac{T}{M}(-\tilde{A}_M)^{l}q$ is bounded and satisfies

$$
0\prec T\underline{\delta}\mathcal{L}_R(-q)1_n\preccurlyeq-\frac{T}{M}\sum_{l=1}^{M}(-\tilde{A}_M)^{l}q\preccurlyeq T\bar{\delta}_{M^*}\left\|-q\right\|1_n.
$$

According Lemma 4, for any Metzler matrix A and a scalar k, there exists a scalar $M^{**}\geq M^*$ such that

$$
-\frac{1}{k}\left\|(-\tilde{A}_M)^{M}-e^{A^{\mathrm{T}}T}\right\|p+T\underline{\delta}\mathcal{L}_R(-q)1_n\succ 0
$$

holds for all $M\geq M^{**}$. When $M\geq M^{**}$, v is bounded and satisfies

$$
0\prec v\preccurlyeq T\bar{\delta}_{M^*}\left\|-q\right\|1_n+T\underline{\delta}\mathcal{L}_R(-q)1_n.
$$

According to (10c) and (10d), p_0 and p_M are positive vectors. Then the positivity of vector p_j, where $j\in\{1,2,\ldots,M-1\}$, is proved in the following. According to equalities (11) and (12), p_j can be written as follows:

$$
p_j=(-\tilde{A}_M)^{M-j}p-\left[\frac{T}{M}\sum_{l=1}^{M-j}(-\tilde{A}_M)^{l}\right]kq,
$$

where $j\in\{1,2,\ldots,M-1\}$. Since $-\tilde{A}_M$ is a full rank nonnegative matrix, $p_j\in\mathbb{R}^n_+$ for all $j\in\{0,1,\ldots,M\}$. By substituting (11) and (12) into the left-hand side of inequality (10a), we have

$$
\begin{aligned}
A^{\mathrm{T}}p_{M-1}-\frac{M(p_{M-1}-p_M)}{T}&=A^{\mathrm{T}}p+\left(I_n-\frac{T}{M}A^{\mathrm{T}}\right)\phi_M\\
&=A^{\mathrm{T}}p-\tilde{A}_M^{-1}\tilde{A}_M(A^{\mathrm{T}}p-kq)\\
&=kq\prec 0.
\end{aligned}
\tag{19}
$$

Furthermore, the relation between $A^{\mathrm{T}}p_{j-1}-\frac{M}{T}p_{j-1}+\frac{M}{T}p_j$ and $A^{\mathrm{T}}p_{j-2}-\frac{M}{T}p_{j-2}+\frac{M}{T}p_{j-1}$, for $j\in\{2,3,\ldots,M\}$ are as follows:

$$
\begin{aligned}
&A^{\mathrm{T}}p_{j-2}-\frac{M\left(p_{j-2}-p_{j-1}\right)}{T}-\left[A^{\mathrm{T}}p_{j-1}-\frac{M\left(p_{j-1}-p_j\right)}{T}\right]\\
&=\left(I_n-\frac{T}{M}A^{\mathrm{T}}\right)\phi_{j-1}-\phi_j\\
&=\left(I_n-\frac{T}{M}A^{\mathrm{T}}\right)(-\tilde{A}_M)^{M+2-j}(kq-A^{\mathrm{T}}p)\\
&\quad-(-\tilde{A}_M)^{M+1-j}(kq-A^{\mathrm{T}}p)\\
&=0.
\end{aligned}
\tag{20}
$$

Combining (19) and (20),

$$
A^{\mathrm{T}}p_{j-1}-\frac{M}{T}p_{j-1}+\frac{M}{T}p_j=kq\prec 0 \tag{21}
$$

holds for all $j\in\{1,2,\ldots,M\}$. By substituting (11) and (21) into the left-hand side of (10b), equation

$$
\begin{aligned}
A^{\mathrm{T}}p_j-\frac{M}{T}p_{j-1}+\frac{M}{T}p_j&=A^{\mathrm{T}}p_j-A^{\mathrm{T}}p_{j-1}+kq\\
&=\frac{T}{M}A^{\mathrm{T}}\phi_j+kq
\end{aligned}
$$

holds for all $j\in\{1,2,\ldots,M\}$. For a given $M\geq M^{**}$, $A^{\mathrm{T}}\phi_j$ satisfies

$$
A^{\mathrm{T}}\phi_j\geq-\bar{\delta}_{M^*}\left\|kA^{\mathrm{T}}q-\left(A^{\mathrm{T}}\right)^2p\right\|1_n, \tag{22}
$$

$$
A^{\mathrm{T}}\phi_j\preccurlyeq\bar{\delta}_{M^*}\left\|kA^{\mathrm{T}}q-\left(A^{\mathrm{T}}\right)^2p\right\|1_n, \tag{23}
$$

where $j\in\{1,2,\ldots,M\}$. Inequalities (22) and (23) show that function $A^{\mathrm{T}}\phi_j$ is bounded and cannot go to infinity when M goes to infinity. In other words, for any Metzler matrix A and scalar $k>0$, there exists a scalar $M^{***}\geq M^{**}$ such that

$$
\frac{T}{M}\bar{\delta}_{M^*}\left\|kA^{\mathrm{T}}q-\left(A^{\mathrm{T}}\right)^2p\right\|1_n+kq\prec 0
$$

holds for all $M \geq M^{***}$. Therefore, for a given $q \prec 0$, when $M \geq M^{***}$, inequality $\frac{T}{M}A^{\mathrm{T}}\phi_j + kq \prec 0$ holds for all $j \in \{1,2,\ldots,M\}$. When $M \geq M^{***}$, there exist a set of vectors $p_j \in \mathbb{R}_+^n$ and a vector $v \in \mathbb{R}_+^n$ such that condition (10) holds, which proves Lemma 8.

Remark 2. According to (13), the value of v is affected by k, $\tilde{A}_M$, p and M. For any k, one can always let $\frac{1}{k}\left[\left(-\tilde{A}_M\right)^M - e^{A^{\mathrm{T}}T}\right]p \prec \Lambda$, where Λ is a given positive vector, by increasing M. Therefore, for any k, the value of v is less than a certain positive vector. In other words, when k goes to 0, v will not go to infinity.

According to Lemma 8, we give the relation between a set of positive vectors p_i and a Metzler matrix A, when the number of the vectors is sufficiently large. By substituting the relation into Theorem 1, the necessity of condition (7) in Proposition 1 is proved when M is sufficiently large, and Theorem 2 is given.

Theorem 2 Stability characterization via system matrices. *Given a periodic piecewise positive system (1) with $u(t)=0$. The system is asymptotically stable if and only if there exist a sufficiently large M and a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$ satisfying condition (7), for $i = 1,2,\ldots,m$, and $j=1,2,\ldots,M$.*

Proof. The sufficiency of condition (7) has been proved in Proposition 1. The necessity of Theorem 2 is proved by contradiction. We start by assuming that the periodic piecewise positive system (1) with $u(t)=0$ is asymptotically stable, and there do not exist a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$ such that condition (7) holds for any $M \in \mathbb{N}_+$. According to Lemma 8, there exist a sufficiently large scalar $M \in \mathbb{N}_+$ and a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$ such that (7a)–(7c) hold, and the vectors $p_{i,0}$ and $p_{i,M}$ satisfy $p_{m,M}=p$, $p_{i,0}=e^{A_i^{\mathrm{T}}T_i}p_{i,M}+kv_i$, $i=1,2,\ldots,m$, for any vector $p \in \mathbb{R}_+^{n_x}$ and any scalar $k>0$, where v_i satisfies that $0 \prec v_i \preceq \bar{v}_i$ and $\bar{v}_i$ is independent of k. Based on the assumption, there do not exist a scalar $k>0$ and a set of vectors $p'_i \in \mathbb{R}_+^{n_x}$ such that

$$e^{A_{m+1-i}^{\mathrm{T}}T_{m+1-i}}p'_i + kv_{m+1-i} = p'_{i+1},\ i=1,2,\ldots,m, \tag{24}$$

$$p'_{m+1} \prec p'_1, \tag{25}$$

where $0 \prec v_i \preceq \bar{v}_i$, $p'_1 = p_{m,M} = p$, and $p'_i = p_{m+2-i,0}$, for $i=2,\ldots,m+1$. When conditions (24) and (25) do not hold, Theorem 1 indicates that $\rho\left(\prod_{i=1}^m e^{A_{m+1-i}^{\mathrm{T}}T_{m+1-i}}\right) \geq 1$ and $\rho\left(\prod_{i=1}^m e^{A_iT_i}\right) \geq 1$. Since system (1) is asymptotically stable, the spectral radius of the state transition matrix $\prod_{i=1}^m e^{A_iT_i}$ is less than 1. It contradicts the assumption, and the necessity of Theorem 2 is proved.

Remark 3. By applying the time segmentation approach and utilizing the positivity of the state transition matrix, a necessary and sufficient stability condition is proposed. Compared with the existing results (e.g., the stability conditions derived by the matrix polynomial [32] and the discontinuous Lyapunov function [36]), our stability condition is less conservative.

3.2 | Spectral radius characterization

Thus far, the asymptotic stability of periodic piecewise positive systems has been investigated. In this subsection, the spectral radius of the state transition matrix, which plays an important role in characterizing the exponential stability and designing iterative stabilization algorithm, is discussed. Based on Theorem 2, two characterizations of the spectral radius of the state transition matrix for the system (1) are given first.

Theorem 3 Spectral radius characterization I. *Given periodic piecewise positive system (1) with $u(t)=0$. The spectral radius of the state transition matrix satisfies $\rho\left(\prod_{i=1}^m e^{A_iT_i}\right) < \gamma$, where $\gamma \in \mathbb{R}_+$, if and only if there exist a sufficiently large $M \in \mathbb{N}_+$ and a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,M$, satisfying*

$$A_i^{\mathrm{T}}p_{i,j-1} - \frac{M}{T_i}p_{i,j-1} + \frac{M}{T_i}p_{i,j} \prec 0, \tag{26a}$$

$$A_i^{\mathrm{T}}p_{i,j} - \frac{M}{T_i}p_{i,j-1} + \frac{M}{T_i}p_{i,j} \prec 0, \tag{26b}$$

$$p_{i,M} = p_{i+1,0},\ i=1,2\ldots,m-1, \tag{26c}$$

$$\gamma p_{m,M} \succ p_{1,0}. \tag{26d}$$

Proof. The proof of the necessity of Theorem 3 is similar to that in the proof of Theorem 2, thus omitted here. For sufficiency, by considering a time-scheduled copositive Lyapunov function,

$$V(t)=x^{\mathrm{T}}(t)p(t), \tag{27}$$

where

$$\begin{aligned} p(t) &= \alpha_{i,j}(t)p_{i,j-1}+\tilde{\alpha}_{i,j}(t)p_{i,j}, \\ \alpha_{i,j}(t) &= \frac{M}{T_i}\left(kT_p+t_{i-1}+\frac{jT_i}{M}-t\right), \\ \tilde{\alpha}_{i,j}(t) &= 1-\alpha_{i,j}(t)=\frac{M}{T_i}\left(t-kT_p-t_{i-1}-\frac{(j-1)T_i}{M}\right), \end{aligned}$$

when $t\in\left[kT_p+t_{i-1}+\frac{j-1}{M}T_i, kT_p+t_{i-1}+\frac{j}{M}T_i\right)$ with $i=1,2,\ldots,m$ and $j=1,2,\ldots,M$. The derivative of the copositive Lyapunov function is

$$\begin{aligned} \dot{V}(t) &= \dot{x}^{\mathrm{T}}(t)p(t)+x^{\mathrm{T}}(t)\dot{p}(t) \\ &= x^{\mathrm{T}}(t)A_i^{\mathrm{T}}p(t)+x^{\mathrm{T}}(t)\left(-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right) \\ &= x^{\mathrm{T}}(t)\left[\alpha_{i,j}(t)\left(A_i^{\mathrm{T}}p_{i,j-1}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right)\right. \\ &\quad \left.+\tilde{\alpha}_{i,j}(t)\left(A_i^{\mathrm{T}}p_{i,j}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right)\right]. \end{aligned} \tag{28}$$

Combining (28) with condition (26), the copositive Lyapunov function (27) satisfies

$$V\left(x\left((k+1)T_p\right)\right)<\gamma V\left(x\left(kT_p\right)\right) \tag{29}$$

for all $x(kT_p)\geq 0$ and $x(kT_p)\neq 0$. According to system (1), the relation between $x\left((k+1)T_p\right)$ and $x\left(kT_p\right)$ is

$$x\left((k+1)T_p\right)=\prod_{i=1}^{m}e^{A_iT_i}x\left(kT_p\right). \tag{30}$$

Combining (29) and (30), inequality

$$\left[\prod_{i=1}^{m}e^{A_iT_i}x\left(kT_p\right)\right]^{\mathrm{T}}p_{1,0}<\gamma x^{\mathrm{T}}(kT_p)p_{1,0}$$

holds for all $x(kT_p)\geq 0$ and $x(kT_p)\neq 0$. Letting $x(kT_p)$ to be a standard basis vector for $\mathbb{R}^{n_x}$ successively yields

$$\left(\prod_{i=1}^{m}e^{A_iT_i}\right)^{\mathrm{T}}p_{1,0}\prec\gamma p_{1,0}.$$

According to Lemma 2, the spectral radius of $\prod_{i=1}^{m}e^{A_iT_i}$ is less than γ. The sufficiency is proved.

Theorem 4 Spectral radius characterization II. *Given periodic piecewise positive system (1) with $u(t)=0$. The spectral radius of the state transition matrix satisfies $\rho\left(\prod_{i=1}^{m}e^{A_iT_i}\right)<e^{-\varepsilon T_p}$, where $\varepsilon\in\mathbb{R}$, if and only if there exist a sufficiently large $M\in\mathbb{N}_+$ and a set of vectors $p_{i,j}\in\mathbb{R}_+^{n_x}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,M$, satisfying*

$$A_i^{\mathrm{T}}p_{i,j-1}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\prec-\varepsilon p_{i,j-1}, \tag{31a}$$

$$A_i^{\mathrm{T}}p_{i,j}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\prec-\varepsilon p_{i,j}, \tag{31b}$$

$$p_{i,M}=p_{i+1,0},\ i=1,2\ldots,m-1, \tag{31c}$$

$$p_{m,M}\succ p_{1,0}. \tag{31d}$$

Proof. Inequality $\rho\left(\prod_{i=1}^{m}e^{A_iT_i}\right)<e^{-\varepsilon T_p}$ is equivalent to $\rho\left(\prod_{i=1}^{m}e^{(A_i+\varepsilon I_{n_x})T_i}\right)<1$. Let $\hat{A}_i=A_i+\varepsilon I_{n_x}$, according to Theorem 2, $\rho\left(\prod_{i=1}^{m}e^{\hat{A}_iT_i}\right)<1$ if and only if there exist a sufficiently large $M\in\mathrm{N}_+$, and a set of vector $p_{i,j}\in\mathbb{R}_+^{n_x}$, $i=1,2\ldots,m$, $j=1,2,\ldots,M$, satisfying

$$\hat{A}_i^{\mathrm{T}}p_{i,j-1}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\prec 0, \tag{32}$$

$$\hat{A}_i^{\mathrm{T}}p_{i,j}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\prec 0, \tag{33}$$

and conditions (31c) and (31d). One can find that (32) and (33) are equivalent to (31a) and (31b); thus, Theorem 4 is proved.

Remark 4. For stability and spectral radius characterization, it indicates that there exists a sufficiently large M such that the corresponding conditions hold. In order to determine whether the given M is sufficiently large, tolerances $\overline{\tau}_1$ and $\overline{\tau}_2$ may be introduced as stopping criteria in the computational algorithm. For a given $\beta \in \{2,3,\ldots\}$, estimated spectral radii for M and βM are defined as $\rho_{E,M}$ and $\rho_{E,\beta M}$, respectively. When the relative error of the estimated spectral radii with M and βM satisfy $\frac{|\rho_{E,M}-\rho_{E,\beta M}|}{\rho_{E,M}} \leq \overline{\tau}_1$ and the absolute error of the estimated spectral radii for M and βM satisfy $|\rho_{E,M}-\rho_{E,\beta M}| \leq \overline{\tau}_2$, one can regard M as a sufficiently large number of segmentation in Theorems 3 and 4. However, as in many numerical algorithms, it is difficult to determine the value of M a priori.

Remark 5 Alternative spectral radius characterization. According to Theorem 3 and Theorem 4, condition (26) is equivalent to condition (31). When introducing scalars γ' and ε' simultaneously, the condition that there exist a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,M$, such that

$$A_i^{\mathrm{T}} p_{i,j-1} - \frac{M}{T_i} p_{i,j-1} + \frac{M}{T_i} p_{i,j} \prec -\varepsilon' p_{i,j-1}, \tag{34a}$$

$$A_i^{\mathrm{T}} p_{i,j} - \frac{M}{T_i} p_{i,j-1} + \frac{M}{T_i} p_{i,j} \prec -\varepsilon' p_{i,j}, \tag{34b}$$

$$p_{i,M} = p_{i+1,0},\ i=1,2\ldots,m-1, \tag{34c}$$

$$\gamma' p_{m,M} \succ p_{1,0} \tag{34d}$$

is still necessary and sufficient condition to characterize the spectral radius of state transition matrix $\left(\rho\left(\prod_{i=1}^m e^{A_i T_i}\right) < \gamma' e^{-\varepsilon' T_p}\right)$, when the scalar $M \in \mathbb{N}_+$ is sufficiently large.

According to Theorem 3 (resp. Theorem 4), when $\gamma = 1$ (resp. $\varepsilon = 0$), conditions in Theorem 3 (resp. Theorem 4) reduce to the asymptotic stability conditions in Theorem 2. When $\gamma < 1$ or $\varepsilon > 0$, the convergence rate can be analyzed and exponential stability can be characterized based on the above two theorems. Before giving the characterization of the convergent rate, the definition of the λ-exponential stability of periodic piecewise positive systems is given.

Definition 3 λ-exponential stability. Periodic piecewise positive system (1) with $u(t)=0$ is said to be λ-exponentially stable that the state of the system satisfies

$$\|x(t)\| \leq \kappa e^{-\lambda t} \|x(0)\|,\ \forall t \geq 0, \tag{35}$$

for some constants $\kappa \geq 1$, $\lambda > 0$.

Based on Definition 3, the relation between the convergent rate λ and the spectral radius of the state transition matrix is discussed.

Theorem 5 λ-exponential stability characterization. *Given periodic piecewise positive system (1) with $u(t)=0$, the following conditions holds:*

(i) *If $\rho\left(\prod_{i=1}^m e^{A_i T_i}\right) < e^{-\lambda T_p}$ or $\prod_{i=1}^m e^{A_i T_i}$ is irreducible and $\rho\left(\prod_{i=1}^m e^{A_i T_i}\right) = e^{-\lambda T_p}$, then the system is λ-exponentially stable;*

(ii) *If the system is λ-exponentially stable, then $\rho\left(\prod_{i=1}^m e^{A_i T_i}\right) \leq e^{-\lambda T_p}$ holds.*

Proof. Since the cyclic permutation of $(\sigma(1),\sigma(2),\ldots,\sigma(m))$ does not affect the spectral radius of matrix $\prod_{i=1}^m e^{A_{\sigma(i)} T_{\sigma(i)}}$, without loss of generality, we assume $\sigma(i)=i$ in the following proofs.*Proof of (i)*: According to Lemma 2 and the Perron–Frobenius theorem, when $\rho\left(\prod_{i=1}^m e^{A_i T_i}\right) < e^{-\lambda T_p}$ or $\prod_{i=1}^m e^{A_i T_i}$ is irreducible and $\rho\left(\prod_{i=1}^m e^{A_i T_i}\right) = e^{-\lambda T_p}$, there exists a vector $p \in \mathbb{R}_+^{n_x}$ such that $\left(\prod_{i=1}^m e^{A_i T_i}\right) p \preccurlyeq e^{-\lambda T_p} p$. For system (1) with initial state $x(0)=p$, one has

$$x(kT_p) = \left(\prod_{i=1}^m e^{A_i T_i}\right)^k p \preccurlyeq e^{-k\lambda T_p} p. \tag{36}$$

Assume $\psi = \max_{t \in [0,T_p]} \|\Phi(t)\|$, where

$$\begin{aligned} \Phi(t) &= e^{A_1 t}, t \in [0,t_1], \\ \Phi(t) &= e^{A_i(t-t_{i-1})} \prod_{l=1}^{i-1} e^{A_l T_l}, t \in [t_{i-1},t_i],\ i=2,3,\ldots,m. \end{aligned}$$

When $t \in \left[kT_p,(k+1)T_p\right]$,

$$\begin{aligned} \|x(t)\| &= \left\|\Phi(t-kT_p)x(kT_p)\right\| \\ &\leq \psi \left\|x(kT_p)\right\| \\ &\leq e^{-\lambda\left(t-kT_p\right)} e^{\lambda T_p} \psi \left\|x(kT_p)\right\|. \end{aligned} \tag{37}$$

Combining inequality (36) with (37), one can obtain

$$\|x(t)\| \leq e^{-\lambda t} e^{\lambda T_p} \psi \|p\|,$$

when $t \in \left[kT_p,(K+1)T_p\right]$. For any non-zero vector v, one can always find a positive scalar $\frac{\|v\|}{\mathcal{L}_R(p)}$ such that $v \preccurlyeq \frac{\|v\|}{\mathcal{L}_R(p)} p$. Therefore, inequality $\|x_1(t)\| \leq \|x_2(t)\|$ holds, where $x_1(t)$

and $x_2(t)$ are the states of system (1) with initial states $x_1(0)=v$ and $x_2(0)=\frac{\|v\|}{\mathcal{L}_R(p)}p$, respectively. In other words, for any nonnegative initial condition $x(0)=v$, $\|x(t)\|$ always satisfies the inequality (35), where $\kappa=\frac{\|p\|}{\mathcal{L}_R(p)}e^{\lambda T_p}\psi$ and system (1) is λ-exponentially stable. This completes the proof.

Proof of (ii): It is proved by contraposition that, when $\rho\left(\prod_{i=1}^{m}e^{A_iT_i}\right)=\gamma>e^{-\lambda T_p}$, the system is not λ-exponentially stable. According to Perron-Frobenius Theorem, we can find a vector $p'\in\mathbb{R}_{0,+}^{n_x}$ satisfying $\left(\prod_{i=1}^{m}e^{A_iT_i}\right)p'=\gamma p'$. Let $x(0)=p'$, $x(kT_p)=\gamma^k p'$. Based on Definition 3, for system (1) to be λ-exponentially stable, there must exist a positive scalar κ such that $\gamma^k\le\kappa e^{-k\lambda T_p}$, which indicates $\ln\kappa\ge k\left(\ln\gamma+\lambda T_p\right)$. Since $\left(\ln\gamma+\lambda T_p\right)>0$, when $t\to$, $k\to$ and $\kappa\to$. Hence, a finite κ does not exist. This completes the proof.

Based on Theorem 3 (resp. Theorem 4) and Theorem 5, linear inequalities can be applied to characterize the convergent rate of the system. When the value of M in Theorem 3 (resp. Theorem 4) goes to infinity, the estimated convergent rate of the system will increase to the greatest one. However, it only indicates that one can find a sufficiently large scalar $M\in\mathbb{N}_+$ to characterize the spectral radius of the state transition matrix and the convergent rate of the system. It does not mean that the infimum of γ in (26d) monotonically decreases with the increase of M. In order to explicitly demonstrate the effect of M on the infimum of γ in (26d), Theorem 6 is given.

Theorem 6 Monotonicity of estimated spectral radius. *Given a periodic piecewise positive system (1) with $u(t)=0$ and scalars $M\in\mathbb{N}_+$, $\gamma\in\mathbb{R}_{0,+}$. When there exist a set of vectors $p_{i,j}\in\mathbb{R}_+^{n_x}$ satisfying condition (26), for any scalar $\beta\in\mathbb{N}_+$, there exist a set of vectors $p_{i,j^*}^*\in\mathbb{R}_+^{n_x}$, $i=1,2,\ldots,m$, $j^*=1,2,\ldots,\beta M$, satisfying*

$$A_i^{\mathrm{T}}p_{i,j^*-1}^*-\frac{\beta M}{T_i}p_{i,j^*-1}^*+\frac{\beta M}{T_i}p_{i,j^*}^*\prec 0, \tag{38a}$$

$$A_i^{\mathrm{T}}p_{i,j^*}^*-\frac{\beta M}{T_i}p_{i,j^*-1}^*+\frac{\beta M}{T_i}p_{i,j^*}^*\prec 0, \tag{38b}$$

$$p_{i,\beta M}^*=p_{i+1,0}^*,\ i=1,2\ldots,m-1, \tag{38c}$$

$$\gamma p_{m,\beta M}^*\succ p_{1,0}^*. \tag{38d}$$

Proof. When a set of vectors $p_{i,j}\in\mathbb{R}_+^{n_x}$ satisfy condition (26), let

$$p_{i,j^*}^*=\frac{\beta-\beta^*}{\beta}p_{i,j-1}+\frac{\beta^*}{\beta}p_{i,j},\ i=1,2,\ldots,m, \tag{39}$$

where $j^*=\beta(j-1)+\beta^*$, $j=1,2,\ldots,M$, and $\beta^*=0,1,\ldots,\beta$. Equation (39) shows that $p_{i,\beta M}^*=p_{i,M}$ and $p_{i,0}^*=p_{i,0}$ for all $i=1,2,\ldots,m$, and conditions (38c) and (38d) hold, obviously. According to (39), we also have

$$\begin{aligned}\frac{\beta M}{T_i}p_{i,j^*}^*-\frac{\beta M}{T_i}p_{i,j^*-1}^* &=\frac{\beta M}{T_i}\left(\frac{1}{\beta}p_{i,j}-\frac{1}{\beta}p_{i,j-1}\right)\\ &=\frac{M}{T_i}p_{i,j}-\frac{M}{T_i}p_{i,j-1},\end{aligned}$$

where $j^*\in\{\beta(j-1)+1,\beta(j-1)+2,\ldots,\beta j\}$, $j=1,2,\ldots,M$, and $i=1,2,\ldots,m$. Furthermore, by substituting (39) into (38a) and (38b), respectively, one can derive

$$\begin{aligned}&A_i^{\mathrm{T}}p_{i,j^*-1}^*-\frac{\beta M}{T_i}p_{i,j^*-1}^*+\frac{\beta M}{T_i}p_{i,j^*}^*\\ &=\frac{\beta+1-\beta^*}{\beta}\left(A_i^{\mathrm{T}}p_{i,j-1}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right)\\ &\quad+\frac{\beta^*-1}{\beta}\left(A_i^{\mathrm{T}}p_{i,j}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right),\end{aligned} \tag{40}$$

$$\begin{aligned}&A_i^{\mathrm{T}}p_{i,j^*}^*-\frac{\beta M}{T_i}p_{i,j^*-1}^*+\frac{\beta M}{T_i}p_{i,j^*}^*\\ &=\frac{\beta-\beta^*}{\beta}\left(A_i^{\mathrm{T}}p_{i,j-1}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right)\\ &\quad+\frac{\beta^*}{\beta}\left(A_i^{\mathrm{T}}p_{i,j}-\frac{M}{T_i}p_{i,j-1}+\frac{M}{T_i}p_{i,j}\right),\end{aligned} \tag{41}$$

where $j^*=\beta(j-1)+\beta^*$, $j=1,2,\ldots,M$, and $\beta^*=1,2,\ldots,\beta$. Combining (26a) and (26b) with (40) and (41), conditions (38a) and (38b) hold for $i=1,2,\ldots,m$, and $j^*=1,2,\ldots,\beta M$; thus, Theorem 6 is proved.

Remark 6. According to Theorem 6, for given γ and M, condition (38) is sufficient condition of those in (26). In other words, the infimum of γ with βM is no larger than the one with M. Theorem 6 gives a way to increase the value of M and guarantees the decrease of the infimum of γ. The stability of system (1) is affected by the spectral radius of the state transition matrix. Theorem 6 also implies that the stability condition becomes less conservative with the increase of the value of M.

3.3 | Controller synthesis

In this subsection, a periodic piecewise state-feedback controller is introduced to stabilize system (1). By introducing a periodic piecewise constant state-feedback controller

$$u(t) = K(t)x(t), \tag{42}$$

where $K(t) = K(t + T_p)$ and $K(t) = K_{\sigma(i)}$ when $t \in [t_{i-1,\sigma(i)-1}, t_{i,\sigma(i)})$, the closed-loop system is given as

$$\dot{x}(t) = (A(t) + B(t)K(t))x(t). \tag{43}$$

Based on Theorem 2, a proposition to check whether the system can be stabilized via the state-feedback controller (42) is given as follows.

Proposition 2. Given a closed-loop periodic piecewise system (43). The closed-loop system is positive and asymptotically stable if and only if there exists a sufficient large scalar $M \in \mathbb{N}_+$, a set of vectors $p_{i,j} \in \mathbb{R}_+^{n_x}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., M$, and a set of matrices $K_i \in \mathbb{R}^{n_u \times n_x}$, $i = 1, 2, ..., m$, satisfying

$$(A_i + B_i K_i)^{\mathrm{T}} p_{i,j-1} - \frac{M}{T_i} p_{i,j-1} + \frac{M}{T_i} p_{i,j} \prec 0, \tag{44a}$$

$$(A_i + B_i K_i)^{\mathrm{T}} p_{i,j} - \frac{M}{T_i} p_{i,j-1} + \frac{M}{T_i} p_{i,j} \prec 0, \tag{44b}$$

$$p_{i,M} = p_{i+1,0},\ i = 1, 2 ..., m-1, \tag{44c}$$

$$p_{m,M} \succ p_{1,0}, \tag{44d}$$

$$A_i + B_i K_i \in \mathbb{M}^{n_x \times n_x}. \tag{44e}$$

Remark 7. When choosing controller gains that depend on both i and j, the closed-loop systems turn into periodic piecewise systems with time-varying subsystems. It is a completely different system from the one in Theorem 2, and the stability criteria are inapplicable to such systems. Thus, an iterative algorithm is proposed to design a piecewise constant control matrix $K(t)$ in our work.

Algorithm SPPPS State-feedback controller design for periodic piecewise positive systems

- **Step 1.** Set initial iteration label $k = 1$, tolerant τ and M. Set initial control matrices $K_{k,i} = 0$ for all $i = 1, 2, \ldots, m$.
- **Step 2.** For fixed $K_{k,i}$, $i = 1, 2, \ldots, m$, solve the following minimization problem for $\hat{\gamma}$ subject to $p_{i,j}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, M$,

  OP1: Minimize $\hat{\gamma}$ subject to

$$\left(A_i^{\mathrm{T}} + K_{k,i}^{\mathrm{T}} B_i^{\mathrm{T}}\right) p_{i,j-1} + \frac{M\left(p_{i,j} - p_{i,j-1}\right)}{T_i} \prec 0, \tag{45a}$$

$$\left(A_i^{\mathrm{T}} + K_{k,i}^{\mathrm{T}} B_i^{\mathrm{T}}\right) p_{i,j} + \frac{M\left(p_{i,j} - p_{i,j-1}\right)}{T_i} \prec 0, \tag{45b}$$

$$p_{i,j} \succ 0, \tag{45c}$$

$$p_{i,M} = p_{i+1,0}, \quad i = 1, 2 \ldots, m-1, \tag{45d}$$

$$\hat{\gamma} p_{m,M} \succ p_{1,0}. \tag{45e}$$

  Let $\gamma_k = \hat{\gamma}$.
- **Step 3.** If $\gamma_k < 1$, then $K_{k,i}$ can be applied to stabilize the system, otherwise go to Step 4.
- **Step 4.** For fixed $p_{i,j}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, M$, solve the following optimization problem for ε and $K_{k+1,i}$, $i = 1, 2, \ldots, m$.

  OP2: Minimize ε subject to

$$\left(A_i^{\mathrm{T}} + K_{k+1,i}^{\mathrm{T}} B_i^{\mathrm{T}}\right) p_{i,j-1} + \frac{M\left(p_{i,j} - p_{i,j-1}\right)}{T_i} \preceq \varepsilon p_{i,j-1}, \tag{46a}$$

$$\left(A_i^{\mathrm{T}} + K_{k+1,i}^{\mathrm{T}} B_i^{\mathrm{T}}\right) p_{i,j} + \frac{M\left(p_{i,j} - p_{i,j-1}\right)}{T_i} \preceq \varepsilon p_{i,j}, \tag{46b}$$

$$A_i + B_i K_{k+1,i} \in \mathbb{M}^{n_x \times n_x}, \tag{46c}$$

- **Step 5.** If $k \neq 1$ and $\left(\gamma_{k-1} - \gamma_k\right)/\gamma_k \leq \tau$, a solution is not found; else set $k = k + 1$ and go to Step 2.

As seen in Proposition 2, there are nonlinear terms $K_i^{\mathrm{T}} B_i^{\mathrm{T}} p_{i,j-1}$ and $K_i^{\mathrm{T}} B_i^{\mathrm{T}} p_{i,j}$. It is not a convex problem and Proposition 2 cannot be directly applied to designing the state-feedback controller. An iterative algorithm is given to design the parameter of the controller. By replacing A_i in (26a) and (26b) with $A_i + B_i K_i$, the spectral radius of the closed-loop system (43) can be characterized based on Corollary 1. For fixed K_i and a sufficiently large M, we can obtain an estimated spectral radius of the closed-loop state transition matrix and a set of $p_{i,j}$. Then fix $p_{i,j}$, we can find a new set of K_i to reduce the value of γ and renew the closed-loop state transition matrix. Then by changing the values of K_i and $p_{i,j}$ iteratively, the estimated spectral radius of the state transition matrix is monotonically decreasing. Based on this idea, an algorithm of state-feedback controller design for periodic piecewise positive, named Algorithm SPPPS, is given. In the algorithm, two linear programming problems are introduced to design parameters $p_{i,j}$ and $K_{i,j}$ and minimize estimated spectral radius. The linear programming problems are solved by CVX toolbox in Matlab [44] involving $(M+1)n$ unknown parameters. Compared with the conditions derived from the discontinuous copositive Lyapunov function, which have $2Mn$ unknown parameters, our conditions will have lower computational complexity where $M>1$.

Different from the algorithm in Zhu et al. [36], in which the set of vectors $p_{i,j}$ are obtained by solving a feasibility problem, the Algorithm SPPPS solves an optimization problem to obtain vectors $p_{i,j}$. It guarantees that the objective function is optimized in Step 2.

Remark 8. Due to the number of the time-scheduled intervals M being fixed, Algorithm SPPPS can only reach a local minimum. With the increase of M, $\hat{\gamma}$ in Step 2 converges to the spectral radius of the closed-loop transition matrix. M is chosen based on the maximum eigenvalue and time interval of each subsystem. If the parameter of the controller cannot be found, one can increase the value of M and apply Algorithm SPPPS again.

Remark 9 Monotonicity of $\gamma(k)$. The fixed vectors $p_{i,j}$ in Step 4 satisfy conditions (45d) and (45e). Based on conditions (46a) and (46b) and Proposition 1, $\dot{V}((k+1)T_p) \prec e^{\varepsilon T_p}\gamma_k V(kT_p)$ holds and the spectral radius of the closed-loop transition matrix is less than $e^{\varepsilon T_p}\gamma_k$. Therefore, the ε in OP2 is less than or equal to 0. By solving OP2 in Step 4, one can guarantee that γ_k in the algorithm is monotonically decreasing.

4 | ILLUSTRATIVE EXAMPLES

4.1 | Practical example

Consider a simplified model for the mitigation of HIV infection in previous research [23, 44]. Under simplifying assumptions, the drug treatment could be seen as a positive system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + \mu M_u x(t)$$

where $x(t)\in\mathbb{R}^n$ stands for the population of n viral genotypes, μ stands of the mutation rate, $A_{\sigma(t)}$ stands for the variation rates (including replication rates and viral clearance rates by different treatments), and M_u stands for the genetic mutation rates between different genotypes. We assume that $\sigma(t)$ is a periodic piecewise function taking value between 1 and 2 with the time intervals for each subsystem that are 1.2 and 1, $\mu=0.05$, and the structures of $A_{\sigma(t)}$ and M_u are given as follows:

$$\begin{aligned} A_1 &= \mathbf{diag}(-1.3,-1.2,0.7,0.6),\\ A_2 &= \mathbf{diag}(0.8,0.5,-1.2,-1),\\ M_u &= \begin{bmatrix} 0&1&1&0\\ 1&0&0&1\\ 1&0&0&1\\ 0&1&1&0 \end{bmatrix}. \end{aligned}$$

By applying Theorem 2 and letting the time segmentation $M=16$, we can find a set of vectors $p_{i,j}$ such that condition (7) holds. Furthermore, we use Theorem 3 to calculate the estimated spectral radius of the state transition matrices, which is equal to 0.8869 and it is close to the true spectral radius $\rho\left(\prod_{i=1}^{2}(A_i+\mu M_u)\right)=0.8400$. Therefore, under the above-scheduled treatment, all viruses with different genotypes are finally eliminated.

4.2 | Numerical example

A periodic piecewise system with two subsystems is given as follows:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{47}$$

where

$$A_1 = \begin{bmatrix} -1&1&0.3\\ 1.2&0.4&0.8\\ 0.3&1.1&-0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.7&1.4&1.2\\ 0.5&-0.5&1.1\\ 0.3&0.5&0.6 \end{bmatrix},$$

$$B_1 = [0.7\ \ 0.4\ \ 0.8]^{\mathrm{T}}, B_2 = [2.4\ \ 0.2\ \ 0.9]^{\mathrm{T}},$$

and $T_1=1$, $T_2=0.6$. Since matrices A_1 and A_2 are Metzler, system (47) is positive when $u(t)=0$. The eigenvalues of matrix $e^{A_2T_2}e^{A_1T_1}$ are 16.0936, 0.5880, and 0.1545. According to Lemma 1, $\rho(e^{A_2T_2}e^{A_1T_1})>1$ and the system is unstable. In what follows, a state-feedback controller is first designed. Then, for the stable closed-loop system, the corresponding copositive Lyapunov function is constructed. Finally, the spectral radius characterization is given and the λ-exponential stability is investigated. Main results obtained in this paper are illustrated by numerical examples as follows:

- **Stability and stabilization:** A state-feedback controller (42) is designed based on Algorithm SPPPS. Let the initial controller $K_{1,i}=0$ for $i=1,2$ and set M to be 128. By using the algorithm, the state-feedback control matrices $K_{k,1}$ and $K_{k,2}$ converge to

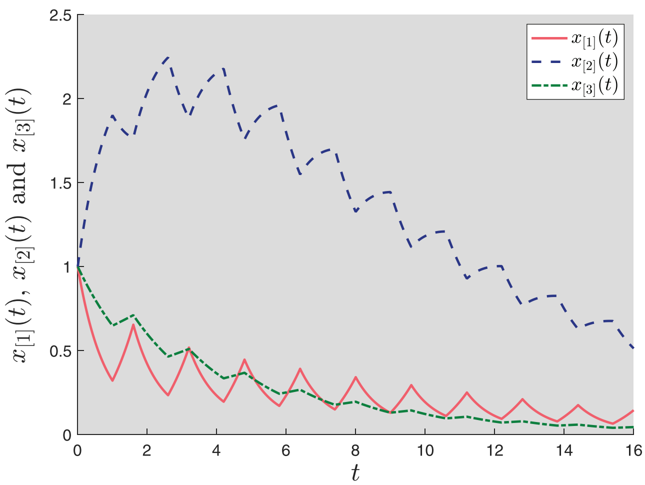

FIGURE 1 Trajectory of the state components [Color figure can be viewed at wileyonlinelibrary.com]

$$K_1=\begin{bmatrix}-0.3737\\-1.3699\\-0.4270\end{bmatrix}^{\mathrm{T}},\ K_2=\begin{bmatrix}-0.3333\\-0.5541\\-0.4995\end{bmatrix}^{\mathrm{T}}, \qquad (48)$$

and γ_k converges to 0.7994, which indicates the closed-loop system is stable. The trajectory of the state of the closed-loop system with initial state $x(0)=[111]^{\mathrm{T}}$ is given in Figure 1. Even though the value of $x_{[3]}(t)$ increases at the beginning, it finally converges to zero. Figure 2 shows the trajectory of time-scheduled copositive Lyapunov function. Since the vector function $p(t)$ in (27) satisfies inequalities (7c) and (7d) in Theorem 2, in each period, the copositive Lyapunov function is continuous, and jump discontinuities only happen at time kT_p.

- **Spectral radius characterization:** Figure 3 shows the relation between the value of estimated spectral radius $\hat{\gamma}$ and z, where $M=2^z$. When $M=1$, $\hat{\gamma}=1.190$, which means that the estimated spectral radius is larger than 1 and the stability cannot be checked by the time-scheduled copositive Lyapunov function with $M=1$. Only when M is larger than 2, we can find a set of vectors $p_{i,j}$ satisfying condition (7). With the increasing of z, $\hat{\gamma}$ is monotonically decreasing to

$$\gamma=\rho\left(e^{(A_2+B_2K_2)T_2}e^{(A_1+B_1K_1)T_1}\right)=0.79758,$$

which verifies Theorem 5 and Theorem 6.

- **Convergent rate:** In order to characterize the convergent rate of system (47) and verify the effectiveness of Theorem 5, Figure 4 is given. The solid line denotes

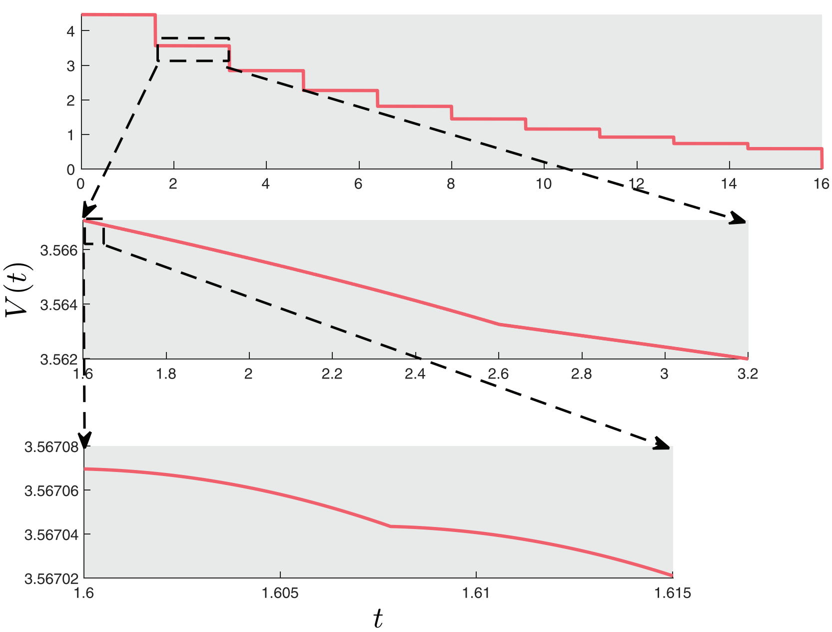

FIGURE 2 The trajectory of a time-scheduled co-positive Lyapunov function with $M=128$ [Color figure can be viewed at wileyonlinelibrary.com]

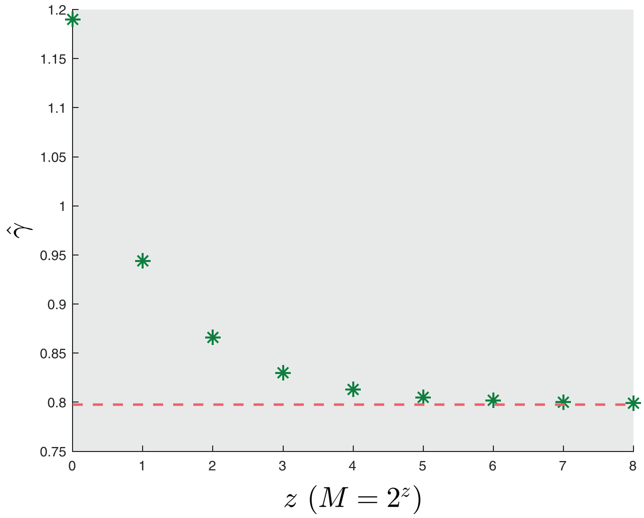

FIGURE 3 Variation of $\hat{\gamma}$ with $z(M = 2^z)$ [Color figure can be viewed at wileyonlinelibrary.com]

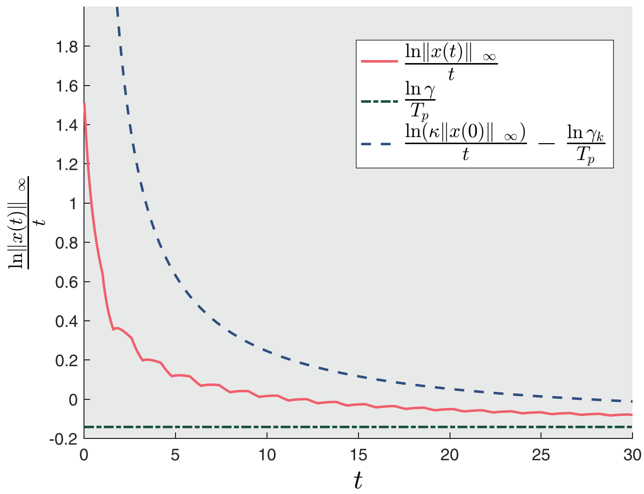

FIGURE 4 Variation of $\frac{\ln\|x(t)\|}{t}$ with time t [Color figure can be viewed at wileyonlinelibrary.com]

the variation of function $\frac{\ln\|x(t)\|}{t}$. Based on inequality (35) in Definition 3, function $\frac{\ln\|x(t)\|}{t}$ satisfies

$$\frac{\ln\|x(t)\|}{t} \leq \frac{\ln\left(\kappa\|x(0)\|\right)}{t} - \frac{\ln\gamma_k}{T_p},$$

where $\gamma_k = 47.1348$, which is shown in Figure 4. The largest convergent rate of the system is $-\frac{\ln\gamma}{T_p} = 0.14136$; with the increase of time t, the value of $\frac{\ln\|x(t)\|}{t}$ will finally converge to it.

- **Stabilizing effectiveness:** One hundred randomly generated three-order single-input single-output stabilizable periodic piecewise positive systems with two subsystems are given. The time intervals of each subsystem are the same and equal to 1. Metzler matrices A_1, A_2 and nonnegative matrices B_1, B_2 are randomly generated. Table 1 demonstrates the effectiveness of different algorithm by giving the number of systems that are stabilized. It shows that with an increase of M, the number of stabilized systems increases. Furthermore, a comparison between Algorithm PPPSSCD in Zhu et al. [36] and Algorithm SPPPS is given. The result shows that the performance of Algorithm PPPSSCD is a better than Algorithm SPPPS with $M = 1$. When M is larger, the performance of Algorithm SPPPS is better.

TABLE 1 Effectiveness of different algorithms

Algorithm	Number of stabilized systems
PPPSSCD in Zhu et al. [36]	14
SPPPS with $M = 1$	12
SPPPS with $M = 2$	22
SPPPS with $M = 4$	52
SPPPS with $M = 8$	95

5 | CONCLUSION

In this paper, the stability condition of linear periodic piecewise positive systems has been discussed. In each time interval of the systems, by utilizing time segmentation approach to partition the copositive Lyapunov function into a given number of segments, a time-scheduled copositive Lyapunov function has been constructed. It is shown that the asymptotic stability of the system can be checked by solving linear inequalities if the number of segments is sufficiently large. Based on the equivalent stability condition, the spectral radius of the state transition matrix is characterized in two different ways. The relation between spectral radius and exponential stability also has been investigated. Furthermore, a state-feedback controller has been designed, and the iterative algorithm has been constructed to minimize the spectral radius of the closed-loop state transition matrix. Finally, numerical examples have been given to illustrate the theoretical results.

The effectiveness of the time segmentation approach in reducing the conservativeness of the stability condition for periodic piecewise positive systems has been shown in Theorem 6. However, the result only shows that the condition with time segmentation βM, where $\beta \in \{1, 2, \ldots\}$, is less conservative than the one with time segmentation M. Whether the stability condition with time segmentation $M+1$ is less conservative than the one with time segmentation M is an open question for further investigation. Moreover, the stability of periodic piecewise positive systems with time delay has been analyzed in Zhu et al. [45]. Our time segmentation approach could be applied to reduce the conservativeness of the proposed sufficient stability condition. Further research includes the consistency analysis for equivalent stability conditions of the periodic piecewise positive systems with delay by using a time segmentation approach.

AUTHOR CONTRIBUTIONS

Bohao Zhu: conceptualization, methodology. **James Lam:** conceptualization, formal analysis. **Xiaoqi Song:** formal analysis; validation. **Hong Lin:** formal analysis, project administration, supervision. **Jason Ying Kuen Chan:** software; validation. **Ka-Wai Kwok:** software, validation.

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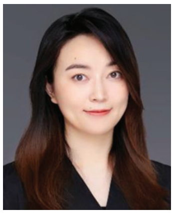

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