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Non-fragile PD control of linear time-delay positive discrete-time systems

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ABSTRACT

This paper addresses the issue of proportional-derivative (PD) controllers design for positive linear systems in the discrete-time domain, which still remains a challenging problem in positive systems theory. The specific aim is to design a PD controller for a system with constant time delay, which simultaneously ensures closed-loop system stability and preserves positivity. Moreover, additive gain variation of the controller is considered in the synthesis process. Systematic formulation and tractable algorithms are developed to find the PD controller gains for positive stabilization. The performance of such methods is validated by numerical examples.

1. Introduction

Positive system is a class of dynamic systems whose state and output variables are positive, or at least non-negative. The research on positive linear systems traces back to David G. Luenberger who, for the first time, introduced the concept of such class of systems in his fundamental book [24]. Since then, positive systems theory has found numerous applications in industrial problems, such as biochemical engineering and traffic control [4,29], to name just a few. The theory is becoming prominent since for many real-world physical systems, the descriptor variables usually have intrinsically positive or non-negative features, otherwise the system state will lose its physical attributes [12]. For example, the amount of electric charges stored in a capacitor must always remain non-negative. Meanwhile, positive systems theory has also been significantly employed in stochastic processes, considering the probability's non-negative characteristic. Markov chains [37] and Poisson processes [16] are representative probabilistic models that can be regarded as special types of positive systems. With the recent progress in non-negative matrices [3,13] and co-positive programming [15], an increasing number of mathematical tools are employed to develop positive systems theory, which identifies its particularity and significance compared with other dynamic systems. Current researches on positive systems, especially positive linear systems, could be roughly divided into three types, i.e., positive controllability and controller design [11,32], positive observability and observer design [8,22], and positive realization [2]. In recent years, positive systems theory has also been combined with other branches of control systems theory, such as cooperative control [35] and time-delay systems [42]. Much of the recent research interest has been distributed on widely different control issues, in particular, robust positive stabilization and system performance [10,14,18,25,30], the Bounded Real Lemma [33] and the Kalman–Yakubovic–Popov Lemma [28] for positive systems, as well as decentralized and distributed control [9,26]. Though much effort has been devoted to tackling various control problems

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Table 1
Notations employed in the paper.

Notation	Type	Description
$\mathbb{R}$	set	set of real numbers
$\mathbb{R}^n$	set	n-dimensional real Euclidean space
$\mathbb{R}_+$	set	set of non-negative real numbers
$\mathbb{M}$	set	set of Metzler matrices
$\mathbb{S}$	set	set of Schur matrices
$\mathbb{H}$	set	set of Hurwitz matrices
$\alpha(X)$	scalar	spectral abscissa of matrix X
$\rho(X)$	scalar	spectral radius of matrix X
X^T	matrix	transpose of X
$\mathrm{sym}(X)$	matrix	symmetric matrix $X^\mathrm{T}+X$
I (or I_n)	matrix	$(n \times n)$ identity matrix
$G(z)$	function	transfer function
$X \succ$ (or $\succeq$) 0	operator	positive (semi-) definite
$X \prec$ (or $\preceq$) 0	operator	negative (semi-) definite
$X >$ (or $\geq$) 0	operator	$\forall i, j,\ [X]_{ij} > 0$ (or ≥ 0)
$X \succ$ (or $\succeq$) Y	operator	$X - Y \succ 0$ (or $\succeq 0$)
$X >$ (or $\geq$) Y	operator	$X - Y > 0$ (or ≥ 0)
$\lvert \cdot \rvert$	operator	Euclidean norm for vectors

of positive systems, the PD controller design, a fundamental methodology in feedback control systems [1,20,21], is still a challenging problem for such kind of systems. The major challenge stems from the difficulty in guaranteeing the difference operator's positivity of PD controllers. More specifically, the input signal of the difference operator is not guaranteed to be monotonic, thus resulting in a sign-indefinite output signal. Therefore, the design of an appropriate gain preserving the system's positivity has become the key issue. This matter is further complicated by the significant coupling between the centralized multivariable proportional and discrete-time derivative gains in the synthesis process.

In the literature, a considerable amount of feedback methodologies have been applied to positive systems control for fulfilling various constraints and performance indices. For example, a finite-time output-feedback controller was designed in Liu et al. [19] for stabilization of positive time-varying discrete-time linear systems. Necessary and sufficient conditions of state-feedback controller design were established in linear programming for positive delay systems with semi-Markov process [27]. Static output-feedback (SOF) controllers were designed for discrete-time and discrete-time positive linear systems [6,34]. Distributed controllers were employed for positive consensus of networked control systems [31]. An implicit assumption inherent in these approaches is that the implementation of controllers can be free of errors. However, in practical applications, a certain degree of errors may be induced due to various factors such as the finite word length in digital systems, the imprecision of analog systems, and the additional parameter fine-tuning after final implementation. Therefore, designing a non-fragile controller which is insensitive to the variations in its gain has received increasing attention [5,7,38–41]. However, up to now, very few results have been dedicated to the design of such controllers for positive systems [37]. This paper investigates the non-fragile PD controller design problem for linear time-delay positive discrete-time systems, where additive gain variation in the controller is considered.

The main results and contributions of this work are summarized as follows:

- The PD controller design problem of positive discrete-time systems with constant delay is tackled in this work;
- A systematic formulation is proposed to design non-fragile PD controllers for such kind of systems;
- Based on the necessary and sufficient conditions derived, two tractable linear-programming-based and semi-definite-programming-based algorithms are developed for calculating the solution.

The rest of this paper is organized as follows. Section 2 presents some mathematical preliminaries and defines the problem to be solved. In Section 3, based on a novel formulation of PD controller design, both analysis and synthesis conditions are derived for such kind of systems. In Section 4, numerical simulations are conducted to verify the obtained results. Section 6 summarizes this paper with remarks.

2. Preliminaries

2.1. Notations

The notations employed in this paper are summarized in the following table (Table 1).

In addition, the symbol $\mathrm{diag}(v_i)$ $(i = 1, 2, \ldots, n)$ denotes a diagonal matrix with diagonal entries being the entries of vector $v := [v_1, v_2, \ldots, v_n]^\mathrm{T}$. For matrix $A \in \mathbb{R}^{n\times n}$, we use a_{ij} to denote the entry located at the ith row and the jth column. Matrix A is called Metzler if all of its off-diagonal entries are non-negative, denoted by $A \in \mathbb{M}$. Matrix A is called Hurwitz if all of its eigenvalues have strictly negative real parts, denoted by $A \in \mathbb{H}$. Matrix A is called Schur if the absolute value of each its eigenvalue is less than one, denoted by $A \in \mathbb{S}$.

2.2. Positive systems theory

Consider a linear discrete-time system with time delay

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-d) + Bu(k), \\ y(k) = Cx(k) + C_d x(k-d), \\ x(\theta) = \varphi(\theta), \theta = -d, -d+1, \dots, 0 \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^p$ is the input, and $y(k) \in \mathbb{R}^q$ is the output. Furthermore, matrices $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$ and $C_d \in \mathbb{R}^{q\times n}$ are known real constant matrices. $d \in \mathbb{N}_+$ is a constant time delay, and $\varphi : \{-d, -d+1, \dots, 0\} \to \mathbb{R}^n$ is the initial condition.

To pave the way for further analysis, some useful results [12,17,23], are provided as follows.

Definition 1 ([17]). The system in (1) is said to be positive, if for every $\theta \in \mathbb{Z}$ $(\theta = -d, -d+1, \dots, 0)$ and every $u(k) \in \mathbb{R}^p_+$, we have $x(k) \in \mathbb{R}^n_+$, $y(k) \in \mathbb{R}^q_+$ for $k \geq 0$.

Lemma 1 ([12]). *The system in (1) is positive if and only if A, A_d, B, C and C_d are nonnegative matrices.*

Lemma 2 ([12]). *For any Metzler matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, if $A \leq B$, then $\alpha(A) \leq \alpha(B)$.*

Lemma 3 ([17,23,36]). *The positive system in (1) with $u(k) = 0$ is asymptotically stable for all $d \in \mathbb{N}_+$, if and only if one of the following conditions hold:*

1) *The following positive system*

$$x(k+1) = (A + A_d)x(k) \quad (2)$$

is asymptotically stable;

2) *Matrix $(A + A_d)$ is Schur stable, that is, $\rho(A + A_d) < 1$;*
3) *Matrix $(A + A_d - I)$ is Hurwitz stable, that is, there exists a vector $\lambda \in \mathbb{R}^n_+$ such that $\lambda^{\mathrm{T}}(A + A_d - I) \prec 0$;*
4) *There exists a diagonal matrix $P \succ 0$ such that*

$$(A + A_d)P(A + A_d)^{\mathrm{T}} - P \prec 0;$$

5) *There exists a diagonal matrix $P \succ 0$ such that*

$$\begin{bmatrix} -P & (A + A_d)^{\mathrm{T}}P \\ \# & -P \end{bmatrix} \prec 0; \quad (3)$$

6) *There exist diagonal matrices $P_0 \succ 0$ and $P_n \succ 0$ such that*

$$\begin{bmatrix} A^{\mathrm{T}}P_0A + P_n - P_0 & A^{\mathrm{T}}P_0A_d \\ \# & A_d^{\mathrm{T}}P_0A_d - P_n \end{bmatrix} \prec 0. \quad (4)$$

Through using the above fundamental results on matrix theory and positive systems theory, the PD controller design of the linear time-delay positive discrete-time system in (1) will be investigated in the following sections.

3. Main results

In this section, we propose a formulation for PD controller design of linear time-delay positive discrete-time systems (1), and then provide several positivity and stability results. Based on positive systems theory and Lyapunov theory, the positivity and stability design of PD controllers is derived and the corresponding semi-definite programming algorithm is developed.

3.1. Formulation of non-fragile PD controller

The main objective of this subsection is to provide a systematic framework for the tuning of the gains of the following multi-variable PD controller:

$$u(k) = K_{\mathrm{P}}y(k) + K_{\mathrm{D}}\hat{y}(k) \quad (5)$$

where $K_{\mathrm{P}} \in \mathbb{R}^{p\times q}$, $K_{\mathrm{D}} \in \mathbb{R}^{p\times q}$, and $\hat{y}(k)$ is the output signal of the difference operator. Using the z-transformation, the multi-input multi-output (MIMO) PD controller can be described as

$$U(z) = K_{\mathrm{P}}Y(z) + K_{\mathrm{D}}H_{\mathrm{D}}(z)Y(z)$$

where

$$H_{\mathrm{D}}(z) := I_q \otimes \frac{z-1}{z} \quad (6)$$

denotes the transfer function matrix. The symbols $U(z)$ and $Y(z)$ here are the z-transforms of input $u(k)$ and output $y(k)$ of system (1), respectively.

To transform the tuning of controller parameters into an SOF control problem, we reformulate the transfer function matrix $H_{\mathrm{D}}(z)$ into the following state-space form:

$$\begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}y(k) \\ \hat{y}(k) = \hat{C}\hat{x}(k) + \hat{D}y(k) \end{cases} \tag{7}$$

where

$$\hat{A} = 0, \quad \hat{B} = I_q, \quad \hat{C} = -I_q, \quad \hat{D} = I_q.$$

Notice that system (7) is a state-space realization given in an explicit form. Therefore, the closed-loop system in (1) with the PD controller in (5) can be reformulated as

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} A + BK_{\mathrm{P}}C + BK_{\mathrm{D}}C & -BK_{\mathrm{D}} \\ C & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} A_d + BK_{\mathrm{P}}C_d + BK_{\mathrm{D}}C_d & 0 \\ C_d & 0 \end{bmatrix} \begin{bmatrix} x(k-d) \\ \hat{x}(k-d) \end{bmatrix}. \tag{8}$$

Hence, the tuning of the PD controller parameters for the positive linear system in (1) is reduced to find an SOF controller gain matrix $K = \begin{bmatrix} K_{\mathrm{P}} & K_{\mathrm{D}} \end{bmatrix}$ for the overall closed-loop system in (8). In order to formulate a systematic procedure to determine the SOF controller gain K, we further define that

$$\tilde{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ C & -I \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} A_d & 0 \\ C_d & 0 \end{bmatrix}, \quad \tilde{C}_d = \begin{bmatrix} C_d & 0 \\ C_d & 0 \end{bmatrix},$$

and

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}, \quad \tilde{x}(k-d) = \begin{bmatrix} x(k-d) \\ \hat{x}(k-d) \end{bmatrix},$$

then the closed-loop system in (8) can be represented in the following compact form:

$$\tilde{x}(k+1) = (\tilde{A} + \tilde{B}K\tilde{C})\tilde{x}(k) + (\tilde{A}_d + \tilde{B}K\tilde{C}_d)\tilde{x}(k-d). \tag{9}$$

In the above derivations, the gain variations of the PD controller are not considered, while in real situations, they always exist [37]. With the additive gain variations, the control input in (5) becomes

$$u(k) = (K_{\mathrm{P}} + \Delta_{\mathrm{P}})y(k) + (K_{\mathrm{D}} + \Delta_{\mathrm{D}})\tilde{y}(k) \tag{10}$$

where Δ_{P} and Δ_{D} denote the gain variations of K_P and K_D, respectively. The gain variations can be described as

$$-\underline{\Delta}_{\mathrm{P}} \le \Delta_{\mathrm{P}} \le \overline{\Delta}_{\mathrm{P}} \quad \text{and} \quad -\underline{\Delta}_{\mathrm{D}} \le \Delta_{\mathrm{D}} \le \overline{\Delta}_{\mathrm{D}} \tag{11}$$

where $\underline{\Delta}_{\mathrm{P}}$, $\overline{\Delta}_{\mathrm{P}}$, $\underline{\Delta}_{\mathrm{D}}$ and $\overline{\Delta}_{\mathrm{D}}$ are non-negative matrices with compatible dimensions. Define $\Delta := \begin{bmatrix} \Delta_{\mathrm{P}} & \Delta_{\mathrm{D}} \end{bmatrix}$. Using the non-fragile PD controller in (10) to the system (9) leads to the following system:

$$\tilde{x}(k+1) = (\tilde{A} + \tilde{B}(K+\Delta)\tilde{C})\tilde{x}(k) + (\tilde{A}_d + \tilde{B}(K+\Delta)\tilde{C}_d)\tilde{x}(k-d). \tag{12}$$

Based on the above discussions, the problem to be solved in this paper is presented as follows.

Problem PDTDS (PD Controller Design of Linear Time-Delay Positive Discrete-time Systems): Design the non-fragile PD controller gain in (10), that is, K_{P} and K_{D}, for the positive discrete-time system in (1) such that the closed-loop system in (12) is asymptotically stable under the controller gain variations in (11), and the system state $\tilde{x}(k)$ always stays in the non-negative orthant for all k, that is, $\tilde{x}(k) \ge 0$ for $k \ge 0$.

The key point is how to choose the appropriate gains of PD controllers. The main obstacle of the design is to preserve stability and positivity of the positive linear system in (1) simultaneously. In the following subsections, the **Problem PDTDS** is analyzed and solved utilizing positive systems theory and Lyapunov theory.

3.2. Positivity and stability analysis

Theorem 1. Problem PDTDS *is solvable if and only if the following conditions hold:*

1) $A + B(K_{\mathrm{P}} - \underline{\Delta}_{\mathrm{P}})C + B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}})C \ge 0$;
2) $A_d + B(K_{\mathrm{P}} - \underline{\Delta}_{\mathrm{P}})C_d + B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}})C_d \ge 0$;
3) $-B(K_{\mathrm{D}} + \overline{\Delta}_{\mathrm{D}}) \ge 0$;

4) *The following matrix is Schur stable:*

$$\Gamma := \begin{bmatrix} \begin{Bmatrix} A + A_d + B(K_P + \overline{\Delta}_P)(C + C_d) \\ +B(K_D + \overline{\Delta}_D)(C + C_d) \end{Bmatrix} & -B(K_D - \underline{\Delta}_D) \\ (C + C_d) & 0 \end{bmatrix}, \tag{13}$$

that is, $\rho(\Gamma) < 1$.

Proof. Regarding the proof, the necessary part is obvious, since the above four conditions always hold if **Problem PDTDS** is solvable. We give the proof of sufficiency as follows.

We first prove the positivity of the system in (12). Since $-\underline{\Delta}_P \le \Delta_P \le \overline{\Delta}_P$ and $-\underline{\Delta}_D \le \Delta_D \le \overline{\Delta}_D$, along with 1), $A + B(K_P + \Delta_P)C + B(K_D + \Delta_D)C \ge A + B(K_P - \underline{\Delta}_P)C + B(K_D - \underline{\Delta}_D)C \ge 0$; along with 3), $-B(K_D + \Delta_D) \ge -B(K_D + \overline{\Delta}_D) \ge 0$. Also note that $C > 0$ due to Lemma 1, then we have

$$\begin{bmatrix} A + B(K_P + \Delta_P)C + B(K_D + \Delta_D)C & -B(K_D + \Delta_D) \\ C & 0 \end{bmatrix} \ge 0.$$

Hence, we prove $(\tilde{A} + \tilde{B}(K + \Delta)\tilde{C})$ is nonnegative. Analogously, we can prove that $(\tilde{A}_d + \tilde{B}(K + \Delta)\tilde{C}_d)$ is also nonnegative. Then by Lemma 1, the system (12) is positive. Moreover, we can prove that

$$\Gamma \ge (\tilde{A} + \tilde{B}(K + \Delta)\tilde{C}) + (\tilde{A}_d + \tilde{B}(K + \Delta)\tilde{C}_d) \ge 0.$$

Hence, $(\tilde{A} + \tilde{B}(K + \Delta)\tilde{C}) + (\tilde{A}_d + \tilde{B}(K + \Delta)\tilde{C}_d) - I$ is Metzler. By 4) and Lemma 2, we have $\rho(\tilde{A} + \tilde{B}(K + \Delta)\tilde{C} + \tilde{A}_d + \tilde{B}(K + \Delta)\tilde{C}_d) < 1$. By Lemma 3 2), we prove the stability of system (12). □

3.3. Positivity and stability design

Considering the interval gain variations in the design, we will derive the conditions for positivity and stability analysis in the following theorems. First, a useful lemma, which is necessary in the proof of theorems in the following subsections, is presented.

Lemma 4. *The condition 4) in Theorem 1 is equivalent to the following condition: There exist a diagonal matrix $P \succ 0$, scalar $\gamma > 0$, and a matrix $K = \begin{bmatrix} K_P & K_D \end{bmatrix}$ such that*

$$\Theta(\gamma, P, K_P, K_D) := \begin{bmatrix} \overline{A}P\overline{A}^T - P - \gamma \overline{B}KK^T\overline{B}^T & \overline{A}P\overline{C}^T + \gamma \overline{B}K \\ \# & \overline{C}P\overline{C}^T - \gamma I \end{bmatrix} \prec 0 \tag{14}$$

where

$$\overline{A} = \begin{bmatrix} A + A_d + B(\overline{\Delta}_P + \overline{\Delta}_D)(C + C_d) & B\underline{\Delta}_D \\ (C + C_d) & 0 \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \text{and} \quad \overline{C} = \begin{bmatrix} C + C_d & 0 \\ C + C_d & -I \end{bmatrix}.$$

Proof. We first prove the necessity of the theorem. Assuming the problem is solvable, it is obvious that conditions 1) to 3) hold. Note that

$$\Gamma = \overline{A} + \overline{B}K\overline{C} = \begin{bmatrix} A + A_d + B(\overline{\Delta}_P + \overline{\Delta}_D)(C + C_d) & B\underline{\Delta}_D \\ (C + C_d) & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K_P & K_D \end{bmatrix} \begin{bmatrix} C + C_d & 0 \\ C + C_d & -I \end{bmatrix}.$$

By Lemma 3 4), $\Gamma = (\overline{A} + \overline{B}K\overline{C})$ is Schur stable. There exists a diagonal matrix P such that $(\overline{A} + \overline{B}K\overline{C})P(\overline{A} + \overline{B}K\overline{C})^T - P \prec 0$. Hence there exists a scalar $\gamma > 0$ such that $\overline{C}P\overline{C}^T - \gamma I \prec 0$ and $(\overline{A} + \overline{B}K\overline{C})P(\overline{A} + \overline{B}K\overline{C})^T - P + (\overline{A} + \overline{B}K\overline{C})P(\gamma I - \overline{C}P\overline{C}^T)^{-1}P^T(\overline{A} + \overline{B}K\overline{C})^T \prec 0$. Through Schur complement equivalence, it follows that

$$\sigma = \begin{bmatrix} (\overline{A} + \overline{B}K\overline{C})P(\overline{A} + \overline{B}K\overline{C})^T - P & (\overline{A} + \overline{B}K\overline{C})P \\ \# & \overline{C}P\overline{C}^T - \gamma I \end{bmatrix} \prec 0. \tag{15}$$

Define the non-singular matrix T as follows:

$$T = \begin{bmatrix} I & -\overline{B}K \\ 0 & I \end{bmatrix}.$$

Pre- and post-multiplying Γ by T and T^T, then we have $T\Gamma T^T = \Theta \prec 0$. Hence, condition 4) is necessary for solving the problem.

We then prove the sufficiency of the theorem. We decompose Θ in the following form:

$$\Theta = \begin{bmatrix} I & -\bar{B}K \\ 0 & I \end{bmatrix} \begin{bmatrix} (\bar{A}+\bar{B}K\bar{C})P(\bar{A}+\bar{B}K\bar{C})^{\mathrm{T}} - P & (\bar{A}+\bar{B}K\bar{C})P \\ \# & \bar{C}P\bar{C}^{\mathrm{T}} - \gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ -K^{\mathrm{T}}\bar{B}^{\mathrm{T}} & I \end{bmatrix} \prec 0.$$

By Schur complement equivalence,

$$\begin{bmatrix} (\bar{A}+\bar{B}K\bar{C})P(\bar{A}+\bar{B}K\bar{C})^{\mathrm{T}} - P & (\bar{A}+\bar{B}K\bar{C})P \\ \# & \bar{C}P\bar{C}^{\mathrm{T}} - \gamma I \end{bmatrix} \prec 0.$$

Hence, $(\bar{A}+\bar{B}K\bar{C})P(\bar{A}+\bar{B}K\bar{C})^{\mathrm{T}} - P \prec 0$, hence by Lemma 3 3), the system (12) is stable. The proof is completed. □

Theorem 2. Problem PDTDS *is solvable if and only if there exist diagonal matrices $P_1 \succ 0$, $P_2 \succ 0$, scalar $\gamma > 0$, and matrices L_{P}, L_{D}, M_{P} and M_{D} such that the following conditions hold:*

1) $\gamma A + B(L_{\mathrm{P}} - \gamma\underline{\Delta}_{\mathrm{P}})C + B(L_{\mathrm{D}} - \gamma\underline{\Delta}_{\mathrm{D}})C \geq 0$;
2) $\gamma A_d + B(L_{\mathrm{P}} - \gamma\underline{\Delta}_{\mathrm{P}})C_d + B(L_{\mathrm{D}} - \gamma\underline{\Delta}_{\mathrm{D}})C_d \geq 0$;
3) $-B(L_{\mathrm{D}} + \gamma\overline{\Delta}_{\mathrm{D}}) \geq 0$;
4) $\Lambda(\gamma, P_1, P_2, L_{\mathrm{P}}, L_{\mathrm{D}}, M_{\mathrm{P}}, M_{\mathrm{D}}) :=$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ \# & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \# & \# & \Lambda_{33} & \Lambda_{34} \\ \# & \# & \# & \Lambda_{44} \end{bmatrix} \prec 0 \tag{16}$$

where

$$\begin{aligned}
\bar{A}_{11} &= A + A_d + B(\overline{\Delta}_{\mathrm{P}} + \overline{\Delta}_{\mathrm{D}})(C + C_d), \\
\Lambda_{11} &= \bar{A}_{11}P_1\bar{A}_{11}^{\mathrm{T}} + B\overline{\Delta}_{\mathrm{D}}P_2\overline{\Delta}_{\mathrm{D}}^{\mathrm{T}}B^{\mathrm{T}} - P_1 - BL_{\mathrm{P}}M_{\mathrm{P}}^{\mathrm{T}} - M_{\mathrm{P}}L_{\mathrm{P}}^{\mathrm{T}}B^{\mathrm{T}} \\
&\quad + \gamma M_{\mathrm{P}}M_{\mathrm{P}}^{\mathrm{T}} - BL_{\mathrm{D}}M_{\mathrm{D}}^{\mathrm{T}} - M_{\mathrm{D}}L_{\mathrm{D}}^{\mathrm{T}}B^{\mathrm{T}} + \gamma M_{\mathrm{D}}M_{\mathrm{D}}^{\mathrm{T}}, \\
\Lambda_{12} &= \bar{A}_{11}P_1(C + C_d)^{\mathrm{T}}, \\
\Lambda_{13} &= \bar{A}_{11}P_1(C + C_d)^{\mathrm{T}} + BL_{\mathrm{P}}, \\
\Lambda_{14} &= \bar{A}_{11}P_1(C + C_d)^{\mathrm{T}} - B\Delta_{\mathrm{D}}P_2 + BL_{\mathrm{D}}, \\
\Lambda_{22} &= (C + C_d)P_1(C + C_d)^{\mathrm{T}} - P_2, \\
\Lambda_{23} &= (C + C_d)P_1(C + C_d)^{\mathrm{T}}, \\
\Lambda_{24} &= (C + C_d)P_1(C + C_d)^{\mathrm{T}}, \\
\Lambda_{33} &= (C + C_d)P_1(C + C_d)^{\mathrm{T}} - \gamma I, \\
\Lambda_{34} &= (C + C_d)P_1(C + C_d)^{\mathrm{T}}, \\
\Lambda_{44} &= (C + C_d)P_1(C + C_d)^{\mathrm{T}} + P_2 - \gamma I.
\end{aligned}$$

When the above conditions hold, the PD controller gains can be obtained by $K_{\mathrm{P}} = (1/\gamma)L_{\mathrm{P}}$ and $K_{\mathrm{D}} = (1/\gamma)L_{\mathrm{D}}$.

Proof. In the following, we will give a proof to show that Theorem 2 is equivalent to Theorem 1. Substituting $L_{\mathrm{P}} = \gamma K_{\mathrm{P}}$ and $L_{\mathrm{D}} = \gamma K_{\mathrm{D}}$ into Theorem 2 1)–3), since $\gamma > 0$, it follows that Theorem 1 1)–3) are equivalent to Theorem 2 1)–3). By Lemma 4, it suffices to show $\Lambda \prec 0$ is equivalent to $\Theta \prec 0$.

Let $\overline{\Theta}$ denote the expanded form of Θ in terms of A, B, C, etc, we notice that $\overline{\Theta}$ is 4×4 in block matrix form.

$$\Theta = \overline{\Theta} = \begin{bmatrix} \overline{\Theta}_{11} & \overline{\Theta}_{12} & \overline{\Theta}_{13} & \overline{\Theta}_{14} \\ \# & \overline{\Theta}_{22} & \overline{\Theta}_{23} & \overline{\Theta}_{24} \\ \# & \# & \overline{\Theta}_{33} & \overline{\Theta}_{34} \\ \# & \# & \# & \overline{\Theta}_{44} \end{bmatrix}. \tag{17}$$

Assume $\Lambda \prec 0$. Substituting $L_{\mathrm{P}} = \gamma K_{\mathrm{P}}$ and $L_{\mathrm{D}} = \gamma K_{\mathrm{D}}$ into (16), we have $\Lambda_{ij} = \overline{\Theta}_{ij}$ for all i, j except $i, j = 1$. We also notice that

$$\overline{\Theta}_{11} - \Lambda_{11} = -\gamma(BK_{\mathrm{P}} - M_{\mathrm{P}})(BK_{\mathrm{P}} - M_{\mathrm{P}})^{\mathrm{T}} - \gamma(BK_{\mathrm{D}} - M_{\mathrm{D}})(BK_{\mathrm{D}} - M_{\mathrm{D}})^{\mathrm{T}} \prec 0$$

which implies that $\overline{\Theta}_{11} \prec 0$ hence $\overline{\Theta} \prec 0$.

On the other hand, we assume $\overline{\Theta} \prec 0$. If $\overline{\Theta} \prec 0$, there exist $M_{\mathrm{P}} = BK_{\mathrm{P}}$ and $M_{\mathrm{D}} = BK_{\mathrm{D}}$, such that $\Lambda = \overline{\Theta} \prec 0$. This proves $\overline{\Theta} \prec 0$ implies $\Lambda \prec 0$. The proof is completed. □

Remark 1. Inequality (14) is of quadratic matrix form and inequality (16) is of bi-linear matrix form. Solving these inequalities is not an easy task since both of them are nonlinear, but one can develop a sequential algorithm to solve the inequality (16) by fixing some variables.

In particular, when the system in (1) is a single-input positive system, a linear-programming-based condition without conservatism can be developed in the following theorem for solution.

Theorem 3. *Assume that system (1) has only one input.* ***Problem PDTDS*** *is solvable if and only if there exist vectors* $p_1 \succ 0$, $p_2 \succ 0$, *and matrices* K_{P} *and* K_{D} *such that the following linear program holds:*

1) $A + B(K_{\mathrm{P}} - \underline{\Delta}_{\mathrm{P}})C + B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}})C \geq 0$;
2) $A_d + B(K_{\mathrm{P}} - \underline{\Delta}_{\mathrm{P}})C_d + B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}})C_d \geq 0$;
3) $-B(K_{\mathrm{D}} + \overline{\Delta}_{\mathrm{D}}) \geq 0$;
4) $p_1^{\mathrm{T}}(A + A_d) + (K_{\mathrm{P}} + \overline{\Delta}_{\mathrm{P}})(C + C_d) + (K_{\mathrm{D}} + \overline{\Delta}_{\mathrm{D}})(C + C_d) - p_1^{\mathrm{T}} I + p_2^{\mathrm{T}}(C + C_d) \leq 0$;
5) $-(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}}) - p_2^{\mathrm{T}} \leq 0$;
6) $p_1^{\mathrm{T}} B = 1$.

When the linear program holds, the PD controller gains K_{P} *and* K_{D} *can be obtained.*

Proof. In the following, we will give a proof that conditions 4)–6) in Theorem 3 are equivalent to Condition 4) in Theorem 1 since the conditions 1)–3) in Theorems 1 and 3 are identical.

In Theorem 3, since 6) $p_1^{\mathrm{T}} B = 1$ is a scalar, we substitute it into 4) and 5) in the following way:

$$p_1^{\mathrm{T}}(A + A_d) + p_1^{\mathrm{T}} B(K_{\mathrm{P}} + \overline{\Delta}_{\mathrm{P}})(C + C_d) + p_1^{\mathrm{T}} B(K_{\mathrm{D}} + \overline{\Delta}_{\mathrm{D}})(C + C_d) - p_1^{\mathrm{T}} I + p_2^{\mathrm{T}}(C + C_d) \leq 0, \tag{18}$$

$$-p_1^{\mathrm{T}} B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}}) - p_2^{\mathrm{T}} \leq 0. \tag{19}$$

We write (18) and (19) into the compact form:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Gamma_{11} - I & -B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}}) \\ (C + C_d) & -I \end{bmatrix} < 0$$

where

$$\Gamma_{11} = A + A_d + B(K_{\mathrm{P}} + \overline{\Delta}_{\mathrm{P}})(C + C_d) + B(K_{\mathrm{D}} + \overline{\Delta}_{\mathrm{D}})(C + C_d). \tag{20}$$

By Lemma 3 3), we can conclude that $\tilde{A} + \tilde{A}_d + \tilde{B}(K + \Delta)(\tilde{C} + \tilde{C}_d) - I$ is Hurwitz, which is equivalent to the matrix in (13) being Schur stable.

On the other hand, if the matrix in (13) is Schur stable, according to Lemma 3, $[\tilde{A} + \tilde{A}_d + \tilde{B}(K + \Delta)(\tilde{C} + \tilde{C}_d) - I]$ is Hurwitz. Hence, there exist $\tilde{p}_1 > 0$ and $\tilde{p}_2 > 0$ such that

$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Gamma_{11} - I & -B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}}) \\ (C + C_d) & -I \end{bmatrix} < 0.$$

Noticing that $\tilde{p}_1^{\mathrm{T}} B$ is a positive scalar, the following inequality also holds:

$$\frac{1}{\tilde{p}_1^{\mathrm{T}} B} \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Gamma_{11} - I & -B(K_{\mathrm{D}} - \underline{\Delta}_{\mathrm{D}}) \\ (C + C_d) & -I \end{bmatrix} < 0$$

which means that there exist vectors $p_1 = \tilde{p}_1/(\tilde{p}_1^{\mathrm{T}} B)$ and $p_2 = \tilde{p}_2/(\tilde{p}_1^{\mathrm{T}} B)$ such that conditions 4) to 6) hold. The proof is completed. □

3.4. Algorithmic solution

Based on the discussions and derivations in the previous subsections, in particular, Theorem 2, an iterative algorithm is constructed to design the non-fragile PD controller gains for the positive linear system in (1).

Algorithm: Non-fragile PD Controller Design (NPDCD).

Step 1: Set $k = 1$ and $\epsilon^{(0)} = 0$. Select initial values $M_{\mathrm{P}}^{(1)}$ and $M_{\mathrm{D}}^{(1)}$.

Step 2: For fixed $M_{\mathrm{P}} = M_{\mathrm{P}}^{(k)}$ and $M_{\mathrm{D}} = M_{\mathrm{D}}^{(k)}$, solve the following convex optimization problem with respective to $\gamma > 0$, $P_1 \succ 0$, $P_2 \succ 0$, L_{P} and L_{D}: minimize $\epsilon^{(k)}$ subject to

$$\begin{cases} \gamma A + B(L_{\mathrm{P}} - \gamma \underline{\Delta}_{\mathrm{P}})C + B(L_{\mathrm{D}} - \gamma \underline{\Delta}_{\mathrm{D}})C \geq 0, \\ \gamma A_d + B(L_{\mathrm{P}} - \gamma \underline{\Delta}_{\mathrm{P}})C_d + B(L_{\mathrm{D}} - \gamma \underline{\Delta}_{\mathrm{D}})C_d \geq 0, \\ -B(L_{\mathrm{D}} + \gamma \overline{\Delta}_{\mathrm{D}}) \geq 0, \\ \Lambda(\gamma, P_1, P_2, L_{\mathrm{P}}, L_{\mathrm{D}}, M_{\mathrm{P}}, M_{\mathrm{D}}) \prec \epsilon^{(k)} I. \end{cases}$$

Step 3: If $\epsilon^{(k)} \leq 0$, STOP; a solution is obtained as $K_P^* = (1/\gamma)L_P$ and $K_D^* = (1/\gamma)L_D$. Otherwise, go to next step.

Step 4: If $|\epsilon^{(k)} - \epsilon^{(k-1)}|/\epsilon^{(k)} < \delta$ which is a prescribed tolerance, STOP. Otherwise, update $k = k+1$, $M_P^{(k)} = (1/\gamma)BL_P$, and $M_D^{(k)} = (1/\gamma)BL_D$, then go to Step 2.

Remark 2. In Step 1, the initial values $M_P^{(1)}$ and $M_D^{(1)}$ can be obtained by solving a structured observer gain matrix:

$$M := \begin{bmatrix} M_P^{(1)} & M_D^{(1)} \\ 0 & 0 \end{bmatrix}. \tag{21}$$

This will lead to the solvability of a positive observer design problem [22]. Solving the following LMIs with respect to two diagonal matrices $Q_1 \succ 0$ and $Q_2 \succ 0$, and two matrix variables S_P and S_D:

- $Q_1 A + (S_P + S_D)C - Q_1 B(\underline{\Delta}_P + \underline{\Delta}_D)C \geq 0$;
- $Q_1 A_d + (S_P + S_D)C_d - Q_1 B(\underline{\Delta}_P + \underline{\Delta}_D)C_d \geq 0$;
- $-(S_D + Q_1 B\overline{\Delta}_D) \geq 0$;

$$\begin{bmatrix} Q & \# \\ \Sigma & Q \end{bmatrix} \prec 0 \tag{22}$$

where

$$Q = \begin{bmatrix} -Q_1 & 0 \\ 0 & -Q_2 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \begin{Bmatrix} Q_1 (A + A_d + B(\overline{\Delta}_P + \overline{\Delta}_D)(C + C_d)) \\ +S_P(C + C_d) + S_D(C + C_d) \end{Bmatrix} & -S_D - Q_1 B\underline{\Delta}_D \\ Q_2(C + C_d) & 0 \end{bmatrix}. \tag{23}$$

Then $M_P^{(1)} = Q_1^{-1} S_P$, and $M_D^{(1)} = Q_1^{-1} S_D$.

Remark 3. Although condition 4) of Theorem 2 remains a bi-linear matrix inequality, it becomes a linear matrix inequality when M_P and M_D are given and fixed. With such an idea (see the Step 2 of Algorithm NPDCD), one can develop a sequential minimization algorithm which is represented as the semi-definite programming for solution.

4. Case study

In this section, we will verify the effectiveness of our theoretical results and algorithms through using two illustrative examples.

4.1. Multi-input case

Consider the positive discrete-time system in (1) with the following system matrices:

$$A = \begin{bmatrix} 0.5150 & 0.2908 & 0.5218 \\ 0.0158 & 0.5393 & 0.2442 \\ 0.1289 & 0.4340 & 0.0569 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.0029 & 0.0787 & 0.0771 \\ 0.0514 & 0.0816 & 0.0333 \\ 0.0501 & 0.0730 & 0.0588 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0126 & 0.0342 \\ 0.0133 & 0.0371 \\ 0.0747 & 0.0409 \end{bmatrix}, \quad C = \begin{bmatrix} 0.0806 & 0.0596 & 0.0436 \\ 0.0162 & 0.0947 & 0.0261 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 0.0934 & 0.0993 & 0.0529 \\ 0.0178 & 0.0722 & 0.0281 \end{bmatrix}, \quad \overline{\Delta}_P = \overline{\Delta}_D = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \quad \underline{\Delta}_P = \underline{\Delta}_D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The spectral radius of this system is $1.0110 > 1$, which implies that it is unstable. By *Remark 1*, the initial value can be calculated that

$$M_P^{(1)} = \begin{bmatrix} 0.6131 & -1.4229 \\ 0.5518 & -1.4604 \\ -0.0324 & -0.4049 \end{bmatrix}, \quad M_D^{(1)} = \begin{bmatrix} -0.0789 & -0.0756 \\ -0.0643 & -0.0578 \\ -0.0729 & -0.0953 \end{bmatrix}.$$

By implementing Algorithm NPDCD using the YALMIP along with MATLAB R2020b, a feasible solution is obtained as

$$K_P = \begin{bmatrix} -2.7308 & 9.9992 \\ 9.9992 & -39.6525 \end{bmatrix}, \quad K_D = \begin{bmatrix} 0.2396 & -0.5673 \\ -0.5673 & 0.1991 \end{bmatrix}. \tag{24}$$

Substituting (24) into the conditions 1)–3) of Theorem 1, we can verify that

$$A + B(K_P - \underline{\Delta}_P)C + B(K_D - \underline{\Delta}_D)C = \begin{bmatrix} 0.5114 & 0.1838 & 0.4980 \\ 0.0120 & 0.4229 & 0.2183 \\ 0.1161 & 0.3357 & 0.0314 \end{bmatrix} \geq 0,$$

$$A_d + B(K_P - \underline{\Delta}_P)C_d + B(K_D - \underline{\Delta}_D)C_d = \begin{bmatrix} 0.0000 & 0.0085 & 0.0528 \\ 0.0483 & 0.0052 & 0.0069 \\ 0.0363 & 0.0012 & 0.0317 \end{bmatrix} \geq 0,$$

$$-B(K_D + \overline{\Delta}_D) = \begin{bmatrix} 0.0778 & 0.0195 \\ 0.0844 & 0.0204 \\ 0.1122 & 0.1522 \end{bmatrix} \geq 0.$$

The matrix in (13) is

$$\Gamma = \begin{bmatrix} 0.5185 & 0.2013 & 0.5554 & 0.0961 & 0.0292 \\ 0.0679 & 0.4378 & 0.2302 & 0.1043 & 0.0308 \\ 0.1650 & 0.3596 & 0.0730 & 0.1401 & 0.1936 \\ 0.1740 & 0.1588 & 0.0965 & 0 & 0 \\ 0.0341 & 0.1669 & 0.0542 & 0 & 0 \end{bmatrix} \tag{25}$$

which preserves the positivity of the system (1). Further, the eigenvalues of the corresponding matrix (25) are $\{-0.1209 - 0.0743i, -0.1209 + 0.0743i, 0.0149, 0.3439, 0.9123\}$. Hence, $\rho(Z) = 0.9123 < 1$, which guarantees the stability according to Theorem 1.

To illustrate the non-fragility of the solution (24) in system (12) with the system matrices mentioned above, one considers

$$\tilde{x}(k+1) = (\tilde{A} + \tilde{B}(K + \Delta^{(n)})\tilde{C})\tilde{x}(k) + (\tilde{A}_d + \tilde{B}(K + \Delta^{(n)})\tilde{C}_d)\tilde{x}(k-d) \tag{26}$$

in which

$$\Delta^{(n)} = \begin{bmatrix} \Delta_P^{(n)} & \Delta_D^{(n)} \end{bmatrix},$$

we randomly generate the system with 100 different gain variations. The variations are evenly distributed in the given interval. For example, $\Delta_P^{(n)} = -\underline{\Delta}_P + (n/100) \times (\overline{\Delta}_P + \underline{\Delta}_P)$ and $\Delta_D^{(n)} = -\underline{\Delta}_D + (n/100) \times (\overline{\Delta}_D + \underline{\Delta}_D)$. We generate the corresponding system matrices, and hence we have 100 different systems for (26).

To simulate, the initial condition is chosen to be

$$\tilde{x}(\theta) = \begin{bmatrix} \sin(2\theta) + 1 \\ 2\cos(0.5\theta) + 2 \\ 3\sin(\theta) + 3 \\ 0 \\ 0 \end{bmatrix}, \qquad \theta = -5, -4, \ldots, 0. \tag{27}$$

The state responses of system (9) with controller (24) subject to gain variations are shown in Fig. 1. We can see that the system state converges to zero for the system with different gain variations, hence the gain matrix in (24) guarantees the positivity and asymptotic stability of the system.

For comparison purpose, a set of solutions for the stabilization of system (9) with PD control without considering positivity is given as follows:

$$K_P = \begin{bmatrix} -10.5078 & -1.9736 \\ -1.9736 & -1.2955 \end{bmatrix}, \qquad K_D = \begin{bmatrix} -5.5756 & -20.4725 \\ -10.4725 & -0.0214 \end{bmatrix}. \tag{28}$$

The state responses of system (9) with controller (28) subject to the gain variations in (26) are shown in Fig. 2.

We can see that the system state has a negative value component during the evolution, and the gain matrix in (28) cannot guarantee the positivity of the system.

4.2. Single-input case

Consider the positive discrete-time system in (1) with the following system matrices:

$$A = \begin{bmatrix} 0.4382 & 0.2688 & 0.1705 \\ 0.5398 & 0.1410 & 0.2590 \\ 0.4830 & 0.0403 & 0.2731 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.0979 & 0.0584 & 0.0943 \\ 0.0927 & 0.0113 & 0.0214 \\ 0.0598 & 0.0399 & 0.0170 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4220 \\ 0.5757 \\ 0.1860 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0347 & 0.0436 & 0.0285 \\ 0.0620 & 0.0416 & 0.0905 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0.0683 & 0.0084 & 0.0919 \\ 0.0104 & 0.0013 & 0.0368 \end{bmatrix},$$

$$\overline{\Delta}_P = \overline{\Delta}_D = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, \qquad \underline{\Delta}_P = \underline{\Delta}_D = \begin{bmatrix} 0 & 0.2 \end{bmatrix}.$$

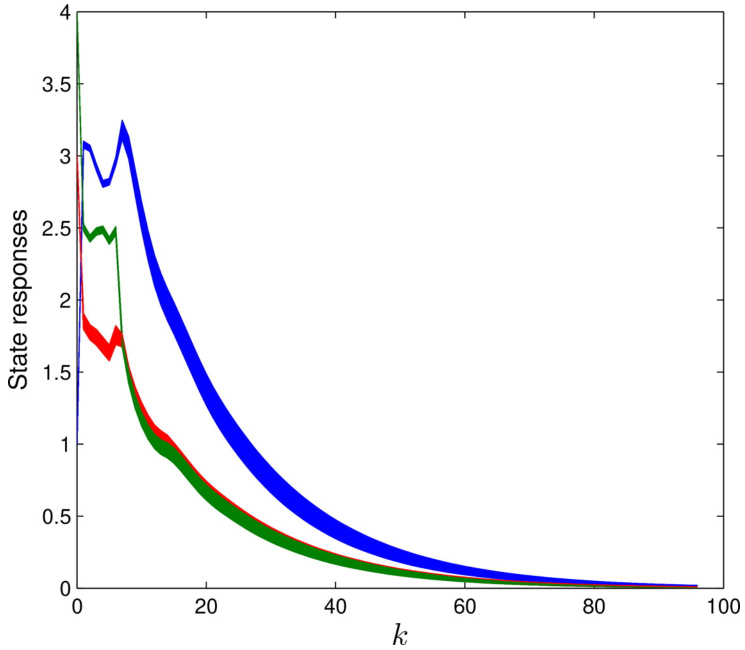

Fig. 1. State responses of system (9) with (24) subject to gain variations.

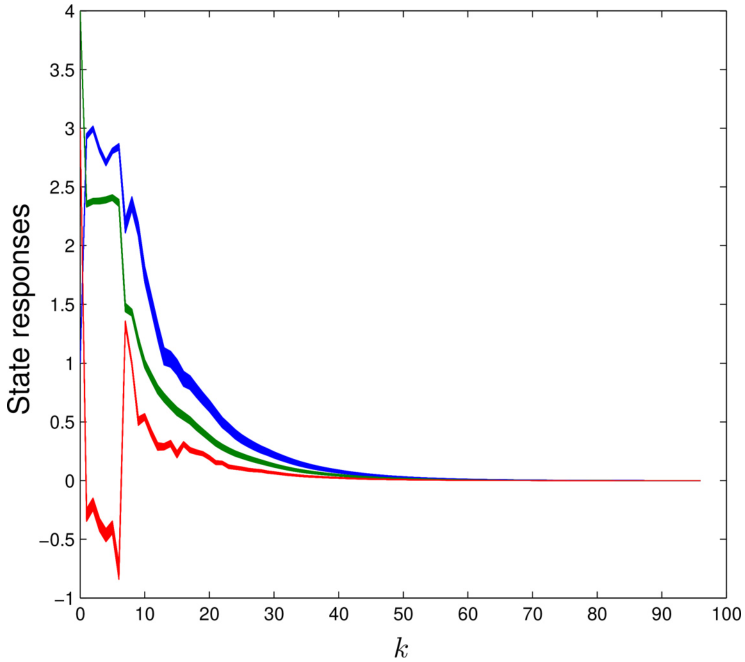

Fig. 2. State responses of system (9) with (28) subject to gain variations.

The spectral radius of this system is $1.0589 > 1$, which implies that it is unstable. By implementing the linear program of Theorem 3, a feasible solution is obtained as

$$K_{\mathrm{P}} = \begin{bmatrix} 2.0178 & -4.0788 \end{bmatrix}, \quad K_{\mathrm{D}} = \begin{bmatrix} -0.4689 & -0.3086 \end{bmatrix}. \tag{29}$$

Substituting (29) into the conditions 1)–3) of Theorem 1, we can verify that

$$A+B(K_{\mathrm{P}}-\underline{\Delta}_{\mathrm{P}})C+B(K_{\mathrm{D}}-\underline{\Delta}_{\mathrm{D}})C=\begin{bmatrix}0.3356 & 0.2133 & 0.0063\\ 0.3998 & 0.0653 & 0.0350\\ 0.4378 & 0.0158 & 0.2007\end{bmatrix}\geq 0,$$

$$A_d+B(K_{\mathrm{P}}-\underline{\Delta}_{\mathrm{P}})C_d+B(K_{\mathrm{D}}-\underline{\Delta}_{\mathrm{D}})C_d=\begin{bmatrix}0.1215 & 0.0612 & 0.0801\\ 0.1248 & 0.0151 & 0.0020\\ 0.0702 & 0.0411 & 0.0107\end{bmatrix}\geq 0,$$

$$-B(K_{\mathrm{D}}+\overline{\Delta}_{\mathrm{D}})=\begin{bmatrix}0.1135 & 0.1302\\ 0.1548 & 0.1777\\ 0.0500 & 0.0574\end{bmatrix}\geq 0.$$

The matrix in (13) is

$$\Gamma=\begin{bmatrix}0.4867 & 0.2904 & 0.1282 & 0.1979 & 0.2146\\ 0.5650 & 0.1021 & 0.0940 & 0.2699 & 0.2928\\ 0.5210 & 0.0640 & 0.2299 & 0.0872 & 0.0946\\ 0.1030 & 0.0519 & 0.1204 & 0 & 0\\ 0.0725 & 0.0429 & 0.1273 & 0 & 0\end{bmatrix} \tag{30}$$

which has preserved the positivity of system (1). Further, the eigenvalues of the corresponding matrix (30) are $\{0, 0.0320+0.1135i, 0.0320-0.1135i, -0.2070, 0.9617\}$. Hence, $\rho(Z)=0.9617<1$, which guarantees the stability according to Theorem 1.

5. Conclusion

This paper studied the problem of PD controllers design for positive linear systems in the discrete-time domain. It aims at designing a PD controller for the system with constant time delay, which can ensure closed-loop system stability as well as preserve positivity. Moreover, the PD controller has additive gain variations in the synthesis process. A systematic formulation was developed to find the PD controller gains for positive stabilization. The methodology and algorithm were provided in the study, and the performance of such methods was validated by numerical examples. In our future work, we intend to expand upon this study by exploring the design of PI controllers for positive systems with time-varying delays.

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