Reachable Set-Based Consensus of Positive Multiagent Systems

Chenchen Fan, James Lam, *Fellow, IEEE*, Xiujuan Lu, Jason J. R. Liu, *Member, IEEE*, Xiaomei Wang, *Member, IEEE*, and Ka-Wai Kwok, *Senior Member, IEEE*

***Abstract*—This work addresses the problem of reachable set-based consensus for positive multiagent systems affected by typical classes of bounded disturbances. In the presence of disturbances, a reachable set-based consensus with positivity preservation is proposed to ensure that the state of the closed-loop system remains positive while enclosing the reachable set of the defined consensus error with an ellipsoidal bounding region. Sufficient conditions are established to achieve the reachable set-based positive consensus under the energy-bounded or peak-bounded disturbance input. Equivalent design conditions are provided, for which a heuristic algorithm is proposed for computing and optimizing the bounding region. Simulations are conducted to validate the obtained results.**

***Index Terms*—Consensus, multiagent systems, positivity, reachable set.**

I. Introduction

A. Research Background and Literature Review

Multiagent systems, which consist of multiple agents via network communication protocols to accomplish designated control goals, have attracted increasing attention in recent decades. The great research interest is aroused by the wide application of multiagent systems in smart transportation technology [1], distributed mobile robotics [2], and power system restoration [3]. Among various research topics in multiagent systems, consensus as a fundamental coordination problem is the main focus. The objective of the consensus problem in the control field is to develop control protocols to ensure all agents reach an agreement. It is well known that the Kronecker product is a basic tool for dealing with the consensus problems of multiagent systems. The Kronecker product is used to represent the governing dynamics of all agents in a compact form, which shares a similar idea with [4] using an augmented matrix approach to compactly represent all possible working modes. Additionally, the properties of the Kronecker product can facilitate the consensus analysis of multiagent systems. Research on consensus problems has been conducted from different perspectives, such as consensus for multiagent systems under various types of communication topologies [5], [6], [7], consensus for many types of systems, like hybrid multiagent systems [8], time-delay multiagent systems [9], [10], etc, and consensus over communication constraints via event-triggered schemes [11], [12]. Note that the aforementioned literature has reported the consensus of all agents without considering the influence of external disturbances. For multiagent systems subject to inevitable disturbances, robust consensus with several kinds of performance indexes like H_2, H_∞, weighted H_∞ and dissipativity performance has been reported in [13], [14], [15], and [16], where the consensus can be achieved with satisfying the corresponding disturbance attenuation levels.

Positive systems are a class of systems whose state and output variables always stay in the non-negative orthant with non-negative inputs [17], [18]. Extensive studies on positive systems have been carried out mainly for two reasons: 1) many practical dynamics such as compartmental epidemic model [19], vehicular traffic flow model [20], and chemical reactions contain non-negative physical quantities; 2) using the positive feature has theoretical advantages in reducing complexity and conservatism in system analysis and synthesis. Apart from many studies on single positive systems, multiple positive systems interconnected via communication protocols are common and have gradually gained attention. Positivity-preserving consensus for multiagent systems with single input and multiple inputs have been investigated in [6], [21], and [22]. Observer-based output feedback consensus for positive multiagent systems has also been reported in [23] and [24], where both controller and observer gains are designed to maintain positivity and achieve consensus simultaneously. However, the effect of disturbances on the positive multiagent systems is rarely taken into account in the previous work.

Reachable set estimation and synthesis are fundamental topics in control theory, widely applied in safety verification [25],

Manuscript received 8 May 2023; revised 12 September 2023; accepted 8 December 2023. This work was supported in part by the Research Grants Council (RGC) under Grant 17205721, Grant 17207020, Grant 17209021, and Grant T42-409/18-R; and in part by the Innovation and Technology Commission (ITC) of the HKSAR Government under the InnoHK Initiative, Hong Kong, through Multi-Scale Medical Robotics Center Ltd. and Centre for Garment Production Ltd. under Grant MRP/029/20X. This article was recommended by Associate Editor Z. G. Wu. *(Corresponding author: Ka-Wai Kwok)*.

Chenchen Fan, Xiujuan Lu, Xiaomei Wang, and Ka-Wai Kwok are with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong (e-mail: fancc@connect.hku.hk; u3007423@connect.hku.hk; wangxmei@connect.hku.hk; kwokkw@hku.hk).

James Lam is with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong, and also with the HKU Shenzhen Institute of Research and Innovation, Shenzhen 518057, China (e-mail: james.lam@hku.hk).

Jason J. R. Liu is with the Department of Electromechanical Engineering, Faculty of Science and Technology, University of Macau, Macau, China (e-mail: jasonliu@um.edu.mo).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TSMC.2023.3342905.

Digital Object Identifier 10.1109/TSMC.2023.3342905

motion planning [26], and collision avoidance [27]. When dynamic systems are affected by bounded disturbance, most studies focus on disturbance attenuation. However, sometimes it needs to know where a physical quantity of the system can reach in the presence of disturbances. The concept of reachable set provides a tool to measure the evolution of system state under bounded disturbances. Many reachable set estimation and synthesis results have been reported to find bounding regions for various kinds of systems subject to bounded disturbances, like genetic regulatory systems [28], switched systems [29], [30], periodic systems [31], [32], and positive systems [33]. For multiagent systems that have recently become a popular research focus, reachable set is also a very useful concept to verify the motion of all agents in the presence of disturbance. However, the reachable set problem of multiagent systems has not been fully addressed. A related work [34] without positivity constraint has been found, of which only peak-bounded disturbances are addressed. The peak-bounded disturbances often characterize some persistent disturbances whose magnitude is always bounded. It is noted that some disturbances can possess a significantly large magnitude while acting over a very short duration, and may be appropriately characterized as energy-bounded disturbances. For energy-bounded disturbances, robust problems are often studied based on H_∞ and dissipativity performance indexes. The reachable set-based robust synthesis in the presence of energy-bounded disturbances is rarely investigated.

B. Research Motivations

This work will investigate the reachable set-based consensus problem for positive multiagent systems, which aims to design a distributed control protocol to preserve positivity constraint and enclose the reachable set of the defined consensus error with an ellipsoidal region. Compared with the reachable set synthesis of single systems, the main challenges arise from dealing with the Laplacian matrix of the communication graph among agents to obtain controller parameters and establish bounding ellipsoids. The research motivations are summarized as follows:

1) *Significance of Reachable Set-Based Consensus:* For multiagent systems affected by disturbances, a kind of robust consensus is desired. The reachable set-based consensus can characterize the bounding region of consensus error to achieve a robust consensus in the presence of disturbances.
2) *Exploration of Novel Theoretical Techniques:* Reachable set-based consensus for positive multiagent systems needs to establish conditions satisfying positivity constraints and reachable set bounding simultaneously, which are more complex and hard to compute.

C. Novelties and Contributions

The primary novelties and contributions of this work are outlined below.

1) Establish the framework of reachable set-based consensus for positive linear multiagent systems in the presence of disturbances, which is different from other results on the consensus of positive multiagent systems without dealing with disturbances [6], [21], [22].
2) Develop the controller design conditions to ensure reachable set-based consensus with positivity preservation for both peak-bounded and energy-bounded disturbances, while the related work [34] has no positivity constraint and only considers peak-bounded disturbances.
3) Propose a heuristic algorithm to simultaneously compute the controller gain and optimize the ellipsoidal bounding region.

The remainder of this work is organized in the following manner. Section II presents useful preliminaries and formulates the reachable set-based consensus problem of multiple positive agents. In Section III, the main results of developing design conditions for reachable set-based positive consensus are presented. Section IV proposes the optimization method to minimize the bounding region. Section V includes illustrative examples that showcase the effectiveness of the proposed techniques, while Section VI serves as the concluding section.

Notations: $\mathbb{R}^n$ and $\mathbb{R}^n_+$ represent the set of n-dimensional vectors with real and non-negative real components, respectively. $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{m\times n}_+$ denote the collection of $m \times n$ matrices with all real and non-negative real entries, respectively. Matrix $A \in \mathbb{R}^{n\times n}$ is a Metzler matrix if its off-diagonal entries are non-negative. I_n represents a $n \times n$ identity matrix. The dimension 0 is appropriate for it to serve as a zero matrix. $\mathbf{1}$ represents the column vector of appropriate dimensions with all elements being 1. The superscript T indicates the matrix transpose. $\otimes$ represents the Kronecker product. For matrix A, $[A]_{ij}$ means the entry in the ith row and the jth column. $A \succeq 0$ means $[A]_{ij} \geq 0$ for all i and j. The matrix P is real symmetric and positive-definite, denoted as $P > 0$. $L_2[0, \infty)$ represents space of square-integrable vector Lebesque functions over $[0, \infty)$. $\|\cdot\|_2$ denotes the L_2-norm. An integer set is defined as $\overline{1, Z} \triangleq \{1, 2, \ldots, Z\}$, where Z is a positive integer. For convenience, $\mathbf{sym}(M) \triangleq M^T + M$. The notation $diag\{\ldots\}$ represents a block diagonal matrix with the input entries as the diagonal blocks.

II. Preliminaries and Problem Formulation

A. Communication Topology

This work focuses on an undirected, connected communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V} = \{1, 2, \ldots, N\}$ represents the collection of nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the collection of edges. Two nodes i and j are adjacent, in other words, if $(i, j) \in \mathcal{E}$, agents i and j can communicate with each other. The adjacency matrix $\mathcal{A}$ of the graph $\mathcal{G}$ is an $N \times N$ matrix, with entries $[\mathcal{A}]_{ij} = [\mathcal{A}]_{ji} = 1$ if $(i, j) \in \mathcal{E}$, and 0 otherwise. The neighbors of node i can be represented by the set $\mathcal{N}_i = \{j \in \mathcal{V}:(i, j) \in \mathcal{E}\}$. The Laplacian matrix $\mathcal{L}$ of the graph $\mathcal{G}$ is taken as an $N \times N$ matrix with the following entries: $[\mathcal{L}]_{ii} = \sum_{j\in\mathcal{N}_i} \mathcal{A}_{ij}$ and $[\mathcal{L}]_{ij} = -\mathcal{A}_{ij}$ for any $i \neq j$. Regarding the undirected, connected graph $\mathcal{G}$, it is referred to [21] and [35] that $\mathcal{L}$ is symmetric and positive semi-definite.

B. Problem Formulation

Consider a multiagent system consisting of N agents, and each agent is represented by the following positive linear system:

$$\dot{x}_i(t) = Ax_i(t) + B_u u_i(t) + B_\omega \omega_i(t), \quad i \in \overline{1, N} \tag{1}$$

where $x_i(t) = [x_{i1}(t) \;\; x_{i2}(t) \;\; \cdots \;\; x_{in_x}(t)]^T \in \mathbb{R}_+^{n_x}$ is the system state vector, $u_i(t) \in \mathbb{R}^{n_u}$ is the control input vector, and $\omega_i(t) \in \mathbb{R}_+^{n_\omega}$ is the exogenous disturbance vector. $A \in \mathbb{R}^{n_x \times n_x}$ is a Metzler matrix, $B_u \in \mathbb{R}_+^{n_x \times n_u}$ and $B_\omega \in \mathbb{R}_+^{n_x \times n_\omega}$ are non-negative matrices. Based on the definition and lemma of positivity for linear positive systems [17], it is said that system (1) is positive when its state $x(t)$ is in the non-negative orthant for all $t \geq 0$, given any initial conditions $x(0) \succeq 0$ and disturbance $\omega(t) \succeq 0$. Also, system (1) is positive if and only if A is Metzler, B_u and B_ω are non-negative. Moreover, (A, B_u) is assumed to be stabilizable in this work.

A distributed state-feedback control protocol is considered as follows:

$$u_i(t) = K \sum_{j=1}^{N} [\mathcal{A}]_{ij} \big(x_j(t) - x_i(t)\big), \quad i \in \overline{1, N} \tag{2}$$

where K is the controller gain to be determined. It is worth noting that the considered control protocol (2) uses the same controller gain K for all agents, which is quite common given the homogeneity of the positive agents in (1).

By denoting the state vector, the control input vector, and the disturbance vector of the multiagent system as $x(t) = [x_1^T(t) \;\; \cdots \;\; x_N^T(t)]^T$, $u(t) = [u_1^T(t) \;\; \cdots \;\; u_N^T(t)]^T$, $\omega(t) = [\omega_1^T(t) \;\; \cdots \;\; \omega_N^T(t)]^T$, the overall closed-loop dynamics under the control (2) can be described by

$$\dot{x}(t) = \big(I_N \otimes A - \mathcal{L} \otimes B_u K\big)x(t) + (I_N \otimes B_\omega)\omega(t). \tag{3}$$

In this article, reachable set-based positive consensus problem in the presence of disturbance is studied. The defined positive consensus in [6] and [21] gives that the positive consensus of agents is guaranteed if a controller matrix K can be found such that the state trajectories are always in non-negative orthant, and the consensus is reached, i.e.,

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = 0 \quad \forall i, j \in \overline{1, N}. \tag{4}$$

However, due to the effect of disturbances, under the distributed state-feedback control protocol, (4) colorredis hard to achieve. Robust consensus problems should be considered.

Motivated by robust consensus in [13], [34], and [36], consensus error states $\eta_i(t) \triangleq x_i(t) - (1/N)\sum_{j=1}^{N} x_j(t)$, $i \in \overline{1, N}$ are introduced. By denoting $\eta(t) = [\eta_1^T(t) \ldots \eta_N^T(t)]^T$, the compact form of $\eta(t)$ is $\eta(t) = (\mathcal{M} \otimes I_{n_x})x(t)$, where $\mathcal{M} = I_N - (1/N)\mathbf{1}\mathbf{1}^T$. From (3) and $\mathcal{M}\mathcal{L} = \mathcal{L}\mathcal{M} = \mathcal{L}$, the consensus error dynamics of $\eta(t)$ can be obtained as follows:

$$\dot{\eta}(t) = \big(I_N \otimes A - \mathcal{L} \otimes B_u K\big)\eta(t) + (\mathcal{M} \otimes B_\omega)\omega(t). \tag{5}$$

Under the state-feedback control, it is important to focus on assessing the reachable consensus error $\eta(t)$ in the presence of external disturbance input $\omega(t)$. Inspired by positive consensus in [6] and [21] and reachable set problem in [34], this article delves into the problem of reachable set-based positive consensus. This problem aims to design the control protocol to ensure that the system state is always in non-negative orthant and simultaneously confines the reachable set of consensus errors within a specified bounding region.

For each agent i, $i \in \overline{1, N}$, the reachable set of the consensus error $\eta_i(t) \in \mathbb{R}^{n_x}$ is defined as

$$\mathcal{R}_{\eta_i} \triangleq \{\eta_i(t) \mid \eta_i(0) = 0, \eta_i(t), \omega_i(t) \text{ satisfy (5)}, t \geq 0\}. \tag{6}$$

Meanwhile, regarding the bounding region, a common type of ellipsoidal region is adopted in this article. An ellipsoidal bounding region is defined as follows:

$$\Phi(P) \triangleq \{\zeta \in \mathbb{R}^{n_x} \mid \zeta^T P \zeta < 1, P > 0\}. \tag{7}$$

It is obvious that for $\eta_i(t) \in \mathcal{R}_{\eta_i}$, if $\eta_i^T(t)P\eta_i(t) < 1$, $\mathcal{R}_{\eta_i} \subseteq \Phi(P)$.

A reachable set-based positive consensus is defined as follows.

Definition 1: The achievement of reachable set-based positive consensus for system (1) under control (2) is recognized when:

d1) The closed-loop system (3) is positive, i.e., $(I_N \otimes A - \mathcal{L} \otimes B_u K)$ is a Metzler matrix, and $(I_N \otimes B_\omega)$ is a non-negative matrix.

d2) The reachable set-based consensus is achieved, that is, for system (5) with zero initial conditions, the reachable set $\mathcal{R}_{\eta_i}$, $i \in \overline{1, N}$ of consensus error $\eta_i(t)$ can be enclosed by the ellipsoidal region $\Phi(P)$.

Remark 1: It is worth noting that the reachable set-based consensus is a kind of robust consensus. In the presence of bounded disturbances, the consensus error may no longer converge to zero. This work investigates the reachable set of the consensus error, which quantifies the reachability of the consensus error under the effect of disturbances with an ellipsoidal bounding region. Furthermore, it is essential to maintain the positivity of agents. Therefore, the reachable set-based positive consensus is proposed as Definition 1. This provides an effective framework for achieving robust consensus among positive agents while accounting for the potential effects of disturbances.

According to Definition 1, the main research problem of this work is finding a controller matrix K such that for system (1), the reachable set-based positive consensus in terms of d1) and d2) is achieved.

III. Main Results

This section develops results on the reachable set-based positive consensus and establishes the equivalent design conditions.

A. Positivity Constraint

Regarding the positivity requirement of *d1)* in Definition 1, motivated by [6] and [21], the following proposition for the considered system (3) is given.

Proposition 1 [6], [21]: The positivity constraint of the overall dynamic system (3) can be satisfied if and only if there exists a matrix K such that $B_uK \succeq 0$ and $A - \xi_{\max}B_uK$ is Metzler, where $\xi_{\max} \triangleq \max(\xi_i)$ and $\xi_i \triangleq \sum_{j=1}^{N}[\mathcal{A}]_{ij}$, $i \in \overline{1,N}$.

B. Reachable Set-Based Positive Consensus

With the positivity constraint satisfied by the established condition in Proposition 1, this part explores the achievable reachable set-based positive consensus under the influence of two distinct classes of bounded disturbances.

With $\mathcal{M} = I_N - (1/N)\mathbf{1}\mathbf{1}^T$ in (5), by referring to in [37, Lemma 5], there exists an orthogonal matrix U such that $U^T\mathcal{M}U = \begin{bmatrix} I_{N-1} & 0 \\ 0 & 0 \end{bmatrix}$, and $(\mathbf{1}/\sqrt{N})$ is the last column of U. Moreover, since $\mathcal{L}$ is the Laplacian matrix of an undirected, connected graph, then $U^T\mathcal{L}U = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$, and L is positive definite. Let $U = [U_1 \ \ \frac{\mathbf{1}}{\sqrt{N}}]$, and perform the orthogonal transformation as follows:

$$\hat{\eta}(t) = (U^T \otimes I_{n_x})\eta(t) = \begin{bmatrix} (U_1^T \otimes I_{n_x})\eta(t) \\ \left((\frac{\mathbf{1}}{\sqrt{N}})^T \otimes I_{n_x}\right)\eta(t) \end{bmatrix} \tag{8}$$

$$\hat{\omega}(t) = (U^T \otimes I_{n_\omega})\omega(t) = \begin{bmatrix} (U_1^T \otimes I_{n_\omega})\omega(t) \\ \left((\frac{\mathbf{1}}{\sqrt{N}})^T \otimes I_{n_\omega}\right)\omega(t) \end{bmatrix}. \tag{9}$$

Then, based on the error dynamic system (5), it can be obtained that

$$\begin{aligned} \dot{\hat{\eta}}(t) &= (U^T \otimes I_{n_x})\dot{\eta}(t) \\ &= (U^T \otimes I_{n_x})\big(I_N \otimes A - \mathcal{L} \otimes B_uK\big)\eta(t) \\ &\quad + (U^T \otimes I_{n_x})(\mathcal{M} \otimes B_\omega)\omega(t) \\ &= (U^T \otimes I_{n_x})\big(I_N \otimes A - \mathcal{L} \otimes B_uK\big)(U \otimes I_{n_x})\hat{\eta}(t) \\ &\quad + (U^T \otimes I_{n_x})(\mathcal{M} \otimes B_\omega)(U \otimes I_{n_\omega})\hat{\omega}(t) \\ &= \big(I_N \otimes A - U^T\mathcal{L}U \otimes B_uK\big)\hat{\eta}(t) \\ &\quad + \big(U^T\mathcal{M}U \otimes B_\omega\big)\hat{\omega}(t). \end{aligned} \tag{10}$$

By denoting $(U_1^T \otimes I_{n_x})\eta(t) \triangleq \hat{\eta}_1(t)$, $((\mathbf{1}/\sqrt{N})^T \otimes I_{n_x})\eta(t) \triangleq \hat{\eta}_2(t)$, system (10) can be divided into the following systems:

$$\dot{\hat{\eta}}_1(t) = \big(I_{N-1} \otimes A - L \otimes B_uK\big)\hat{\eta}_1(t) + (U_1^T \otimes B_\omega)\omega(t) \tag{11}$$

$$\dot{\hat{\eta}}_2(t) = 0. \tag{12}$$

It can be noticed that (12) does not involve control input and disturbance input.

Note that after transformation, the system (11)–(12) is obtained based on system (5). To investigate the correlation between $\hat{\eta}_1(t)$, $\hat{\eta}_2(t)$ and $\eta_i(t)$, the following lemma is proposed.

Lemma 1: If $\hat{\eta}_1(t)$ in (11) satisfies $\hat{\eta}_1^T(t)(I_{N-1} \otimes P)\hat{\eta}_1(t) < 1$, where $P > 0$, then $\eta_i^T(t)P\eta_i(t) < 1$, $i \in \overline{1,N}$, can be obtained for $\eta_i(t)$ in (5).

Proof: With $\hat{\eta}_2(t) = ((\mathbf{1}/\sqrt{N})^T \otimes I_{n_x})\eta(t)$, it holds that $\hat{\eta}_2^T(t)P\hat{\eta}_2(t) = \hat{\eta}_2^T(t)P((\mathbf{1}/\sqrt{N})^T \otimes I_{n_x})(\mathcal{M} \otimes I_{n_x})x(t)$. Since $(\mathbf{1}/\sqrt{N})^T\mathcal{M} = 0$, $\hat{\eta}_2^T(t)P\hat{\eta}_2(t) = 0 < 1$ is obtained. With $\hat{\eta}_1^T(t)(I_{N-1} \otimes P)\hat{\eta}_1(t) < 1$, $\hat{\eta}^T(t)(I_N \otimes P)\hat{\eta}(t) < 1$ holds. Since $\hat{\eta}(t) = (U^T \otimes I_{n_x})\eta(t)$, thus $\sum_{i=1}^{N}\eta_i^T(t)P\eta_i(t) < 1$. Therefore, for each $i \in \overline{1,N}$, $\eta_i^T(t)P\eta_i(t) < 1$ can be obtained. ■

In the following, two types of disturbance inputs are studied.

1) Under Energy-Bounded Disturbance Input: In this section, a typical class of energy-bounded disturbances is considered. It is assumed that for each agent i, $\omega_i \in L_2[0, \infty)$, and

$$\|\omega_i\|_2 \le \bar{\omega}_{ei}, \quad \bar{\omega}_{ei} > 0. \tag{13}$$

From (13), it can be obtained that $\|\omega\|_2 \le \bar{\omega}_e$, where $\bar{\omega}_e \triangleq \sqrt{\sum_{i=1}^{N}\bar{\omega}_{ei}^2}$. Under the energy-bounded disturbances, reachable set-based positive consensus conditions for system (1) are established in the following theorem.

Theorem 1: Consider system (1) with energy-bounded disturbance $\omega_i(t)$ satisfying (13). If there exist matrix $P > 0$, and controller matrix K satisfying the following conditions:

$$B_uK \succeq 0 \tag{14}$$

$$A - \xi_{\max}B_uK \quad \text{is Metzler,} \tag{15}$$

$$\begin{bmatrix} \mathbf{sym}(PA - \lambda_i PB_uK) & PB_\omega \\ B_\omega^T P & -\frac{1}{\bar{\omega}_e^2}I_{n_\omega} \end{bmatrix} < 0, \ \ i \in \overline{1, N-1} \tag{16}$$

where λ_i are the eigenvalues of L, then the positivity of system (3) can be guaranteed, and under zero initial conditions, the reachable set $\mathcal{R}_{\eta_i}$, $i \in \overline{1,N}$ can be bounded by the ellipsoidal region $\Phi(P)$.

Proof: Based on Proposition 1, (14) and (15) are equivalent to that system (3) is positive.

Then, construct a Lyapunov function $V(t) = \hat{\eta}_1^T(t)(I_{N-1} \otimes P)\hat{\eta}_1(t)$, with $P > 0$. Along the trajectory of system (11), the subsequent result can be derived

$$\begin{aligned} &\dot{V}(t) - \frac{1}{\bar{\omega}_e^2}\omega^T(t)\omega(t) \\ &= \hat{\eta}_1^T(t)\big(I_{N-1} \otimes A^TP - L^T \otimes K^TB_u^TP\big)\hat{\eta}_1(t) + \omega^T\big(U_1 \otimes \\ &\quad B_\omega^TP\big)\hat{\eta}_1(t) + \hat{\eta}_1^T(t)\big(I_{N-1} \otimes PA - L \otimes PB_uK\big)\hat{\eta}_1(t) \\ &\quad + \hat{\eta}_1^T(t)\big(U_1^T \otimes PB_\omega\big)\omega(t) - \frac{1}{\bar{\omega}_e^2}\omega^T(t)\omega(t) \\ &= \begin{bmatrix} \hat{\eta}_1(t) \\ \omega(t) \end{bmatrix}^T \begin{bmatrix} \Gamma & U_1^T \otimes PB_\omega \\ U_1 \otimes B_\omega^TP & -\frac{1}{\bar{\omega}_e^2}I_{n_\omega} \end{bmatrix} \begin{bmatrix} \hat{\eta}_1(t) \\ \omega(t) \end{bmatrix} \end{aligned}$$

where $\Gamma = I_{N-1} \otimes \mathbf{sym}(PA) - L \otimes \mathbf{sym}(PB_uK)$.

To obtain $\dot{V}(t) - (1/\bar{\omega}_e)\omega^T(t)\omega(t) < 0$, the following inequality holds:

$$\begin{bmatrix} \Gamma & U_1^T \otimes PB_\omega \\ U_1 \otimes B_\omega^TP & -\frac{1}{\bar{\omega}_e^2}I_{n_\omega} \end{bmatrix} < 0. \tag{17}$$

By using Schur complement equivalence, (17) can be rewritten as follows:

$$\Gamma + \bar{\omega}_e^2U_1^TU_1 \otimes (PB_\omega B_\omega^TP) < 0. \tag{18}$$

Since $\mathcal{G}$ is undirected and connected, matrix L is positive definite. Therefore, an orthogonal matrix F can be found such that $F^TLF = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{N-1}\} \triangleq \Delta$, where λ_i, $i \in \overline{1, N-1}$ are the eigenvalues of matrix L. Conduct pre- and post-multiplication (18) with $(F \otimes I_{n_x})^T$ and $(F \otimes I_{n_x})$. According to the following results:

$$\begin{aligned} &(F \otimes I_{n_x})^T\Gamma(F \otimes I_{n_x}) \\ &= I_{N-1} \otimes \mathbf{sym}(PA) - \Delta \otimes \mathbf{sym}(PB_uK) \end{aligned} \tag{19}$$

and

$$(F\otimes I_{n_x})^T\big(U_1^TU_1\otimes PB_\omega B_\omega^TP\big)(F\otimes I_{n_x}) = I_{N-1}\otimes(PB_\omega B_\omega^TP) \quad (20)$$

Equation (18) can be rewritten as

$$I_{N-1}\otimes \mathbf{sym}(PA) - \Delta\otimes \mathbf{sym}(PB_uK) + \bar{\omega}_e^2 I_{N-1}\otimes(PB_\omega B_\omega^TP) < 0. \quad (21)$$

It is noted that both (19) and (20) are block-diagonal matrices. Based on the partitioned matrix theory, (18) is satisfied with all $N-1$ block-diagonal matrices being negative definite simultaneously, i.e.,

$$\mathbf{sym}(PA-\lambda_iPB_uK) + \bar{\omega}_e^2PB_\omega B_\omega^TP < 0 \quad (22)$$

hold for $i\in\overline{1,N-1}$. On the basis of Schur complement equivalence, (22) are equivalent to (16).

Since (16) gives $\dot{V}(t)-(1/\bar{\omega}_e^2)\omega^T(t)\omega(t) < 0$, after integrating, it can be obtained that

$$V(t)-V(0) < \frac{1}{\bar{\omega}_e^2}\int_0^t \omega^T(\tau)\omega(\tau)d\tau \quad (23)$$

holds for any $t\geq 0$. Under zero initial conditions, one has $V(0)=0$. Noting that $\|\omega_i\|_2\leq\bar{\omega}_{ei}$, one can get that $\int_0^\infty\omega^T(t)\omega(t)dt = \sum_{i=1}^N\bar{\omega}_{ei}^2$. Moreover, $\bar{\omega}_e = \sqrt{\sum_{i=1}^N\bar{\omega}_{ei}^2}$ holds. Therefore, $(1/[\bar{\omega}_e^2])\int_0^t\omega^T(\tau)\omega(\tau)d\tau\leq 1$ for all $t\geq 0$.

Thus, $V(t)<1$, that is $\hat{\eta}_1^T(t)(I_{N-1}\otimes P)\hat{\eta}_1(t)<1$ for all $t\geq 0$. According to Lemma 1, $\hat{\eta}_1^T(t)(I_{N-1}\otimes P)\hat{\eta}_1(t)<1$ can imply that $\eta_i(t)^TP\eta_i(t)<1$, $i\in\overline{1,N}$. Therefore, the reachable set of $\eta_i(t)$, $i\in\overline{1,N}$ in (5) under energy-bounded disturbance input can be bounded by the ellipsoidal region $\Phi(P)$.

In summary, based on Definition 1, constraints (14)–(16) can guarantee the reachable set-based positive consensus of system (1) under energy-bounded disturbance input. ■

2) Under Peak-Bounded Disturbance Input: In this section, a typical class of peak-bounded disturbances is considered. It is assumed that

$$\omega_i^T(t)\omega_i(t)\leq\bar{\omega}_{pi},\quad \bar{\omega}_{pi}>0. \quad (24)$$

From (24), it is obtained that $\omega^T(t)\omega(t)\leq\bar{\omega}_p$, where $\bar{\omega}_p\triangleq\sum_{i=1}^N\bar{\omega}_{pi}$. The following theorem develops the reachable set-based positive consensus conditions for system (1) under peak-bounded disturbances.

Theorem 2: Consider system (1) with peak-bounded disturbance $\omega_i(t)$ satisfying (24). Given a scalar $\alpha>0$, if there exist matrix $P>0$, and controller matrix K, satisfying (14), (15), and the following conditions:

$$\begin{bmatrix}\mathbf{sym}(PA-\lambda_iPB_uK)+\alpha P & PB_\omega\\ B_\omega^TP & -\frac{\alpha}{\bar{\omega}_p}I_{n_\omega}\end{bmatrix}<0,\ \ i\in\overline{1,N-1} \quad (25)$$

where λ_i represents the eigenvalues of L, then the positivity of system (3) can be guaranteed, and under zero initial conditions, the reachable set $\mathcal{R}_{\eta_i}$, $i\in\overline{1,N}$ can be bounded by the ellipsoidal region $\Phi(P)$.

Proof: Consider the Lyapunov candidate $V(t)=\hat{\eta}_1^T(t)(I_{N-1}\otimes P)\hat{\eta}_1(t)$, with $P>0$. Along the trajectory of (11), one has

$$\dot{V}(t)+\alpha V(t)-\frac{\alpha}{\bar{\omega}_p}\omega^T(t)\omega(t) = \begin{bmatrix}\hat{\eta}_1(t)\\ \omega(t)\end{bmatrix}^T\begin{bmatrix}\Gamma+\alpha(I_{N-1}\otimes P) & U_1^T\otimes PB_\omega\\ U_1\otimes B_\omega^TP & -\frac{\alpha}{\bar{\omega}_p}I_{n_\omega}\end{bmatrix}\begin{bmatrix}\hat{\eta}_1(t)\\ \omega(t)\end{bmatrix}.$$

To guarantee $\dot{V}(t)+\alpha V(t)-(\alpha/\bar{\omega}_p)\omega^T(t)\omega(t)<0$, the following inequality needs to be satisfied:

$$\begin{bmatrix}\Gamma+\alpha(I_{N-1}\otimes P) & U_1^T\otimes PB_\omega\\ U_1\otimes B_\omega^TP & -\frac{\alpha}{\bar{\omega}_p}I_{n_\omega}\end{bmatrix}<0. \quad (26)$$

With the similar manipulation of Proof in Theorem 1, (26) is equivalent to inequalities (25). Thus, $\dot{V}(t)+\alpha V(t)-(\alpha/\bar{\omega}_p)\omega^T(t)\omega(t)<0$ is obtained.

Based on $\dot{V}(t)+\alpha V(t)-(\alpha/\bar{\omega}_p)\omega^T(t)\omega(t)<0$, there exists $\mathcal{V}(t)>0$ such that

$$\dot{V}(t)+\alpha V(t)-\frac{\alpha}{\bar{\omega}_p}\omega^T(t)\omega(t)+\mathcal{V}(t)=0. \quad (27)$$

After integrating (27), one can get that

$$\begin{aligned}V(t) &= e^{-\alpha t}V(0)+\int_0^t e^{-\alpha(t-\tau)}\left(\frac{\alpha}{\bar{\omega}_p}\omega^T(\tau)\omega(\tau)-\mathcal{V}(\tau)\right)d\tau\\ &\leq e^{-\alpha t}V(0)+\frac{\alpha}{\sum_{i=1}^N\bar{\omega}_{pi}}\sum_{i=1}^N\bar{\omega}_{pi}\int_0^t e^{-\alpha(t-\tau)}d\tau\\ &= e^{-\alpha t}V(0)+\frac{\alpha}{\alpha}(1-e^{-\alpha t}).\end{aligned} \quad (28)$$

Under zero initial conditions, $V(0)=0$. Moreover, $e^{-\alpha t}>0$ with $\alpha>0$. Thus, based on (28) $V(t)<1$ can be obtained for all $t\geq 0$, that is $\hat{\eta}_1^T(t)(I_{N-1}\otimes P)\hat{\eta}_1(t)<1$. According to Lemma 1, the reachable set $\mathcal{R}_{\eta_i}$, $i\in\overline{1,N}$ under peak-bounded disturbance input can be bounded by $\Phi(P)$.

Therefore, in terms of Definition 1, constraints (14), (15) and (25) can guarantee the reachable set-based positive consensus of system (1) under peak-bounded disturbance input. ■

C. Equivalent Design Conditions

The reachable set-based positive consensus conditions of multiagent system (1) under energy-bounded and peak-bounded disturbance inputs have been developed in Theorem 1 and Theorem 2, respectively. It can be observed that conditions in Theorem 1 and Theorem 2 are nonconvex and hard to compute directly. To achieve the control protocol design, the following equivalent conditions will be established to promote the computation of controller gain K.

Theorem 3: Theorem 1 is equivalent to the following conditions: if there exist matrices W, Ξ_i, $i\in\overline{1,N-1}$, $Q>0$ and a diagonal matrix $X>0$ such that

$$B_uW\succeq 0, \quad (29)$$

$$AX-\xi_{\max}B_uW\ \ \text{is Metzler}, \quad (30)$$

$$\begin{bmatrix}\Upsilon_i & B_\omega & \lambda_iB_uW-Q\\ B_\omega^T & -\frac{1}{\bar{\omega}_e^2}I_{n_\omega} & 0\\ \lambda_iW^TB_u^T-Q & 0 & -X\end{bmatrix}<0,\quad i\in\overline{1,N-1} \quad (31)$$

where $\Upsilon_i = \mathbf{sym}(AQ - \lambda_i B_u W \Xi_i^T B_u^T) + B_u \Xi_i X \Xi_i^T B_u^T$, are satisfied.

To achieve the reachable set-based positive consensus, under the above-obtained conditions (29)–(31), the controller gain is determined as follows:

$$K = WX^{-1}. \tag{32}$$

Proof: First, we need to prove that (14) and (15) are equivalent to (29) and (30). Since $W = KX$, then it gives that $B_u W = B_u KX$, $AX - \xi_{\max} B_u W = AX - \xi_{\max} B_u KX$. By referring to [21], with X being a diagonal matrix and $X > 0$, it is easy to obtain that $B_u K \succeq 0$ and $A - \xi_{\max} B_u K$ is Metzler, which are equivalent to (29) and (30).

Next, the equivalence between (16) and (31) needs to be clarified.

(16) ⇒ (31): For (16), pre- and post-multiplying with $\mathrm{diag}(Q, I_{n_\omega})$, where $Q = P^{-1} > 0$, it holds that

$$\begin{bmatrix} \mathbf{sym}(AQ - \lambda_i B_u KQ) & B_\omega \\ B_\omega^T & -\frac{1}{\bar{\omega}_e^2} I_{n_\omega} \end{bmatrix} < 0. \tag{33}$$

Based on (33), a diagonal matrix $\mathcal{X} > 0$ and a sufficiently large scalar $\kappa > 0$ can always be found such that

$$\begin{bmatrix} \mathbf{sym}(AQ - \lambda_i B_u KQ) & B_\omega \\ B_\omega^T & -\frac{1}{\bar{\omega}_e^2} I_{n_\omega} \end{bmatrix} - \begin{bmatrix} -Q \\ 0 \end{bmatrix} (-\kappa \mathcal{X})^{-1} \begin{bmatrix} -Q & 0 \end{bmatrix} < 0 \tag{34}$$

since $\begin{bmatrix} -Q \\ 0 \end{bmatrix} (-\kappa \mathcal{X})^{-1} \begin{bmatrix} -Q & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\kappa} Q \mathcal{X}^{-1} Q & 0 \\ 0 & 0 \end{bmatrix}$, and a sufficiently large κ generates $(1/\kappa) \to 0$. By letting $X = \kappa \mathcal{X}$, and applying the Schur complement equivalence, inequality (34) is equivalent to

$$\begin{bmatrix} \mathbf{sym}(AQ - \lambda_i B_u KQ) & B_\omega & -Q \\ B_\omega^T & -\frac{1}{\bar{\omega}_e^2} I_{n_\omega} & 0 \\ -Q & 0 & -X \end{bmatrix} < 0. \tag{35}$$

Pre- and post-multiplying (35) with upper triangular matrix $\begin{bmatrix} I_{n_x} & 0 & -\lambda_i B_u K \\ 0 & I_{n_\omega} & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix}$ and its transpose, one can get that

$$\begin{bmatrix} \mathbf{sym}(AQ) - \lambda_i^2 B_u KXK^T B_u^T & B_\omega & \lambda_i B_u KX - Q \\ B_\omega^T & -\frac{1}{\bar{\omega}_e^2} I_{n_\omega} & 0 \\ \lambda_i XK^T B_u^T - Q & 0 & -X \end{bmatrix} < 0. \tag{36}$$

By letting $\Xi_i = \lambda_i K$ and $W = KX$, it follows that:

$$B_u (\Xi_i - \lambda_i K) X (\Xi_i - \lambda_i K)^T B_u^T = 0 \tag{37}$$

which gives $\lambda_i^2 B_u KXK^T B_u^T = \mathbf{sym}(\lambda_i B_u W \Xi_i^T B_u^T) - B_u \Xi_i X \Xi_i^T B_u^T$. Therefore, (31) can be obtained from (36) and (37).

(31) ⇒ (16): Since (31) holds and $W = KX$, by substituting W with KX in (31), $X > 0$ can always give that $B_u (\Xi_i - \lambda_i K) X (\Xi_i - \lambda_i K)^T B_u^T \geq 0$. Thus, with $\lambda_i^2 B_u KXK^T B_u^T \geq \mathbf{sym}(\lambda_i B_u W \Xi_i^T B_u^T) - B_u \Xi_i X \Xi_i^T B_u^T$, (31) can give that (36) holds. Pre- and post-multiplying (36) with $\begin{bmatrix} I_{n_x} & 0 & \lambda_i B_u K \\ 0 & I_{n_\omega} & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix}$ and its transpose, the inequality (35) can be obtained. After using Schur complement equivalency, (33) holds. Pre- and post-multiplying (33) with $\mathrm{diag}(P, I)$, (16) can be guaranteed.

In summary, the equivalence of conditions in Theorem 3 and conditions in Theorem 1 has been established. ■

Theorem 4: Theorem 2 is equivalent to the following conditions: given a scalar $\alpha > 0$, if there exist matrices W, Ξ_i, $i \in \overline{1, N-1}$, $Q > 0$ and a diagonal matrix $X > 0$ such that (29), (30) and

$$\begin{bmatrix} \Upsilon_i + \alpha Q & B_\omega & \lambda_i B_u W - Q \\ B_\omega^T & -\frac{\alpha}{\bar{\omega}_p} I_{n_\omega} & 0 \\ \lambda_i W^T B_u^T - Q & 0 & -X \end{bmatrix} < 0, \quad i \in \overline{1, N-1} \tag{38}$$

hold. To achieve the reachable set-based positive consensus, under conditions (29), (30) and (38), the controller gain is determined by $K = WX^{-1}$.

The proof of Theorem 4 is analogous to that of Theorem 3 and thus is omitted here.

Remark 2: Theorem 3 and Theorem 4 provide the controller design conditions for achieving the reachable set-based positive consensus in the presence of energy-bounded and peak-bounded non-negative disturbances, respectively. It is noted that the derived positivity conditions (29) and (30) are similar to those in [21], as the non-negative B_ω in the original system (1) naturally guarantees that $(I_N \otimes B_\omega)$ is non-negative in the closed-loop system (3). However, conditions (31) in Theorem 3 and (38) in Theorem 4 can ensure the reachable set-based consensus, which differ from the consensus condition in [21]. Moreover, in the presence of energy-bounded disturbances, the condition (25) cannot be simply converted to a decoupling condition (16) as in [34], as the positivity constraint cannot be incorporated equivalently using the design method of Theorem 1 in [34]. Therefore, the above equivalent design conditions are developed.

IV. Optimization of Bounding Region

The reachable set $\mathcal{R}_{\eta_i}$, $i \in \overline{1, N}$ of the consensus error $\eta_i(t)$ are bounded by the ellipsoidal region $\Phi(P)$, where $P = Q^{-1}$. In this section, the optimization problem to minimize the bounding region $\Phi(P)$ is investigated. By regarding Q as a decision variable, it holds that

$$Q - \epsilon I_{n_x} \leq 0, \quad \epsilon > 0. \tag{39}$$

With $P = Q^{-1}$, and Schur complement equivalence, (39) can be rewritten as $\begin{bmatrix} -Q^{-1} & I_{n_x} \\ I_{n_x} & -\epsilon I_{n_x} \end{bmatrix} < 0$. Then, it holds that $P \geq (1/\epsilon) I_{n_x}$. A smaller bounding region $\Phi(P)$ can be obtained by maximizing $(1/\epsilon)$. The minimization goal of the bounding region can be achieved through the following optimization problem:

$$\text{minimizing } \epsilon \text{ subject to} \\ \begin{cases} (29)-(31),\ (39), & \text{based on Theorem 3} \\ (29),\ (30),\ (38),\ (39), & \text{based on Theorem 4.} \end{cases}$$

Algorithm 1

Step 1: Set the initial iteration label $m = 1$, and $\rho^{(0)} = 0$. Compute a matrix H to guarantee that $A - \lambda_i B_u H$ are Hurwitz. Set $\Xi_i^{(1)} = \lambda_i H$.

Step 2: With fixed $\Xi_i = \Xi_i^{(m)}$, solve the following problems:

C1) Under the energy-bounded disturbance input (Based on Theorem 3): Minimize $\rho^{(m)}$ to solve matrices W, $Q > 0$ and diagonal matrix $X > 0$ subject to (29), (30) and

$$\begin{bmatrix} \Upsilon_i & B_\omega & \lambda_i B_u W - Q \\ B_\omega^T & -\frac{1}{\bar{\omega}_e^2} I_{n_\omega} & 0 \\ \lambda_i W^T B_u^T - Q & 0 & -X \end{bmatrix} < \rho^{(m)} I_{2n_x+n_u}.$$

C2) Under the peak-bounded disturbance input (Based on Theorem 4): Minimize $\rho^{(m)}$ to solve matrices W, $Q > 0$ and diagonal matrix $X > 0$ subject to (29), (30) and

$$\begin{bmatrix} \Upsilon_i + \alpha Q & B_\omega & \lambda_i B_u W - Q \\ B_\omega^T & -\frac{\alpha}{\bar{\omega}_p} I_{n_\omega} & 0 \\ \lambda_i W^T B_u^T - Q & 0 & -X \end{bmatrix} < \rho^{(m)} I_{2n_x+n_u}.$$

Step 3: If $\rho^{(m)} \leq 0$, we find a set of feasible solution, go to **Step** 5. Otherwise, go to the next step.

Step 4: If $|\rho^{(m)} - \rho^{(m-1)}| < \varsigma$, where $\varsigma > 0$ is a prescribed small tolerance, the algorithm fails to find a feasible solution. STOP. Otherwise, update $\Xi_i^{(m+1)}$ with $\Xi_i^{(m+1)} = \lambda_i W X^{-1}$, and set $m = m + 1$, then go back to **Step** 2.

Step 5: With the obtained feasible Ξ_i, solve the following optimization problems:

C1) Under the energy-bounded disturbance input:
Op: minimize ϵ for solving matrices W, $Q > 0$ and diagonal matrix $X > 0$ subject to (29)–(31) of Theorem 3 and (39).

C2) Under the peak-bounded disturbance input:
Op: minimize ϵ for solving matrices W, $Q > 0$ and diagonal matrix $X > 0$ subject to (29), (30), (38) of Theorem 4 and (39).

Step 6: Output the solution of W, Q, X and performance index ϵ. Compute the control gain $K = WX^{-1}$ and the ellipsoidal region $\Phi(Q^{-1})$.

A heuristic algorithm (Algorithm 1) is developed to achieve the computation and optimization of reachable set-based positive consensus problems.

V. Illustrative Examples

This section provides a numerical example in a multiinput case and a positive electric circuit system in a single-input case to validate the efficiency of the proposed method.

Example 1: Consider a positive multiagent system in a multiinput case with three agents. Each agent is given in the form of (1) with the parameters:

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 0.5 & -1 & 1 \\ 1 & 1.5 & -1.5 \end{bmatrix}, \ B_u = \begin{bmatrix} 1 & 2 \\ 0.5 & 1 \\ 2 & 1.5 \end{bmatrix}, \ B_\omega = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

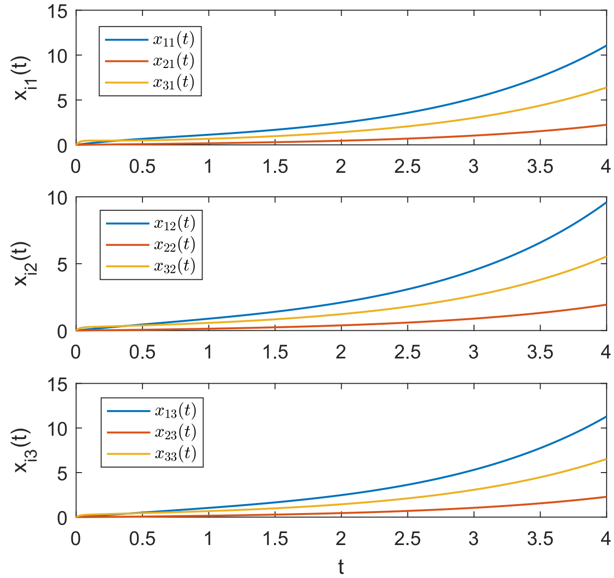

Fig. 1. All agent state components of open-loop system (C1).

The communication graph is supposed to be undirected and connected, and the Laplacian matrix is given below

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

It can be obtained that A is Metzler, B_u and B_ω are nonnegative. Performing the transformation of $\mathcal{L}$, the eigenvalues of L can be obtained as, $\lambda_1 = 1$ and $\lambda_2 = 3$.

In Algorithm 1, the initial values of matrices Ξ_i, $i = 1, 2$, which guarantee $A - \lambda_i B_u H$ are Hurwitz, are obtained as follows:

$$\Xi_1^{(1)} = \begin{bmatrix} 0.0844 & 0.2184 & -0.1753 \\ -0.2274 & 0.8023 & 0.4545 \end{bmatrix}$$
$$\Xi_2^{(1)} = \begin{bmatrix} 0.2533 & 0.6553 & -0.5258 \\ -0.6822 & 2.4068 & 1.3634 \end{bmatrix}.$$

Two kinds of disturbance input including energy-bounded and peak-bounded disturbances are considered for simulation verification.

C1) *Under Energy-Bounded Disturbance Input:* Consider the disturbance input of each agent as: $\omega_1 = 2e^{-1.6t}$, $\omega_2 = 0.2e^{-0.4t}$, $\omega_3 = 20e^{-40t}$, which indicates $\bar{\omega}_e = \sqrt{1.25 + 0.5 + 0.5} = 1.5$. The agent dynamics without control can be illustrated via Fig. 1. It can be seen that the open-loop system is positive but nonconsensus in Fig. 1. After conducting the iteration of Algorithm 1 based on Theorem 3, the following results can be obtained:

$$K = \begin{bmatrix} 0.0693 & -0.0028 & 2.0884 \\ 0.0017 & 0.5006 & -0.7942 \end{bmatrix}$$

and $\epsilon = 1.1994$, $P = Q^{-1} = \begin{bmatrix} 33.2476 & -34.8079 & -13.6362 \\ -34.8079 & 41.4192 & 9.3571 \\ -13.6362 & 9.3571 & 15.2846 \end{bmatrix}$. Under zero

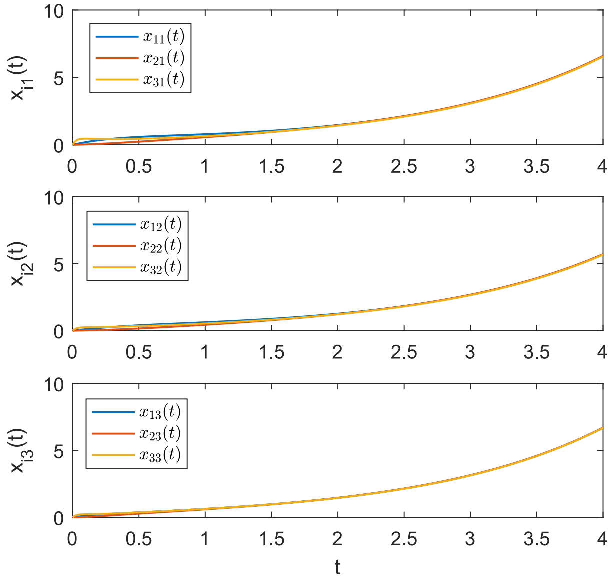

Fig. 2. All agent state components of closed-loop system (C1).

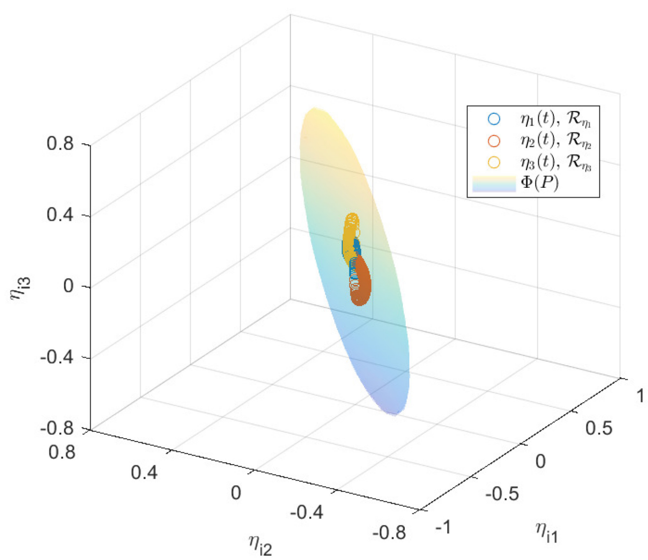

Fig. 3. Reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$ and ellipsoidal region $\Phi(P)$ of closed-loop system (C1).

initial conditions, the closed-loop system dynamics with the obtained K are demonstrated in Figs. 2 and 3. Under the given energy-bounded (exponential decay type) disturbances, it can be observed that with the obtained K, the consensus with positivity preservation of the closed-loop system is achieved in Fig. 2. Moreover, regarding the reachability of the consensus error when the energy-bounded disturbances are present, for each agent i, $i = 1, 2, 3$, the reachable set $\mathcal{R}_{\eta_i}$ (all trajectories) of consensus error $\eta_i(t)$ generated by the error dynamic system is depicted in Fig. 3. The obtained ellipsoidal region $\Phi(P)$ is also included in Fig. 3. It appears that the reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$, is confined within the ellipsoid $\Phi(P)$. Thus, the reachable set-based positive consensus is achieved.

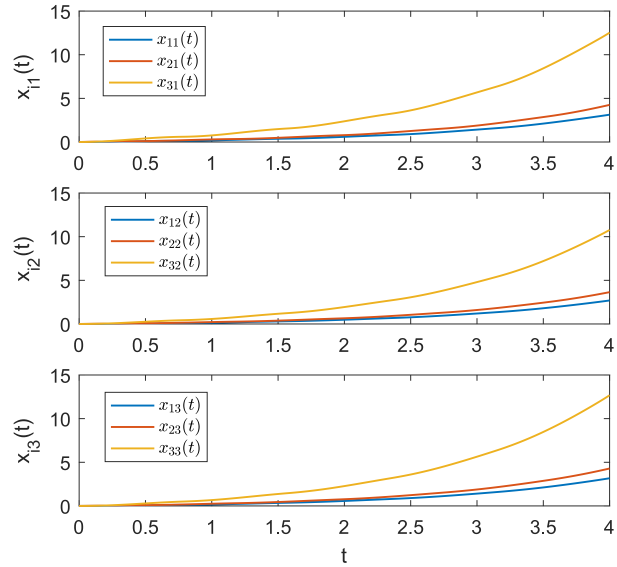

Fig. 4. All agent state components of open-loop system (C2).

C2) *Under Peak-Bounded Disturbance Input:* Consider the disturbance input of each agent as: $\omega_1 = 0.3|\sin 4t|$, $\omega_2 = 0.4|\cos 4t|$, $\omega_3 = 1.2|\sin 4t|$, which indicates $\bar{\omega}_p = 1.69$. The considered disturbance is rectified sinusoidal wave type, which is peak-bounded but not energy-bounded. Without control, the agent dynamics of the open-loop system under peak-bounded disturbances are demonstrated in Fig. 4. It is obvious that all agents are not consensus. Given $\alpha = 1$, with the same initial values of matrices Ξ_i, after iteration of Algorithm 1 based on Theorem 4, the controller gain can be generated as follows:

$$K = \begin{bmatrix} 0.0627 & -0.0013 & 14.1702 \\ 0.0023 & 0.5002 & -6.8351 \end{bmatrix},$$

and $\epsilon = 1.7869$, $P = Q^{-1} = \begin{bmatrix} 12.5110 & -15.0321 & -3.7904 \\ -15.0321 & 19.6835 & 3.2691 \\ -3.7904 & 3.2691 & 12.1161 \end{bmatrix}$. With the obtained K and P, under zero initial conditions, the agent dynamics of the closed-loop system are displayed in Fig. 5. It can be seen that the positivity is preserved, and the consensus is not strictly guaranteed due to the constant fluctuation of rectified sinusoidal wave-type disturbance input. However, after control, the reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$ of the consensus error $\eta_i(t)$ are always resides in a small ellipsoidal region $\Phi(P)$ in Fig. 6. According to the proposed definition, the reachable set-based positive consensus of the considered system has been achieved.

Example 2: Consider a positive multiagent system whose agents are positive electric circuits [24]. It is assumed that the system is affected by non-negative disturbance input. The disturbance input $\omega(t)$ comes into the system from the input.

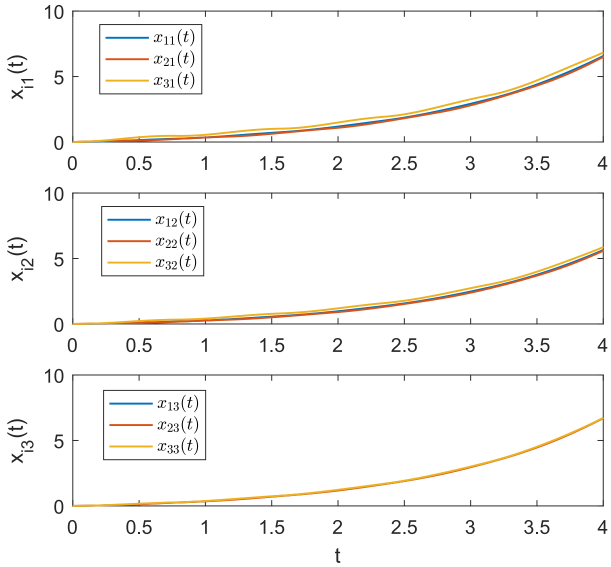

Fig. 5. All agent state components of closed-loop system (C2).

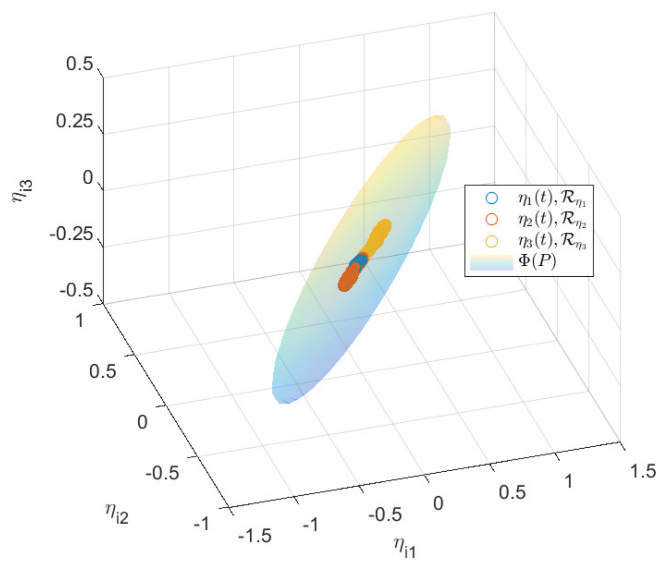

Fig. 6. Reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$ and ellipsoidal region $\Phi(P)$ of closed-loop system (C2).

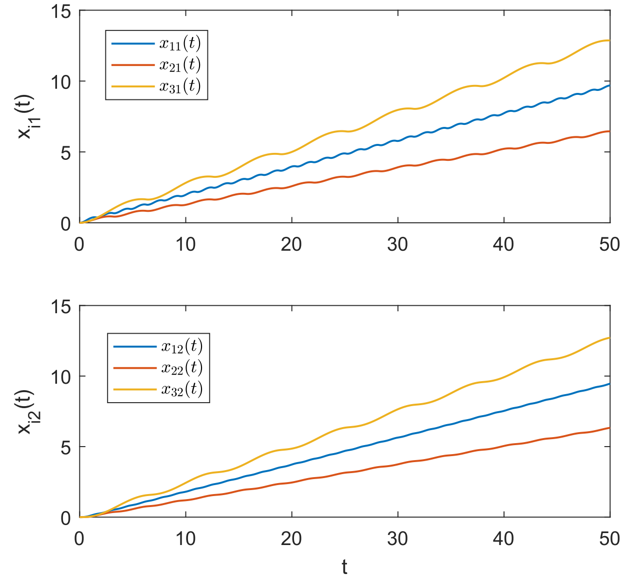

Fig. 7. All agent state components of open-loop system.

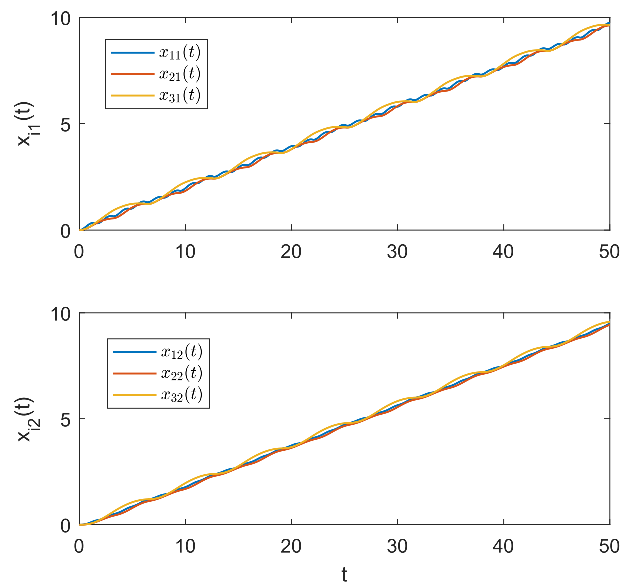

Fig. 8. All agent state components of closed-loop system.

The system parameters are given below

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & \frac{R_1}{L_1} \\ \frac{R_1}{L_2} & -\frac{R_1}{L_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, B_u = B_\omega = \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The considered system contains three positive electric circuit agents. The communication topology over agents is undirected, and the Laplacian matrix is

$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

It is observed that $\xi_{\max} = 2$. Performing the transformation of $\mathcal{L}$, the eigenvalues of L are $\lambda_1 = \lambda_2 = 3$.

In this system, the peak-bounded disturbance input is considered. The disturbance signal of each circuit agent is given as follows: $\omega_1 = 0.6|\sin 2t|$, $\omega_2 = 0.4|\sin t|$, $\omega_3 = 0.8|\sin 0.5t|$, which gives $\bar{\omega}_p = 1.16$. As seen in Fig. 7, the state trajectories of the open-loop system are fluctuating due to the rectified sinusoidal disturbance input, and fail to reach a consensus.

Then, given $\alpha = 1$, by minimizing ϵ via Algorithm 1 based on Theorem 4, the controller gain is obtained as follows:

$$K = \begin{bmatrix} 0.1191 & 0.4195 \end{bmatrix}$$

and $\epsilon = 0.9140$, $P = Q^{-1} = \begin{bmatrix} 2.3402 & -1.1701 \\ -1.1701 & 2.1928 \end{bmatrix}$.

Under the obtained controller, the agent dynamics of the closed-loop system are given in Fig. 8. The state trajectories of closed-loop systems also fluctuate with the rectified sinusoidal disturbance input and distributed state-feedback control protocol. The strict consensus cannot be achieved under such disturbance input. However, the reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$

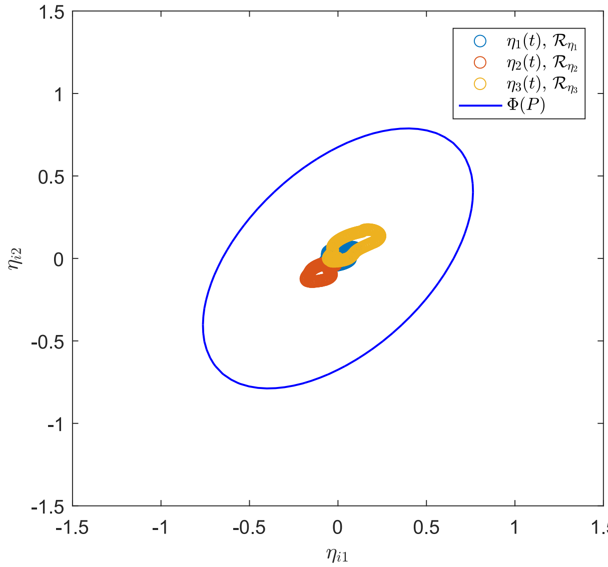

Fig. 9. Reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$ and ellipsoidal region $\Phi(P)$ of closed-loop system.

of the consensus error $\eta_i(t)$, and the obtained ellipsoid $\Phi(P)$ are shown in Fig. 9. Observing Figs. 8 and 9, it is evident that the positivity of the closed-loop system is preserved and the reachable set $\mathcal{R}_{\eta_i}$, $i = 1, 2, 3$ are enclosed by the bounding region $\Phi(P)$. The reachable set-based positive consensus is therefore achieved.

VI. Conclusion

This work has studied the reachable set-based consensus problem for positive multiagent systems subject to bounded external disturbances. A distributed control protocol has been developed to achieve the reachable set-based consensus by ensuring that the closed-loop system remains positive and the consensus error is bounded within an ellipsoidal region. The equivalent conditions are obtained to promote the control design, and a heuristic algorithm is proposed to achieve effective computation and optimization of the ellipsoidal region. Simulation results have been supplied to illustrate the effectiveness of the developed scheme. Future work will extend to reachable set-based synthesis problems for multiagent systems with directed communication topologies or other communication topologies, such as switching topologies.

References

[1] L. Cruz-Piris, D. Rivera, S. Fernandez, and I. Marsa-Maestre, "Optimized sensor network and multi-agent decision support for smart traffic light management," *Sensors*, vol. 18, no. 2, p. 435, 2018.
[2] J. Ota, "Multi-agent robot systems as distributed autonomous systems," *Adv. Eng. Inform.*, vol. 20, no. 1, pp. 59–70, 2006.
[3] T. Nagata and H. Sasaki, "A multi-agent approach to power system restoration," *IEEE Trans. Power Syst.*, vol. 17, no. 2, pp. 457–462, 2002.
[4] X.-P. Xie, Q. Wang, M. Chadli, and K. Shi, "Relaxed observer-based state estimation of discrete-time Takagi–Sugeno fuzzy systems based on an augmented matrix approach," *IEEE Trans. Fuzzy Syst.*, vol. 30, no. 9, pp. 4025–4030, 2021.
[5] K. You, Z. Li, and L. Xie, "Consensus condition for linear multi-agent systems over randomly switching topologies," *Automatica*, vol. 49, no. 10, pp. 3125–3132, 2013.
[6] M. E. Valcher and I. Zorzan, "On the consensus of homogeneous multiagent systems with positivity constraints," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5096–5110, 2017.
[7] K. Liu, Z. Ji, G. Xie, and L. Wang, "Consensus for heterogeneous multi-agent systems under fixed and switching topologies," *J. Franklin Inst.*, vol. 352, no. 9, pp. 3670–3683, 2015.
[8] Y. Zheng, J. Ma, and L. Wang, "Consensus of hybrid multi-agent systems," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 4, pp. 1359–1365, 2017.
[9] W. Hou, M. Y. Fu, and H. Zhang, "Consensusability of linear multi-agent systems with time delay," *Int. J. Robust Nonlin. Control*, vol. 26, no. 12, pp. 2529–2541, 2016.
[10] Z. Ahmed, M. M. Khan, M. A. Saeed, and W. Zhang, "Consensus control of multi-agent systems with input and communication delay: A frequency domain perspective," *ISA Trans.*, vol. 101, pp. 69–77, 2020.
[11] L. Ding, Q.-L. Han, X. Ge, and X.-M. Zhang, "An overview of recent advances in event-triggered consensus of multiagent systems," *IEEE Trans. Cybern.*, vol. 48, no. 4, pp. 1110–1123, 2017.
[12] T.-Y. Zhang, D. Ye, and X. Zhao, "Completely event-triggered consensus for multiagent systems with directed switching topologies," *IEEE Trans. Cybern.*, vol. 52, no. 8, pp. 7865–7874, 2021.
[13] Y. Ao and Y. Jia, "Robust H_2/H_∞ group consensus control for linear clusters over signed digraphs," *J. Franklin Inst.*, vol. 357, no. 12, pp. 7556–7580, 2020.
[14] J. Wang, G. Wen, Z. Duan, Y. Hu, and W. He, "Distributed H_∞ robust control of multiagent systems with uncertain parameters: Performance-region-based approach," *IEEE Trans. Syst., Man, Cybern.: Syst.*, vol. 52, no. 5, pp. 2888–2898, 2021.
[15] J. Zhou, Y. Wang, X. Zheng, Z. Wang, and H. Shen, "Weighted H_∞ consensus design for stochastic multi-agent systems subject to external disturbances and ADT switching topologies," *Nonlinear Dyn.*, vol. 96, no. 2, pp. 853–868, 2019.
[16] P. Shi and J. Yu, "Dissipativity-based consensus for fuzzy multiagent systems under switching directed topologies," *IEEE Trans. Fuzzy Syst.*, vol. 29, no. 5, pp. 1143–1151, 2020.
[17] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. New York, NY, USA: Wiley & Sons, 2000, vol. 50.
[18] X. Song, J. Lam, B. Zhu, and C. Fan, "Interval observer-based fault-tolerant control for a class of positive Markov jump systems," *Inf. Sciences*, vol. 590, pp. 142–157, 2022.
[19] A. Khanafer, T. Başar, and B. Gharesifard, "Stability of epidemic models over directed graphs: A positive systems approach," *Automatica*, vol. 74, pp. 126–134, 2016.
[20] Y. Li et al., "A sliding mode controller for vehicular traffic flow," *Physica A Stat. Mechan. Appl.*, vol. 462, pp. 38–47, 2016.
[21] J. J. Liu, J. Lam, and Z. Shu, "Positivity-preserving consensus of homogeneous multiagent systems," *IEEE Trans. Autom. Control*, vol. 65, no. 6, pp. 2724–2729, 2019.
[22] J. J. Liu, J. Lam, B. Zhu, X. Wang, Z. Shu, and K.-W. Kwok, "Nonnegative consensus tracking of networked systems with convergence rate optimization," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 33, no. 12, pp. 7534–7544, 2021.
[23] H. Wu and H. Su, "Observer-based consensus for positive multiagent systems with directed topology and nonlinear control input," *IEEE Trans. Syst., Man, Cybern.: Syst.*, vol. 49, no. 7, pp. 1459–1469, 2018.
[24] J. J. Liu, J. Lam, Y. Wang, Y. Cui, and Z. Shu, "Robust and nonfragile consensus of positive multiagent systems via observer-based output-feedback protocols," *Int. J. Robust Nonlin. Control*, vol. 30, no. 14, pp. 5386–5403, 2020.
[25] W. Xiang, H.-D. Tran, and T. T. Johnson, "Output reachable set estimation for switched linear systems and its application in safety verification," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5380–5387, 2017.
[26] J. Ding, E. Li, H. Huang, and C. J. Tomlin, "Reachability-based synthesis of feedback policies for motion planning under bounded disturbances," in *Proc. IEEE Int. Conf. Robot. Autom.*. IEEE, 2011, pp. 2160–2165.
[27] Y. Lin and S. Saripalli, "Collision avoidance for UAVs using reachable sets," in *2015 Int. Conf. Unmanned Aircr. Syst. (ICUAS)*. IEEE, 2015, pp. 226–235.
[28] Y. Xue, C. Liu, and X. Zhang, "State bounding description and reachable set estimation for discrete-time genetic regulatory networks with time-varying delays and bounded disturbances," *IEEE Trans. Syst., Man, Cybern.: Syst.*, vol. 52, no. 10, pp. 6652–6661, 2022.
[29] Y. Chen, J. Lam, and B. Zhang, "Estimation and synthesis of reachable set for switched linear systems," *Automatica*, vol. 63, pp. 122–132, 2016.

[30] W.-W. Che, H. Su, P. Shi, and Z.-G. Wu, "Nonfragile and nonsynchronous synthesis of reachable set for bernoulli switched systems," *IEEE Trans. Syst., Man, Cybern.: Syst.*, vol. 50, no. 2, pp. 726–731, 2017.

[31] X. Xie, C. Fan, K.-W. Kwok, and J. Lam, "A Bernstein polynomial approach to estimating reachable set of periodic piecewise polynomial systems," *IEEE Trans. Autom. Control*, vol. 66, no. 10, pp. 4812–4819, 2020.

[32] C. Fan, J. Lam, X. Xie, and X. Song, "Observer-based output reachable set synthesis for periodic piecewise time-varying systems," *Inf. Sciences*, vol. 571, pp. 246–261, 2021.

[33] P. T. Nam, H. Trinh, and P. N. Pathirana, "Minimization of state bounding for perturbed positive systems with delays," *SIAM J. Control Optim.*, vol. 56, no. 3, pp. 1739–1755, 2018.

[34] W. Hu, W. Huang, Y. Huang, S. Chen, and A.-G. Wu, "On reachable set estimation of multi-agent systems," *Neurocomputing*, vol. 401, pp. 69–77, 2020.

[35] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Math. J.*, vol. 23, no. 2, pp. 298–305, 1973.

[36] Z. Li, Z. Duan, and F. L. Lewis, "Distributed robust consensus control of multi-agent systems with heterogeneous matching uncertainties," *Automatica*, vol. 50, no. 3, pp. 883–889, 2014.

[37] P. Lin, Y. Jia, and L. Li, "Distributed robust H_∞ consensus control in directed networks of agents with time-delay," *Syst. Control Lett.*, vol. 57, no. 8, pp. 643–653, 2008.

Chenchen Fan received the B.E. degree in automation and the M.E. degree in control science and engineering from the Harbin Institute of Technology, Harbin, China, in 2014 and 2016, respectively, and the Ph.D. degree in control engineering from The University of Hong Kong, Hong Kong, in 2021.

She is currently a Postdoctoral Fellow with the Department of Mechanical Engineering, The University of Hong Kong, and the Centre for Transformative Garment Production, Hong Kong. Her research interests include robust control and filtering, reachable set, periodic systems, and intelligent systems.

Dr. Fan was awarded the Hong Kong Ph.D. Fellowship in 2016. She also serves as an Associate Editor for ICRA 2024.

James Lam (Fellow, IEEE) received the B.Sc. degree (First-Class Hons.) in mechanical engineering from the University of Manchester, Manchester, U.K., in 1983, and the M.Phil. degree in control engineering & operational research and Ph.D. degree in control engineering from the University of Cambridge, Cambridge, U.K., in 1985 and 1988, respectively.

He is a Croucher Scholar, a Croucher Fellow, and a Distinguished Visiting Fellow of the Royal Academy of Engineering, London, U.K., and a Cheung Kong Chair Professor. Prior to joining The University of Hong Kong, Hong Kong, in 1993, where he is currently a Chair Professor of Control Engineering. He was a Faculty Member with the City University of Hong Kong, Hong Kong, and the University of Melbourne, Parkville, VIC, Australia. His research interests include model reduction, robust synthesis, delay, singular systems, stochastic systems, multidimensional systems, positive systems, networked control systems, and vibration control.

Prof. Lam was awarded the Ashbury Scholarship, the A.H. Gibson Prize, and the H. Wright Baker Prize for his academic performance. He is a Highly Cited Researcher in Engineering from 2014 to 2020, Cross-Field in 2021, and Computer Science in 2015. He is an Editor-in-Chief of *IET Control Theory and Applications*, *Journal of The Franklin Institute*, and *Proceedings of the Institution of Mechanical Engineers—Part I: Journal of Systems and Control Engineering*, a Subject Editor of *Journal of Sound and Vibration*, an Editor of *Asian Journal of Control*, a Senior Editor of *Cogent Engineering*, a Section Editor of *IET Journal of Engineering*, a Consulting Editor of *International Journal of Systems Science*, and an Associate Editor of *Automatica* and *Multidimensional Systems and Signal Processing*. He is a Chartered Mathematician, a Chartered Scientist, a Chartered Engineer, and a Fellow of the Institution of Engineering and Technology, Institute of Mathematics and Its Applications, Institution of Mechanical Engineers, and Hong Kong Institution of Engineers.

Xiujuan Lu received the B.S. degree in automation from Anhui University, Hefei, China, in 2016, and the M.S. degree in control science and engineering from the University of Science and Technology of China, Hefei, in 2019. She is currently pursuing the Ph.D. degree in mechanical engineering with The University of Hong Kong, Hong Kong.

Her research interests include multiagent systems and cone-invariant systems.

Jason J. R. Liu (Member, IEEE) received the B.Eng. degree in automation from Sun Yat-sen University, Guangzhou, China, in 2013, and the Ph.D. degree in control engineering from The University of Hong Kong, Hong Kong, in 2018.

For his doctoral and postdoctoral research, he was supported by the HKU Postgraduate Fellowship and the PH-ITF Fellowship, respectively. He is currently an Assistant Professor with the Department of Electromechanical Engineering, University of Macau, Macau, China. His research interests include networked control systems, multiagent systems, positive systems, intelligent systems, and learning-based control.

Prof. Liu was an early career Editorial Board Member for *Franklin Open*. More information is available on his webpage: https://www.fst.um.edu.mo/personal/jasonliu/.

Xiaomei Wang (Member, IEEE) received the B.E. degree in automation from the Harbin Institute of Technology, Harbin, China, in 2014, the M.E. degree in control science and engineering from the Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen, China, in 2016, and the Ph.D. degree in robotics from The University of Hong Kong, Hong Kong, in 2020.

She is currently a Postdoctoral Fellow with the Department of Mechanical Engineering, The University of Hong Kong, and Multiscale Medical Robotics Center, Hong Kong. Her research interests include learning-based robot control and sensing, surgical robotics, and continuum robot control.

Dr. Wang serves as an Associate Editor for ICRA 2022–2024 and RoboSoft 2022–2023, and an Area Chair for IPCAI 2022.

Ka-Wai Kwok (Senior Member, IEEE) received the B.Eng. and M.Phil. degrees in automation and computer-aided engineering from the Chinese University of Hong Kong, Hong Kong, in 2003 and 2005, respectively, and the Ph.D. degree in computing from the Hamlyn Center for Robotic Surgery, Department of Computing, Imperial College London, London, U.K., in 2012.

He is currently an Associate Professor with the Department of Mechanical Engineering, The University of Hong Kong (HKU), Hong Kong. Prior to joining HKU in 2014, he worked as a Postdoctoral Fellow with Imperial College London in 2012 for surgical robotics research. To date, he has coauthored 152 publications with >50 clinical fellows, >90 engineering scientists, and 6 out of 14 invention patents licensed/transferred to industrial partners in support for their commercialization. His research focuses on surgical robotics, intraoperative image processing, and their uses of intelligent and control systems.

Prof. Kwok was awarded the Croucher Foundation Fellowship, which supported his research jointly supervised by advisors from the University of Georgia, Athens, GA, USA, and Brigham and Women's Hospital, Harvard Medical School, Boston, MA, USA, in 2013. His multidisciplinary work has been recognized by >10 international publication awards, mostly under IEEE, particularly in the largest flagship conferences of robotics: e.g., the ICRA Best Conference Paper Award in 2018 and the IROS Toshio Fukuda Young Professional Award in 2020. He also serves as an Associate Editor for *Journal of Systems and Control Engineering*, *IEEE Robotics and Automation Magazine*, and *Annals of Biomedical Engineering*. He is the Principal Investigator of Research Group for Interventional Robotic and Imaging Systems (IRIS), HKU.