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Switched systems approach to state bounding for time delay systems[☆]

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ABSTRACT

This paper investigates the problems of reachable set estimation for dynamic systems with time-varying delay and random delay. Different from the traditional Lyapunov-Krasovskii functional method, a switched systems approach is adopted to study the reachable set estimation problem for time-varying delay systems. Through augmentation, time-varying delay systems can be transformed into switched delay-free systems. The reachable set of a time-varying delay system is estimated based on its corresponding augmented switched system. The number of decision variables in the obtained reachable set estimation condition is small compared with the Lyapunov-Krasovskii functional method. In addition, the problem of mean square estimation of the system state for random delay systems is also studied. The effectiveness of the theoretical findings is verified through several simulation examples.

1. Introduction

Time delay, whether occurs in the system state, the control input, or the measurement, is often inevitable in practical systems and can be a source of instability and poor performance. The future evolution of the system state of a time delay system depends not only on its current value, but also on its past values. Many processes have time delay characteristics in their dynamics. For example, it takes time for the reactants in slow chemical processes to produce the required products. In networked control systems, the components are connected over networked media, the signal transmission from one place to another takes time. In addition, sensors and actuators in feedback loops usually introduce time delay even sometimes the influence of such delay can be ignored. For readers who are interested in time delay systems, please see monographs [1–4].

In the literature, the research results concerning time delay systems can be classified into two types. One is delay-independent conditions, the other is delay-dependent conditions. Delay-dependent conditions are less conservative compared with the delay-independent conditions because they incorporate the information of time delays. Various techniques have been developed in the literature to derive delay-dependent conditions, such as the Lyapunov–Razumikhin method and Lyapunov–Krasovskii functional method [2]. In [5], the authors proposed the equivalence between stability conditions for switched systems and the Lyapunov–Krasovskii functional stability conditions for discrete-time delay systems. This provides us another method to investigate time delay systems.

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As a basic concept in control theory, the reachable set of dynamic systems has received growing attention in the literature these years. The reachable set of a dynamic system is defined as the set of all the system states that can be reached from the origin under a prescribed class of inputs. It is generally difficult to characterise the exact region of the reachable set. An alternative approach to studying the reachable set is to determine a region that contains the reachable set, and this region should be as small as possible. This is referred to the reachable set estimation problem in the literature. The reachable set estimation conditions for linear time-invariant systems under unit-energy inputs, component-wise unit-energy inputs, and unit-peak inputs were proposed in monograph [6]. In that work, the reachable set of the systems under investigation is bounded by some ellipsoids that can be determined by solving linear matrix inequalities (LMIs). Due to the wide presence of time delay, some researchers took it into consideration when investigating the reachable set estimation problem. The Lyapunov–Razumikhin approach was adopted in [7] to study the reachable set estimation for time-varying delay systems. The Lyapunov–Krasovskii functional approach was utilized in [8] to tackle the reachable set estimation for time-varying delay systems. The number of tunable parameters in the reachable set estimation conditions obtained by using the Lyapunov–Krasovskii functional approach is smaller than that obtained by using Lyapunov–Razumikhin approach. Thus, the reachable set estimation conditions obtained in [8] can be easily checked compared with the conditions obtained in [7]. The authors in [9,10] proposed improved and relaxed results on estimating the reachable set of time-delay systems. In addition, the reachable set estimation conditions for linear systems with both time delays and polytopic uncertainties were presented in [11–13]. Furthermore, some researches have extended the reachable set estimation results to more complicated systems. For example, linear systems with distributed delay [14], singular systems [15], positive systems [16], switched systems [17–19], neural networks [20], Markov jump systems [21,22], periodic systems [23], and nonlinear systems [24]. The authors in [25] studied the reachable set bounding problem for linear discrete-time systems with time delay. However, the number of decision matrices in the obtained reachable set estimation conditions in [25] is relatively large. This motivates us to study the reachable set estimation conditions for discrete-time systems with time delay by using the switched systems approach.

The organisation of this paper is as follows. The problems investigated are formulated in Section 2. The main results obtained for the reachable set estimation problems of time-varying delay systems and random delay systems are presented in Section 3. In that section, time delay systems are augmented into delay-free switched systems. The reachable set of time delay systems is estimated based on the switched delay-free switched systems. The effectiveness of the obtained reachable set estimation conditions is verified through simulation examples in Section 4. Finally, a conclusion of this paper is given in Section 5.

Notation: The notations used in this paper are standard. $\mathbb{R}$ and $\mathbb{R}^n$ represent the set of real numbers, and n-dimensional Euclidean space. $\mathbb{E}(m)$ represents the expected value of random variable m. The superscript "T" represents matrix transpose. The notation "$*$" is used to represent terms implied by symmetry in symmetric matrices. Vectors and matrices are assumed to have compatible dimensions for algebraic operations if their dimensions are not explicitly stated.

2. Problem formulation

The discrete-time systems with time delay considered can be described as follows:

$$\begin{cases} x_{k+1} = Ax_k + A_D x_{k-d_k} + B\omega_k \\ x_k = 0, \ \ \forall k \in [-d_M, 0], \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $\omega_k \in \mathbb{R}^{n_\omega}$ is the bounded peak disturbance vector satisfying

$$\omega_k^T \omega_k \le \overline{\omega}^2, \ \ \forall k \ge 0, \tag{2}$$

$\overline{\omega} > 0$ is a scalar. A, A_D and B_ω are constant system matrices. Time-varying delay d_k belongs to the interval $[d_m, d_M]$, where d_m and d_M are positive integers.

The reachable set of system (1) subject to bounded peak disturbances (2) is defined as

$$\mathfrak{R}_x \triangleq \{x_k \mid x_0 = 0, \ x_k, \ \omega_k \text{ satisfy } (1), \ (2), \ k \ge 0\}. \tag{3}$$

Our aim in this paper is to determine a hyper-sphere to bound the reachable set of system (1). The adopted bounding hyper-sphere can be described in the following form:

$$S(r) \triangleq \{\nu \in \mathbb{R}^{n_x} \mid \nu^T \nu = r, \ r > 0\}. \tag{4}$$

Remark 1. The proposed reachable set estimation problem has been considered in [25] by using the traditional Lyapunov–Krasovskii functional method. However, it is difficult to check the feasibility of the reachable set bounding conditions obtained in [25] because the number of decision matrices is large. To solve this problem, we try to propose a novel reachable set estimation condition by using the switched systems method in this paper.

In some circumstances, the variation of the time delay d_k is random. Suppose the random delay d_k can be modelled as a homogeneous Markov chain. The transition probabilities of the time delay are given by

$$\Pr\{d_{k+1} = j | d_k = i\} = \pi_{ij}, \tag{5}$$

where $\pi_{ij} \ge 0$ and $\sum_{j=d_m}^{d_M} \pi_{ij} = 1$ for all $i \in [d_m, d_M]$. Denote the transition probability matrix by $\Pi = \{\pi_{ij}\}$.

For the reachable set of random delay systems, a hyper-sphere-like mean square estimate is given in the following form:

$$\mathcal{S}(r) \triangleq \{v \in \mathbb{R}^{n_x} | \ \mathbb{E}\{v^T v\} = r, \ \ r > 0\}. \tag{6}$$

Remark 2. As pointed out in [22], the reachable set of a dynamic system that is bounded by a hyper-sphere in the mean square sense can be referred to as a mean square estimate of the reachable set. The vector x_k at any time of k is a random variable, it may diverge to infinity; however, it may be bounded in the mean square sense.

3. Main results

Motivated by the technique utilized in [26], dynamic systems with time delay can be augmented into delay-free switched systems. Therefore, the reachable set of time delay systems can be estimated by the method proposed recently for switched systems in [17]. In the following, through the augmentation method, sufficient conditions for the reachable set bound of system (1) will be established.

At time k, if we augment the state variable as

$$X_k = [x_k^T \ x_{k-1}^T \dots \ x_{k-d_M}^T]^T, \tag{7}$$

one can obtain the following augmented system:

$$X_{k+1} = (\overline{A} + \overline{A}_D \overline{E}_{d_k}) X_k + \overline{B}\omega_k, \tag{8}$$

where

$$\overline{A} = \begin{bmatrix} A & 0 & \cdots & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \ \overline{A}_D = \begin{bmatrix} A_D \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \overline{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\overline{E}_{d_k} = \begin{bmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix}, \tag{9}$$

and $\overline{E}_{d_k}$ has all the elements being zero except for the $(1 + d_k)$th block being identity. It can be seen that the augmented system (8) is a delay-free switched system. Let

$$\overline{A}_{d_k} = \overline{A} + \overline{A}_D \overline{E}_{d_k}, \tag{10}$$

then we have

$$X_{k+1} = \overline{A}_{d_k} X_k + \overline{B}\omega_k. \tag{11}$$

The above augmented system is a typical switched system modelled in [17] with $d_M - d_m + 1$ subsystems. The switching signal σ is determined by the time delay d_k. Now we are in the stage to state our results for finding the reachable set bound of time delay systems.

3.1. Time-Varying delay systems

With the above preliminaries, a sufficient condition for finding the bound of the reachable set (3) can be obtained.

Lemma 1. *Consider the time-varying delay system (1) under bounded peak disturbances (2) and zero initial conditions. If there exist matrices* $P_i > 0$, $i \in [d_m, d_M]$, *and scalars* $\alpha_{j,i} \in [0, 1]$, *such that for all* $(i, j) \in [d_m, d_M] \times [d_m, d_M]$,

$$\begin{bmatrix} (\overline{A} + \overline{A}_D \overline{E}_i)^T P_j (\overline{A} + \overline{A}_D \overline{E}_i) - \alpha_{j,i} P_i & (\overline{A} + \overline{A}_D \overline{E}_i)^T P_j \overline{B} \\ * & \overline{B}^T P_j \overline{B} - \frac{1-\alpha_{j,i}}{\varpi^2} I \end{bmatrix} \le 0, \tag{12}$$

then we have

$$X_k^T P_i X_k \le 1, \ \ i \in [d_m, d_M]. \tag{13}$$

Proof. According to Theorem 1 obtained in [17], the proof of this lemma is straightforward if the augmented system (8) is regarded as a switched system. There are $d_M - d_m + 1$ subsystems represented by system matrices $(\overline{A}_{d_m}, \overline{B})$, $(\overline{A}_{d_m+1}, \overline{B})$, $\dots$, $(\overline{A}_{d_M}, \overline{B})$ in the obtained augmented system (8). Note that time delay d_k is independent of d_{k-1} at any time k, thus the governing switching signal is arbitrarily switched. Based on Theorem 1 obtained in [17], if the following condition

$$\begin{bmatrix} \overline{A}_i^T P_j \overline{A}_i - \alpha_{j,i} P_i & \overline{A}_i^T P_j \overline{B} \\ * & \overline{B}^T P_j \overline{B} - \frac{1-\alpha_{j,i}}{\varpi^2} I \end{bmatrix} \le 0, \ \ (i, j) \in [d_m, d_M] \times [d_m, d_M] \tag{14}$$

holds, then we have $X_k^T P_i X_k \le 1, i \in [d_m, d_M]$. Note that condition (14) is equivalent to condition (12) as $\overline{A}_i = \overline{A} + \overline{A}_D \overline{E}_i$. □

The above lemma gives a sufficient condition for finding the reachable set bound of the augmented system (8). However, our target is to determine the reachable set bound of the time-varying delay system (1). This purpose can be achieved by using the following theorem.

Theorem 1. *Consider the time-varying delay system (1) under bounded peak disturbances (2) and zero initial conditions. If there exist matrices* $P_i > 0$, $i \in [d_m, d_M]$, *and scalars* $\delta_i > 0$, $\alpha_{j,i} \in [0, 1]$, *such that for all* $(i, j) \in [d_m, d_M] \times [d_m, d_M]$,

$$\begin{bmatrix} (\bar{A}+\bar{A}_D\bar{E}_i)^T P_j (\bar{A}+\bar{A}_D\bar{E}_i) - \alpha_{j,i}P_i & (\bar{A}+\bar{A}_D\bar{E}_i)^T P_j \bar{B} \\ * & \bar{B}^T P_j \bar{B} - \frac{1-\alpha_{j,i}}{\varpi^2} I \end{bmatrix} \le 0, \tag{15}$$

$$\begin{bmatrix} I_{n_x} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \le \delta_i P_i, \tag{16}$$

then the reachable set $\mathfrak{R}_x$ *is bounded by the hyper-sphere* $S(r)$ *with* $r = \min(\delta_i)$.

Proof. The proof of Theorem 1 can be obtained based on Lemma 1. Condition (15) implies $X_k^T P_i X_k \le 1$, $i \in [d_m, d_M]$. With condition (16), we have $x_k^T x_k \le \delta_i$, $i \in [d_m, d_M]$. Then the reachable set $\mathfrak{R}_x$ is bounded by a hyper-sphere $S(r)$ with $r = \min(\delta_i)$. □

Remark 3. In practical applications, the bounding hyper-sphere is expected to be as small as possible. If $\alpha_{j,i}$ in condition (15) are fixed, this turns out to be a generalized eigenvalue minimization problem, which can be easily solved by using standard software. In addition, condition (15) in Theorem 1 is not an LMI because it has the nonlinear terms $\alpha_{j,i}P_i$. As proposed in [17], various stochastic optimization methods can be utilized to search for the optimal decision variables, such as Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) method.

Remark 4. Compared with Theorem 3.1 obtained in [25], the reachable set estimation condition obtained in Theorem 1 is simpler than that in Theorem 3.1 in [25]. Thus it is easier to check the feasibility of the reachable set estimation condition in Theorem 1.

Next, let us consider two special cases. The first case is $d_m = d_M$, the time-varying delay d_k reduces to a constant delay d. The second case is $|d_{k+1} - d_k| \le h$, where $h \ge 1$ is an integer.

For the first case, the number of subsystem of the augmented system (8) is 1. The augmented system becomes a time-invariant system. We have the following corollary for constant delay systems. The proof of the proposed corollary is quite straightforward and thus omitted here.

Corollary 1. *Consider the constant delay system*

$$\begin{cases} x_{k+1} = Ax_k + A_D x_{k-d} + B\omega_k, \\ x_k \equiv 0, \ \forall k \in [-d, 0], \end{cases} \tag{17}$$

under bounded peak disturbances (2) and zero initial conditions. If there exist matrix $P > 0$ *and scalars* $\delta > 0$, $\alpha \in [0, 1]$, *such that*

$$\begin{bmatrix} (\bar{A}+\bar{A}_D\bar{E})^T P (\bar{A}+\bar{A}_D\bar{E}) - \alpha P & (\bar{A}+\bar{A}_D\bar{E})^T P \bar{B} \\ * & \bar{B}^T P \bar{B} - \frac{1-\alpha}{\varpi^2} I \end{bmatrix} \le 0, \tag{18}$$

$$\begin{bmatrix} I_{n_x} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \le \delta P, \tag{19}$$

where

$$\bar{E} = \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix}, \tag{20}$$

then the reachable set $\mathfrak{R}_x$ *is bounded by the hyper-sphere* $S(r)$ *with* $r = \delta$.

For the second case, $|d_{k+1} - d_k| \le h$ means there are some constraints in the switching signal of the augmented system (8). To ease our analysis, we assume $h = 1$ in the following. The obtained reachable set estimation condition for $h = 1$ can be extended to $h > 1$ in a straightforward manner. We have the following corollary for the case $|d_{k+1} - d_k| \le 1$.

Corollary 2. *Consider the time-varying delay system (1) with* $|d_{k+1} - d_k| \le 1$ *under bounded peak disturbances (2) and zero initial conditions. If there exist matrices* $P_i > 0$, $i \in [d_m, d_M]$, *and scalars* $\delta > 0$, α_i, α_i^+, $\alpha_i^- \in [0, 1]$, *such that*

$$\begin{bmatrix} (\bar{A}+\bar{A}_D\bar{E}_i)^T P_{i+1} (\bar{A}+\bar{A}_D\bar{E}_i) - \alpha_i^+ P_i & (\bar{A}+\bar{A}_D\bar{E}_i)^T P_{i+1} \bar{B} \\ * & \bar{B}^T P_{i+1} \bar{B} - \frac{1-\alpha_i^+}{\varpi^2} I \end{bmatrix} \le 0, \ \ i \in [d_m, d_M - 1], \tag{21}$$

$$\begin{bmatrix} (\overline{A}+\overline{A}_D\overline{E}_i)^T P_i(\overline{A}+\overline{A}_D\overline{E}_i)-\alpha_i P_i & (\overline{A}+\overline{A}_D\overline{E}_i)^T P_i\overline{B} \\ * & \overline{B}^T P_i\overline{B}-\frac{1-\alpha_i}{\overline{\omega}^2}I \end{bmatrix} \le 0,\ \ i\in[d_m,d_M], \tag{22}$$

$$\begin{bmatrix} (\overline{A}+\overline{A}_D\overline{E}_i)^T P_{i-1}(\overline{A}+\overline{A}_D\overline{E}_i)-\alpha_i^- P_i & (\overline{A}+\overline{A}_D\overline{E}_i)^T P_{i-1}\overline{B} \\ * & \overline{B}^T P_{i-1}\overline{B}-\frac{1-\alpha_i^-}{\overline{\omega}^2}I \end{bmatrix} \le 0,\ \ i\in[d_m+1,d_M], \tag{23}$$

$$\begin{bmatrix} I_{n_x} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \le \delta P_i,\ \ i\in[d_m,d_M], \tag{24}$$

then the reachable set $\mathfrak{R}_x$ is bounded by the hyper-sphere $S(r)$ with $r=\delta$.

Proof. Choose a Lyapunov function for system (8) as

$$V(X_k)=V_{d_k}(X_k)=X_k^T P_{d_k} X_k. \tag{25}$$

If we can ensure $V(X_{k+1})\le 1$ from $V(X_k)\le 1$, then we can conclude that $V(X_k)\le 1$ holds for all $k\ge 0$ as $V(X_0)=0\le 1$.

Suppose $d_k=i$, then $d_{k+1}\in\{i-1,\ i,\ i+1\}$ due to the restriction $|d_{k+1}-d_k|\le 1$. Actually, conditions (21), (22), and (23) ensure $V_{i+1}(X_{k+1})\le 1$, $V_i(X_{k+1})\le 1$, and $V_{i-1}(X_{k+1})\le 1$, respectively, for $V_i(X_k)\le 1$. Taking condition (21) for example. From condition (21), we have

$$\begin{bmatrix} X_k \\ \omega_k \end{bmatrix}^T \begin{bmatrix} (\overline{A}+\overline{A}_D\overline{E}_i)^T P_{i+1}(\overline{A}+\overline{A}_D\overline{E}_i)-\alpha_i^+ P_i & (\overline{A}+\overline{A}_D\overline{E}_i)^T P_{i+1}\overline{B} \\ * & \overline{B}^T P_{i+1}\overline{B}-\frac{1-\alpha_i^+}{\overline{\omega}^2}I \end{bmatrix} \begin{bmatrix} X_k \\ \omega_k \end{bmatrix} \le 0. \tag{26}$$

This means

$$X_k^T\overline{A}_i^T P_{i+1}\overline{A}_i X_k-\alpha_i^+ X_k^T P_i X_k+X_k^T\overline{A}_i^T P_{i+1}\overline{B}\omega_k+\omega_k^T\overline{B}^T P_{i+1}\overline{A}_i X_k+\omega_k^T\overline{B}^T P_{i+1}\overline{B}\omega_k-\frac{1-\alpha_i^+}{\overline{\omega}^2}\omega_k^T\omega_k\le 0. \tag{27}$$

With $X_{k+1}=\overline{A}_i X_k+\overline{B}\omega_k$ and $\omega_k^T\omega_k\le\overline{\omega}^2$, the above inequality implies

$$X_{k+1}^T P_{i+1} X_{k+1}-\alpha_i^+ X_k^T P_i X_k\le 1-\alpha_i^+. \tag{28}$$

That is,

$$\alpha_i^+(1-V_i(X_k))\le 1-V_{i+1}(X_k). \tag{29}$$

Then condition (21) ensures $V_{i+1}(X_{k+1})\le 1$ for $V_i(X_k)\le 1$. Conditions $V_i(X_{k+1})\le 1$ and $V_{i-1}(X_{k+1})\le 1$ can be derived in the same way. As $d_{k+1}\in\{i-1,i,i+1\}$, we can conclude that the conditions in Corollary 2 ensure $V(X_{k+1})\le 1$ for $V(X_k)\le 1$. Therefore, $V(X_k)\le 1$ holds for all $k>0$. In addition, condition (24) means $x_k^T x_k\le\delta X_k^T P_i X_k\le\delta$, then we can conclude that the reachable set $\mathfrak{R}_x$ is bounded by the hyper-sphere $S(r)$. □

3.2. Random delay systems

By using the same augmentation technique, random delay systems can be transformed into delay-free Markov jump systems. Before presenting the reachable set estimation condition, a lemma that is used to derive the reachable set estimation condition will be introduced.

Lemma 2. *Assume there exist functional $V(X_k,d_k)$ and scalar $\alpha\in[0,1]$ such that $V(X_0,d_0)=0$ for any initial d_0 and*

$$\mathbb{E}[V(X_{k+1},d_{k+1})|X_k,d_k]-\alpha V(X_k,d_k)-\frac{1-\alpha}{\overline{\omega}^2}\omega_k^T\omega_k\le 0. \tag{30}$$

Then we have

$$\mathbb{E}[V(X_k,d_k)|X_0,d_0]\le 1 \tag{31}$$

for system (8) under bounded peak disturbances (2) and zero initial condition $X_0=0$.

Proof. As $\omega_k^T\omega_k\le 1$ and $0\le\alpha\le 1$, the condition in (30) indicates

$$\mathbb{E}[V(X_{k+1},d_{k+1})|X_k,d_k]-1\le\alpha(V(X_k,d_k)-1). \tag{32}$$

Taking the expectation of both sides of the above inequality yields

$$\mathbb{E}[V(X_{k+1},d_{k+1})]-1\le\alpha(\mathbb{E}[V(X_k,d_k)-1). \tag{33}$$

Suppose

$$\mathbb{E}[V(X_l, d_l)] \leq 1 \tag{34}$$

holds for time l, then we can get

$$\mathbb{E}[V(X_{l+1}, d_{l+1})] \leq 1. \tag{35}$$

In addition, under zero initial condition $X_0 = 0$, we have

$$\mathbb{E}[V(X_0, d_0)] = 0 \leq 1. \tag{36}$$

Then by using mathematical induction, we can conclude that

$$\mathbb{E}[V(X_k, d_k)|X_0, d_0] \leq 1 \tag{37}$$

holds for any $k \geq 1$. The proof of Lemma 2 is completed. □

Based on Lemma 2, we have the following reachable set estimation theorem for random delay systems.

Theorem 2. *Consider the random delay system (1) under bounded peak disturbances (2) and zero initial conditions. If there exist matrices $P_i > 0$, $i \in [d_m, d_M]$, and scalars $\delta > 0$, $\alpha \in [0, 1]$, such that for all $i \in [d_m, d_M]$,*

$$\begin{bmatrix} (\bar{A} + \bar{A}_D\bar{E}_i)^T P_i(\bar{A} + \bar{A}_D\bar{E}_i) - \alpha P_i & (\bar{A} + \bar{A}_D\bar{E}_i)^T P_i\bar{B} \\ * & \bar{B}^T P_i\bar{B} - \frac{1-\alpha}{\bar{\omega}^2}I \end{bmatrix} \leq 0, \tag{38}$$

$$\begin{bmatrix} I_{n_x} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \leq \delta P_i, \tag{39}$$

where $\bar{P}_i = \sum_{j=d_m}^{d_M} \pi_{ij}P_j$, then the reachable set $\mathcal{R}_x$ is bounded by the hyper-sphere $\mathcal{S}(r)$ with $r = \delta$ in the mean square sense.

Proof. Choose the following Lyapunov function

$$V(X_k, d_k) = X_k^T P_{d_k} X_k. \tag{40}$$

Assume the mode at time k is $i \in [d_m, d_M]$. Then at time $k+1$, the system may jump to any mode $j \in [d_m, d_M]$ with probability π_{ij}. Denote

$$J = \mathbb{E}[V(X_{k+1}, d_{k+1})|X_k, d_k] - \alpha V(X_k, d_k) - \frac{1-\alpha}{\bar{\omega}^2}\omega_k^T\omega_k, \tag{41}$$

then

$$\begin{aligned} J &= \mathbb{E}[X_{k+1}^T P_{d_{k+1}} X_{k+1}|X_k, i] - \alpha X_k^T Pi_i X_k - \frac{1-\alpha}{\bar{\omega}^2}\omega_k^T\omega_k \\ &= \sum_{j=d_m}^{d_M} \pi_{ij} X_{k+1}^T P_j X_{k+1} - \alpha X_k^T P_i X_k - \frac{1-\alpha}{\bar{\omega}^2}\omega_k^T\omega_k \\ &= X_{k+1}^T \bar{P}_i X_{k+1} - \alpha X_k^T P_i X_k - \frac{1-\alpha}{\bar{\omega}^2}\omega_k^T\omega_k \\ &= (\bar{A}_i X_k + \bar{B}_\omega\omega_k)^T \bar{P}_i(\bar{A}_i X_k + \bar{B}_\omega\omega_k) - \alpha X_k^T P_i X_k - \frac{1-\alpha}{\bar{\omega}^2}\omega_k^T\omega_k \\ &= \begin{bmatrix} X_k \\ \omega_k \end{bmatrix}^T \begin{bmatrix} (\bar{A} + \bar{A}_D\bar{E}_i)^T \bar{P}_i(\bar{A} + \bar{A}_D\bar{E}_i) - \alpha P_i & (\bar{A} + \bar{A}_D\bar{E}_i)^T \bar{P}_i\bar{B} \\ * & \bar{B}^T \bar{P}_i\bar{B} - \frac{1-\alpha}{\bar{\omega}^2}I \end{bmatrix} \begin{bmatrix} X_k \\ \omega_k \end{bmatrix} \\ &\leq 0. \end{aligned} \tag{42}$$

According to Lemma 2, we have

$$\mathbb{E}[V(X_k, d_k)|X_0, d_0] \leq 1 \tag{43}$$

In addition, condition (39) ensures

$$\mathbb{E}[x_k^T, x_k] \leq \delta\mathbb{E}[V(X_k, d_k)|X_0, d_0], \tag{44}$$

therefore,

$$\mathbb{E}[x_k^T, x_k] \leq \delta. \tag{45}$$

This means the reachable set $\mathcal{R}_x$ is bounded by the hyper-sphere $\mathcal{S}(r)$ with $r=\delta$ in the mean square sense. □

In some circumstances, the mode transition probabilities are unknown. Next, we will extend the reachable set estimation condition to random delay systems with partially known transition probabilities. In this case, the transition probabilities matrix may take the following form:

$$\begin{bmatrix} \pi_{d_m d_m} & ? & \cdots & \pi_{d_m d_M} \\ ? & ? & \cdots & \pi_{d_m+1 d_M} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{d_M d_m} & 0 & \cdots & \pi_{d_M d_M} \end{bmatrix}, \tag{46}$$

where "?" denotes an unknown transition probability. For simplicity, $\forall i \in [d_m, d_M]$, let

$$L_K^i = \{j: \text{ if } \pi_{ij} \text{isknown}\}, \; L_{UK}^i = \{j: \text{ if } \pi_{ij} \text{isunknown}\}. \tag{47}$$

The following theorem gives a reachable set estimation condition for random delay system (1) with partially known transition probabilities.

Theorem 3. *Consider the random delay system (1) under bounded peak disturbances (2) and zero initial conditions. If there exist matrices $P_i > 0$, $i \in [d_m, d_M]$, and scalars $\delta > 0$, $\alpha \in [0, 1]$, such that for all $j \in L_{UK}^i$,*

$$\begin{bmatrix} (\bar{A}+\bar{A}_D\bar{E}_i)^T\left(\bar{P}_i^K + (1-\pi_i^K)P_j\right)(\bar{A}+\bar{A}_D\bar{E}_i) - \alpha P_i & (\bar{A}+\bar{A}_D\bar{E}_i)^T\left(\bar{P}_i^K + (1-\pi_i^K)P_j\right)\bar{B} \\ * & \bar{B}^T\left(\bar{P}_i^K + (1-\pi_i^K)P_j\right)\bar{B} - \frac{1-\alpha}{\varpi^2} I \end{bmatrix} \le 0, \tag{48}$$

$$\begin{bmatrix} I_{n_x} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \le \delta P_i, \tag{49}$$

where $\bar{P}_i^K = \sum_{j \in L_K^i} \pi_{ij} P_j$, $\pi_i^K = \sum_{j \in L_K^i} \pi_{ij}$, then the reachable set $\mathcal{R}_x$ is bounded by the hyper-sphere $\mathcal{S}(r)$ with $r=\delta$ in the mean square sense.

Proof. It is obvious that $0 \le \pi_i^K \le 1$. For $\pi_i^K = 1$, condition (48) becomes condition (38) in Theorem 2. We will consider $\pi_i^K < 1$ in the following. The left-hand side of condition (38) can be rewritten as

$$\begin{aligned} \Delta &= \begin{bmatrix} \bar{A}_i^T\left(\bar{P}_i^K + \sum_{j\in L_{UK}^i} \pi_{ij}P_j\right)\bar{A}_i - \alpha P_i & \bar{A}_i^T\left(\bar{P}_i^K + \sum_{j\in L_{UK}^i} \pi_{ij}P_j\right)\bar{B} \\ * & \bar{B}^T\left(\bar{P}_i^K + \sum_{j\in L_{UK}^i} \pi_{ij}P_j\right)\bar{B} - \frac{1-\alpha}{\varpi^2} I \end{bmatrix}, \\ &= \begin{bmatrix} \bar{A}_i^T\left(\bar{P}_i^K + (1-\pi_i^K)\sum_{j\in L_{UK}^i} \frac{\pi_{ij}}{1-\pi_i^K}P_j\right)\bar{A}_i - \alpha P_i & \bar{A}_i^T\left(\bar{P}_i^K + (1-\pi_i^K)\sum_{j\in L_{UK}^i} \frac{\pi_{ij}}{1-\pi_i^K}P_j\right)\bar{B} \\ * & \bar{B}^T\left(\bar{P}_i^K + (1-\pi_i^K)\sum_{j\in L_{UK}^i} \frac{\pi_{ij}}{1-\pi_i^K}P_j\right)\bar{B} - \frac{1-\alpha}{\varpi^2} I \end{bmatrix}. \end{aligned}$$

Since $0 \le \pi_{ij}/(1-\pi_i^K) \le 1$, $\forall j \in L_{UK}^i$ and $\sum_{j\in L_{UK}^i} \pi_{ij}/(1-\pi_i^K) = 1$, we have

$$\Delta = \sum_{j\in L_{UK}^i} \frac{\pi_{ij}}{1-\pi_i^K} \begin{bmatrix} \bar{A}_i^T\left(\bar{P}_i^K + (1-\pi_i^K)P_j\right)\bar{A}_i - \alpha P_i & \bar{A}_i^T\left(\bar{P}_i^K + (1-\pi_i^K)P_j\right)\bar{B} \\ * & \bar{B}^T\left(\bar{P}_i^K + (1-\pi_i^K)P_j\right)\bar{B} - \frac{1-\alpha}{\varpi^2} I \end{bmatrix}.$$

Therefore, condition (48) ensures $\Delta \le 0$. Then, according to Theorem 2, the reachable set $\mathcal{R}_x$ is bounded by the hyper-sphere $\mathcal{S}(r)$ with $r=\delta$ in the mean square sense. □

Remark 5. If the transition probabilities are completely unknown, then the treatment of Markov jump systems resembles to that of arbitrarily switched systems. In this case, the reachable set estimation condition (48) is the same as condition (15) in Theorem 1 because $\bar{P}_i^K = 0$ and $\pi_i^K = 0$.

4. Illustrative examples

4.1. Time-Varying delay system

Consider the time-varying delay system (1) with the following parameters:

$$A = \begin{bmatrix} 0.40 & -0.02 \\ -0.30 & 0.10 \end{bmatrix}, \; A_D = \begin{bmatrix} -0.05 & 0.00 \\ -0.01 & -0.04 \end{bmatrix}, \; B = \begin{bmatrix} 0.10 \\ 0.50 \end{bmatrix}, \tag{50}$$

Table 1
Different minimal values of δ for different $[d_m, d_M]$.

Method & NoV	$[d_m, d_M]$			
	[1, 2]	[1, 3]	[1, 4]	[2, 4]
[25]	0.5660	0.6360	0.8402	0.7299
NoV	145	145	145	145
Theorem 1	0.5360	0.5499	0.5700	0.5771
NoV	42	108	220	165

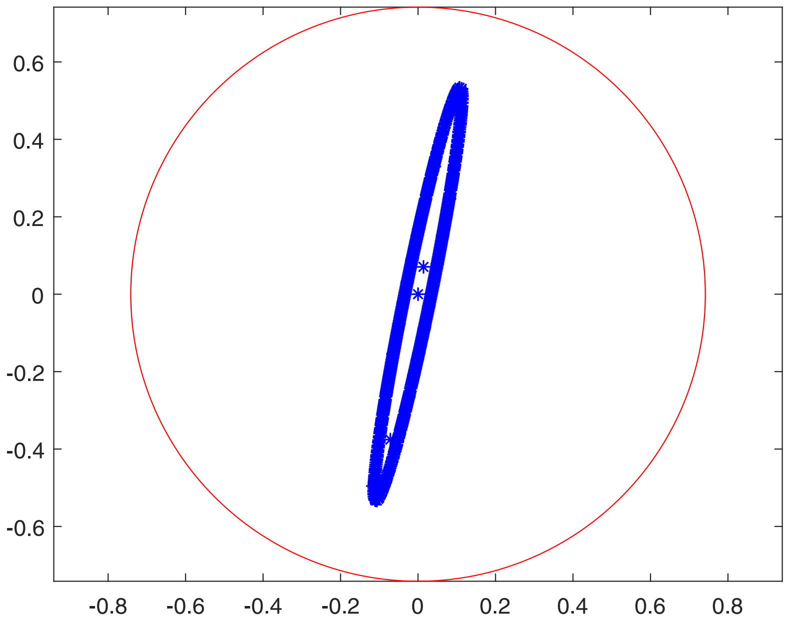


Fig. 1. Bounding hyper-sphere for the reachable set with $d_k \in [1, 3]$.

The peak of the disturbances ω_k is assumed to be $\overline{\omega} = 1$. Next, we will use the proposed reachable set estimation method to determine the bound of the reachable set of the above system.

The parameters of the augmented system can be obtained as

$$\bar{A} = \begin{bmatrix} 0.40 & -0.02 & 0 & 0 & 0 & 0 \\ -0.30 & 0.10 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \ \bar{A}_D = \begin{bmatrix} -0.05 & 0 \\ -0.01 & -0.04 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 0.10 \\ 0.50 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{E}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \ \bar{E}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

We use the GA to search for an optimal set of values of the decision variables in Theorem 1 when minimizing $\min\{\delta_1, \delta_2\}$. To compare with the result in [25], the minimal values of δ for different delay intervals given by different methods are listed in Table 1, where NoV stands for the number of decision variables. From Table 1, it can be seen that the minimal values of δ obtained in this paper are significantly smaller than those obtained in [25]. Moreover, the number of decision variables do not increase a lot. The obtained bounding hyper-sphere is shown in Fig. 1. In this example, the disturbance ω_k is assumed to be $\omega_k = \sin(k)$. Therefore, the effectiveness of the reachable set estimation condition has been verified.

4.2. Random delay system

Consider the random delay system (1) with the following parameters:

$$A = \begin{bmatrix} 0.40 & -0.02 \\ -0.30 & 0.10 \end{bmatrix}, \ A_D = \begin{bmatrix} -0.05 & 0.00 \\ -0.01 & -0.04 \end{bmatrix}, \ B = \begin{bmatrix} 0.10 \\ 0.50 \end{bmatrix}. \tag{51}$$

Next, the completely known and partially known transition probabilities of the system delay will be considered respectively.

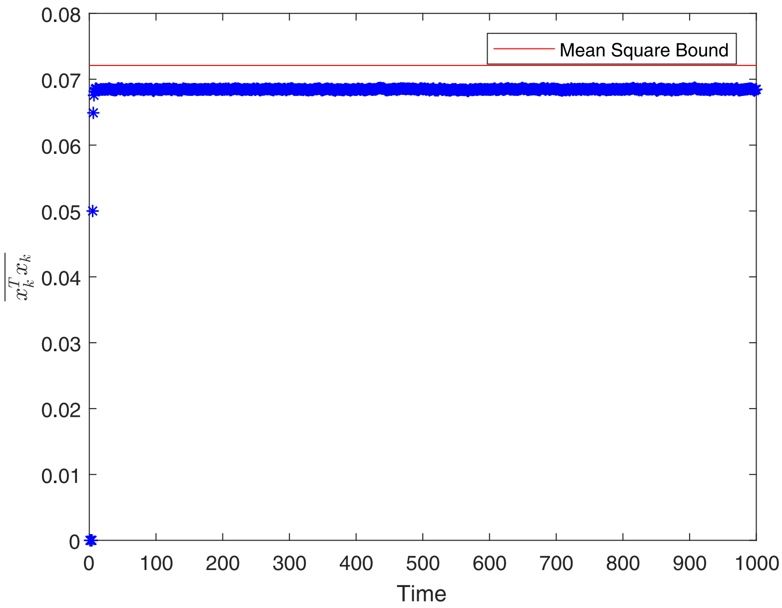


Fig. 2. Mean square of the system state.

4.2.1. Completely known transition probabilities

Suppose the time delay d_k takes its value in the set {1, 2}. The transition probabilities matrix of the time delay $d_k = 1$ and $d_k = 2$ is given by

$$\Pi = \begin{bmatrix} 0.70 & 0.30 \\ 0.20 & 0.80 \end{bmatrix}. \tag{52}$$

The peak of the disturbances ω_k is assumed to be $\overline{\varpi} = 1$. Next, we will use the proposed reachable set estimation method to determine the mean square bound of the above system.

The parameters of the augmented system can be obtained as

$$\overline{A} = \begin{bmatrix} 0.21 & 0.12 & 0 & 0 & 0 & 0 \\ 0.05 & -0.14 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \ \overline{A}_D = \begin{bmatrix} 0.04 & 0 \\ 0.05 & 0.01 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \overline{B} = \begin{bmatrix} 0.10 \\ -0.20 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\overline{E}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \ \overline{E}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

By searching for the optimal value of a in Theorem 2 with a constant step size of 0.0001, we obtain the following optimal result:

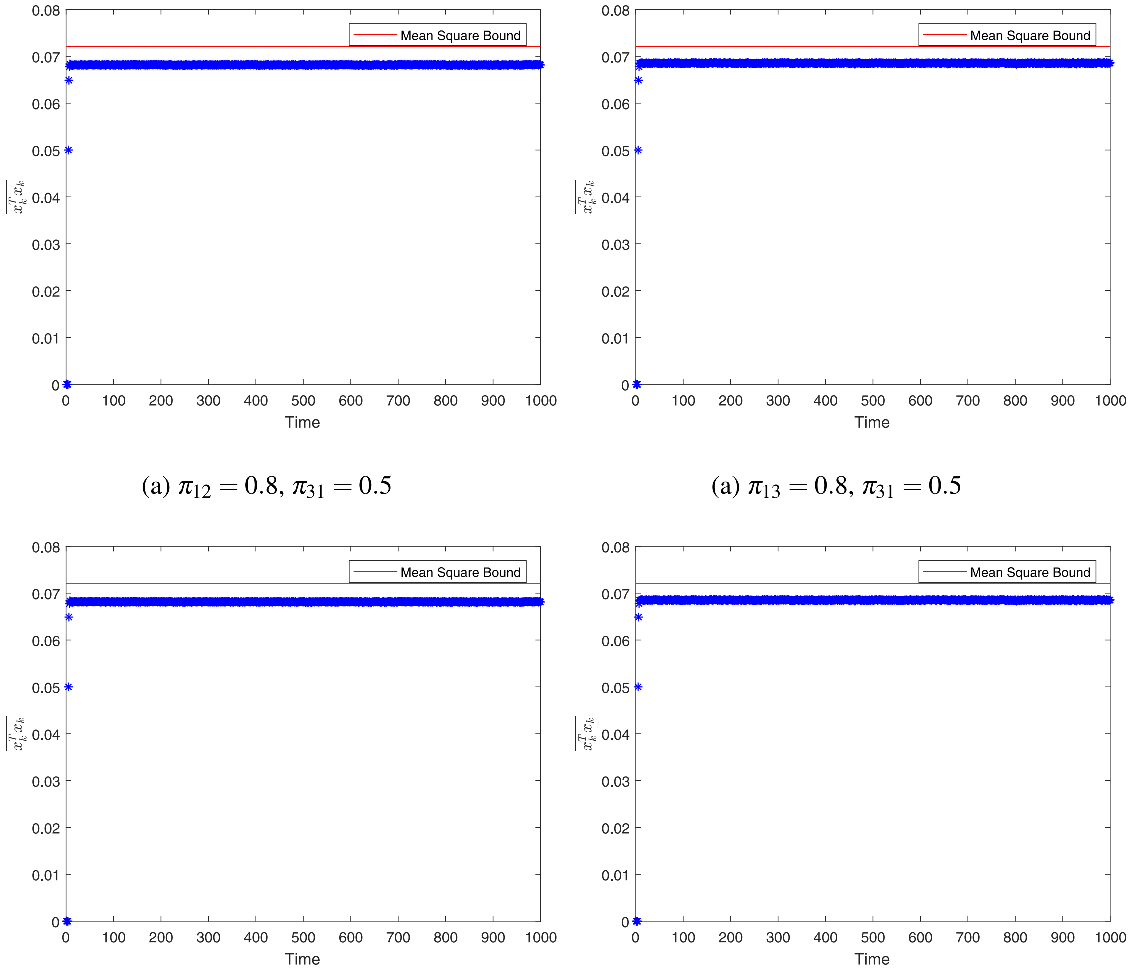
$$\alpha = 0.2152, \quad \delta = 0.0714.$$

The mean square values of the system state are shown in Fig. 2. The disturbance ω_k is assumed to be $\omega_k = (-1)^k$. We ran the Markov jump system for 100 switching realisations. Each * in the figure represents the mean square value of the state for each realisation sequence, that is, $\frac{\sum x_k^T x_k}{100}$. As can be seen from this figure, the mean square value of the system state is bounded by the obtained δ. Therefore, the effectiveness of the reachable set estimation condition for random delay systems has been verified through this numerical example.

4.2.2. Partially known transition probabilities

Suppose the time delay d_k takes its value in the set {1, 2, 3}. The transition probabilities matrix of the time delay $d_k = 1$, $d_k = 2$, and $d_k = 3$ is given by

$$\Pi = \begin{bmatrix} 0.20 & ? & ? \\ 0.30 & 0.10 & 0.60 \\ ? & ? & 0.50 \end{bmatrix}. \tag{53}$$



(a) $\pi_{12} = 0.8$, $\pi_{31} = 0.5$

(a) $\pi_{13} = 0.8$, $\pi_{31} = 0.5$

(a) $\pi_{13} = 0.8$, $\pi_{32} = 0.5$

(a) $\pi_{12} = 0.8$, $\pi_{32} = 0.5$

Fig. 3. Mean square of the system state under different transition probabilities scenarios.

The peak of the disturbances ω_k is assumed to be $\overline{\omega} = 1$. Next, we will use the proposed reachable set estimation method to determine the mean square bound of the above system.

The parameters of the augmented system can be obtained as

$$\bar{A} = \begin{bmatrix} 0.21 & 0.12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.05 & -0.14 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \bar{A}_D = \begin{bmatrix} 0.04 & 0 \\ 0.05 & 0.01 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0.10 \\ -0.20 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{E}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{E}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\bar{E}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.,$$

By searching for the optimal value of α in Theorem 3 with a constant step size of 0.0001, we obtain the following optimal result:

$$\alpha = 0.2213,\ \ \delta = 0.0721.$$

The mean square values of the system state under different transition probabilities scenarios are shown in Fig. 3. The disturbance ω_k and the switching realisations are described in the completely known transition probabilities case. The effectiveness of the reachable set estimation for random delay systems under partially known transition probabilities has been verified through this numerical example.

5. Conclusions

In this paper, the problems of reachable set estimation for dynamic systems with time-varying delay and random delay have been studied. By using the augmentation technique, time delay systems can be transformed into delay-free switched systems. The switching signal of the transformed switched system is determined by the time delay d_k. Sufficient conditions for determining the bounding hyper-sphere for the reachable set have been derived. This paper has proposed different reachable set estimation conditions for different delay types (including time-varying delay and Markov jump time delay). The usefulness of the reachable set estimation conditions has been verified through numerical examples.

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