Positive Consensus of Directed Multiagent Systems

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***Abstract*—This technical note investigates the positive consensus problem of multiagent systems with directed communication topologies, where all the agents have identical continuous-time positive linear dynamics. Existing works of such a problem mainly focus on the case, where networked communication topologies are of either undirected and incomplete graphs, or strongly connected directed graphs. In contrast to them, we study this problem in which the communication topologies of the multiagent system are described by directed graphs each containing a spanning tree, which is a more general and new scenario due to the interplay between the eigenvalues of the Laplacian matrix and the controller gains. Based on the existing results in spectral graph theory and positive linear systems theory, several necessary, and sufficient conditions on positive consensus of the directed multiagent system are derived through using linear matrix inequality techniques. A primal-dual iterative algorithm is developed for the computation of solutions. Finally, several numerical simulations are provided to illustrate the effectiveness of the proposed theoretical results.**

***Index Terms*—Cooperative control, directed graphs, multiagent systems, positive consensus, positive linear systems.**

I. Introduction

AMONG various classes of dynamic systems, there is a special type of systems named *positive dynamic systems*, which can be traced back to a book [1] on fundamental systems theory published by David Luenberger in 1979. Broadly speaking, a positive system can be regarded as a dynamic system whose states and outputs are constrained to be nonnegative given that its inputs and initial states are nonnegative [2]. During the past decades, there has been a large quantity of research devoted to the investigation of positive systems from various engineering and scientific communities for its broad applications in systems biology, pharmacokinetics, and electric circuits [3]–[5]. An important motivation behind positive systems theory is that, in the physical world, many descriptor variables are usually constrained to be nonnegative due to their intrinsic characteristics or physical laws, such as the material flows in a compartmental network [6]. Meanwhile, positive systems theory also finds its way in modeling stochastic or probabilistic processes, since probabilities are intrinsically nonnegative quantities, such as Markov chains [7].

Recently, the research on collective behaviors in a group of positive dynamic systems, referred to as a positive multiagent system, has attracted much attention due to the development and investigation of networked systems, such as Internet of Things (IoT) [8], smart grid [9], as well as dynamical buffer networks, and epidemic spreading processes [10]. Moreover, simple dynamic models such as integrators and first-order lags with positive gains, as well as their series/parallel connections, are all positive, which often represent some typical systems of moving objects. Although their dynamics is simple, the behavior of a multiagent system consisting of them is complicated, and deserves investigation especially in the study area of multiagent systems [11], [12]. Despite the research on positive consensus of undirected multiagent systems in [13] and [14], positive consensus of directed multiagent systems (PCDMASs) has seen very limited progress. This is because conventional positive systems theory, which is used to effectively analyze the positive consensus problem with undirected topologies, would unfortunately fail in the case of directed topologies.

Analysis and synthesis of networked positive systems using positive systems theory have attracted much attention in recent years [15]–[20], and particularly one hot issue of them is the positive consensus problem of multiagent systems [11], [21]–[23]. In [23], positive consensus of a group of single-input single-output systems was investigated and necessary and sufficient conditions were derived. This problem was further discussed in [13] where the dynamics of agents were described by single-input state-space models and several sufficient or necessary conditions for positive consensus were obtained. In [14], the case of multiple-input multiple-output agents was studied and some sufficient conditions for the existence of a solution for positive consensus were derived. This problem was further discussed in [24], where necessary and sufficient conditions of positive consensus were given.

From a practical point of view, a large number of real-world processes, such as the traffic flow networks, the epidemic spreading of networks, the social networks, and the gene regulatory networks, are usually modeled by directed topologies [25]. All the existing results on positive consensus were obtained assuming that the communication topology among the agents is undirected. Since the eigenvalues of the Laplacian matrix of a directed graph are in general complex numbers [26], the analyses and approaches proposed for positive consensus of undirected multiagent systems will fail in the case of directed topologies. In [27] and [28], the PCDMASs was investigated and sufficient conditions were derived. However, these results only give necessary or sufficient conditions for the existence of a solution to the PCDMASs, and the complete answer is still missing.

This technical article studies the positive consensus problem of directed multiagent systems. Compared with the existing results in [24] and [14], which have only investigated the undirected case, this technical note considers a more general case, where the agents' topology is assumed to be directed. Compared with the existing results in [27] and [28], which only present sufficient conditions, this technical note

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gives *necessary and sufficient conditions* for PCDMASs. Moreover, a *primal-dual iterative algorithm* is developed for computation. Numerical simulations are provided to verify the effectiveness of results.

Notations: The notation $\otimes$ means Kronecker product. The imaginary unit is denoted as j ($j^2 = -1$). The upper triangular elements of a symmetric matrix is denoted by $*$. For symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, the notation $A > B$ (respectively, $A \geq B$) means that $A - B$ is positive definite (respectively, positive semidefinite). For matrices $A, B \in \mathbb{R}^{m\times n}$, the notation $A \succ B$ (respectively, $A \succeq B$) means that $A - B$ is positive (respectively, nonnegative). For matrix A, $A \in \mathbb{M}$ means that A is Metzler, and $A \in \mathbb{H}$ means that A is Hurwitz. The n by n matrix of ones is denoted by J_n. Matrices are assumed in compatible dimensions if not stated specifically.

II. Preliminaries and Problem Formulation

A. Preliminaries

1) Graph Theory: The topology of a multiagent system can be well described by its graph. A graph is called directed if all its edges are directed from one vertex to another, and thus undirected graphs can be regarded as a special type of directed graphs. In this technical note, we investigate the multiagent system with directed communication topology that can be described by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{1, 2, \ldots, N\}$ is the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. For any $k, l \in \mathcal{V}$, we assume that $(k, l) \in \mathcal{E}$ if and only if agent l is able to access the full state of agent k. A path of graph $\mathcal{G}$ is defined as a sequence $\{k, l, m, \ldots, o, p, q\}$ such that all the successive tuples $(k, l), (l, m), \ldots, (o, p), (p, q) \in \mathcal{E}$. Graph $\mathcal{G}$ is assumed to contain a spanning tree, that is, there is a root $k \in \mathcal{V}$ such that there exists a path from k to any other vertex $l \in \mathcal{V}$. The adjacency matrix of graph $\mathcal{G}$ is defined and denoted as an $N \times N$ matrix Λ where $[\Lambda]_{kl} = 1$ if $(l, k) \in \mathcal{E}$ and $[\Lambda]_{kl} = 0$ otherwise. It is assumed that graph $\mathcal{G}$ contains no self-loops, that is, $[\Lambda]_{kk} = 0, k \in \mathcal{V}$. Define the neighbor set for any vertex $k \in \mathcal{V}$ as $\aleph_k := \{l \in \mathcal{V} : (l, k) \in \mathcal{E}\}$. The Laplacian matrix of graph $\mathcal{G}$ is defined and denoted as an $N \times N$ matrix Γ such that, $k, l \in \mathcal{V}$

$$[\Gamma]_{kl} = \begin{cases} \sum_{m=1}^{N} [\Lambda]_{km} & \text{if } k = l \\ -[\Lambda]_{kl} & \text{if } k \neq l. \end{cases} \tag{1}$$

By the well-known results in [29], the eigenvalues, $\lambda_i, i = 1, 2, \ldots, N$, of Laplacian matrix Γ are in general complex, which can be ordered as $0 = \mathrm{Re}(\lambda_1(\Gamma)) < \mathrm{Re}(\lambda_2(\Gamma)) \leq \ldots \leq \mathrm{Re}(\lambda_N(\Gamma))$.

2) Positive Systems: For a continuous-time linear system

$$\dot{\xi}(t) = E\xi(t) + F\tau(t) \tag{2}$$

where E, F are system matrices with appropriate dimensions, $\xi(t)$ is the state and $\tau(t)$ is the control input. The system in (2) is called positive if for any initial state $\xi(0) \succeq 0$ and $\tau(t) \succeq 0$ for $t \succeq 0$, we have $\xi(t) \succeq 0$ for $t \succeq 0$.

Lemma 1: The system in (2) is a continuous-time positive linear system if and only if $E \in \mathbb{M}$ and $F \succeq 0$.

Lemma 2: Any matrix M is Hurwitz if and only if there is a matrix $D > 0$ such that

$$DM + M^{\mathrm{T}}D < 0 \text{ or } MD + DM^{\mathrm{T}} < 0.$$

Lemma 3: $\forall M_1, M_2 \in \mathbb{M},\ M_1 \preceq M_2 \Rightarrow \alpha(M_1) \leq \alpha(M_2)$.

Lemma 4: Any matrix $M \in \mathbb{M}$ is Hurwitz if and only if there is a diagonal matrix $D > 0$ such that

$$DM + M^{\mathrm{T}}D < 0 \text{ or } MD + DM^{\mathrm{T}} < 0.$$

The above results [2], [29] pave the way for further analysis on PCDMASs.

B. Problem Formulation

Consider a homogeneous multiagent system constituting of N identical agents in a directed communication topology, and the dynamics of each agent can be described by the following positive linear system

$$\dot{x}_k(t) = Ax_k(t) + Bu_k(t),\ k = 1, 2\ldots, N \tag{3}$$

where $x_k(t) := [x_{k1}, x_{k2}, \ldots, x_{kp}]^{\mathrm{T}} \in \mathbb{R}^p$ is the state of agent k, and $u_k(t) \in \mathbb{R}^r$ is the input of agent k. The pair (A, B) is assumed to be stabilizable. By *Lemma 1*, the system in (3) is positive if and only if $A \in \mathbb{R}^{p\times p}$ is Metzler and $B \in \mathbb{R}^{p\times r}$ is nonnegative.

The following static state-feedback protocol is utilized:

$$U.K.t) = K \sum_{l=1}^{N} [\Lambda]_{kl}(x_l - x_k),\ k = 1, 2\ldots, N \tag{4}$$

where K is the controller gain matrix to be determined. For convenience of expression, define the global state $x(t) := [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{pN}$. Then, the overall multiagent system in (3) can be described as

$$\dot{x}(t) = \Omega x(t) \tag{5}$$

where $\Omega = \mathbb{I}_N \otimes A - \Gamma \otimes BK$.

The positive consensus problem of multiagent systems with directed communication topology is investigated in this technical article. Based on the above model settings, the problem to be solved is formulated and defined as follows.

Problem PCDMAS: Considering the multiagent system in (3) with the state-feedback protocol in (4), given any nonnegative initial states $x_k(0) \succeq 0$, $k = 1, 2, \ldots, N$, determine gain matrix K such that the consensus of the multiagent system in (3) is achievable, that is, $\lim_{t\to\infty}(x_l(t) - x_k(t)) = 0$, $\forall l, k = 1, \ldots, N$, and the state trajectory of each agent remains nonnegative, that is, $x_k(t) \succeq 0, k = 1, 2, \ldots, N$ for $t \succeq 0$.

III. Positive Consensus Analysis

In this section, several necessary and sufficient conditions and analyses on the solvability of *Problem PCDMAS* are derived using graph theory and positive systems theory.

Theorem 1: Problem PCDMAS is solved by gain matrix K if and only if all the following conditions are satisfied:

1) $BK \succeq 0$.
2) $A - \beta(\Gamma)BK \in \mathbb{M}$.
3) For any eigenvalue $\lambda_k(\Gamma)$, $k = 2, 3\ldots, N$

$$\tilde{A}_k := \begin{bmatrix} A - \mathrm{Re}(\lambda_k(\Gamma))BK & -\mathrm{Im}(\lambda_k(\Gamma))BK \\ \mathrm{Im}(\lambda_k(\Gamma))BK & A - \mathrm{Re}(\lambda_k(\Gamma))BK \end{bmatrix} \in \mathbb{H}. \tag{6}$$

Proof: Notice that, $\Omega = \mathbb{I}_N \otimes A - \Gamma \otimes BK =$

$$\begin{bmatrix} A - \sum_{m=1}^{N}[\Lambda]_{1m}BK & [\Lambda]_{12}BK & \cdots & \cdots & [\Lambda]_{1(N-1)}BK & [\Lambda]_{1N}BK \\ [\Lambda]_{21}BK & A - \sum_{m=1}^{N}[\Lambda]_{2m}BK & \cdots & \cdots & [\Lambda]_{2(N-1)}BK & [\Lambda]_{2N}BK \\ \vdots & \vdots & \ddots & \cdots & \ddots & \vdots \\ [\Lambda]_{N1}BK & [\Lambda]_{N2}BK & \cdots & \cdots & [\Lambda]_{N(N-1)}BK & A - \sum_{m=1}^{N}[\Lambda]_{Nm}BK \end{bmatrix}.$$

By *Lemma 1*, the positivity of the dynamics of all the agents, that is, the positivity of system (5) is preserved if and only if system matrix Ω is Metzler. By definition, $\Omega \in \mathbb{M}$ if and only if BK is nonnegative and $A - \sum_{m=1}^{N}[\Lambda]_{km}BK, k = 2, 3, \ldots, N$, are Metzler. As $\beta(\Gamma) = \max\{\sum_{m=1}^{N}[\Lambda]_{km} : k = 2, 3, \ldots, N\}$, it is easy to see that, $A - \sum_{m=1}^{N}[\Lambda]_{km}BK, k = 2, 3, \ldots, N$, are Metzler if and only if $A - \beta(\Gamma)BK$ is Metzler. So, the positivity of the multiagent system in (5) is preserved if and only if $BK \succeq 0$ and $A - \beta(\Gamma)BK \in \mathbb{M}$. By the well-known result in [30], a necessary and sufficient condition for the consensus of the multiagent system in (3) is that, $A - \lambda_k(\Gamma)BK$, $k = 2, 3, \ldots, N$, are Hurwitz. As the communication topology $\mathcal{G}$ is directed here, the Laplacian eigenvalues are in general complex, that is, $\lambda_k(\Gamma) \in \mathbb{C}$, $k = 2, 3, \ldots, N$. Hence, the analyses and approaches in [13], [24] will fail to solve *Problem PCDMAS*. To tackle this challenge, we define the following matrices:

$$S := \frac{1}{\sqrt{2}}\begin{bmatrix} -j\mathbb{I}_p & -j\mathbb{I}_p \\ \mathbb{I}_p & -\mathbb{I}_p \end{bmatrix}, S^* = \frac{1}{\sqrt{2}j}\begin{bmatrix} -\mathbb{I}_p & j\mathbb{I}_p \\ -\mathbb{I}_p & -j\mathbb{I}_p \end{bmatrix} \tag{7}$$

$$S^*\tilde{A}_k S = \hat{A}_k := \begin{bmatrix} A - \lambda_k(\Gamma)BK & 0 \\ 0 & \overline{A - \lambda_k(\Gamma)BK} \end{bmatrix}. \tag{8}$$

So $\hat{A}_k$ is similar to $\tilde{A}_k$, which implies that $\hat{A}_k \in \mathbb{H} \Leftrightarrow \tilde{A}_k \in \mathbb{H}$. Since $A - \lambda_k(\Gamma)BK$ and $\overline{A - \lambda_k(\Gamma)BK}$ are conjugate to each other, we have that, $A - \lambda_k(\Gamma)BK$ is Hurwitz if and only if $\overline{A - \lambda_k(\Gamma)BK}$ is Hurwitz, which implies the fact that $A - \lambda_k(\Gamma)BK \in \mathbb{H} \Leftrightarrow \tilde{A}_k \in \mathbb{H}$. That is, a necessary and sufficient condition for the consensus of the multiagent system in (3) is that, $\tilde{A}_k, k = 2, 3, \ldots, N$, are Hurwitz. The whole proof is completed. □

Corollary 1: Problem PCDMAS is solved by gain matrix $K = QB^{\mathrm{T}}W^{-1}$ if there exist a diagonal matrix $W > 0$ and $Q > 0$ such that all the following conditions are satisfied:

1) $BQB^{\mathrm{T}} \succeq 0$.
2) $AW - \beta(\Gamma)BQB^{\mathrm{T}} \in \mathbb{M}$.
3) $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(\Gamma))BQB^{\mathrm{T}} < 0$.

Proof: Taking $K = QB^{\mathrm{T}}W^{-1}$, then $BQB^{\mathrm{T}} \succeq 0$ is equivalent to $BKW \succeq 0$, which implies that $BK \succeq 0$ as $W > 0$ is a diagonal matrix. Similarly, $AW - BQB^{\mathrm{T}} \in \mathbb{M}$ is equivalent to $A - \beta(\Gamma)BQB^{\mathrm{T}}W^{-1} \in \mathbb{M}$, which implies that $A - \beta(\Gamma)BK \in \mathbb{M}$. If condition 3) holds, as $\mathrm{Re}(\lambda_k(\Gamma)) \succeq \mathrm{Re}(\lambda_2(\Gamma))$, $k = 2, 3, \ldots, N$, by *Lemma 3* we have that $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_k(\Gamma))BQB^{\mathrm{T}} < 0$, $k = 2, 3, \ldots, N$. Obviously, there exists a diagonal matrix

$$\bar{W} := \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} > 0 \tag{9}$$

such that

$$\bar{W}\tilde{A}_k^{\mathrm{T}} + \tilde{A}_k\bar{W} = \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} < 0 \tag{10}$$

where $Y := WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_k(\Gamma))BQB^{\mathrm{T}}$. By *Lemma 4*, we can conclude that matrices $\tilde{A}_k$, $k = 2, 3, \ldots, N$, are Hurwitz. The whole proof is completed. □

Corollary 2: Assume that matrices $\check{A} \in \mathbb{M}$ and $\check{B} \succeq 0$, and system matrices $A \in \mathbb{M}$ and $B \succeq 0$ are unknown where $A \in [\check{A}, \hat{A}]$, $B \in [\check{B}, \hat{B}]$, then *Problem PCDMAS* is solved by gain matrix $K = QB^{\mathrm{T}}W^{-1}$ if there exist a diagonal matrix $W > 0$ and $Q > 0$ such that all the following conditions hold:

1) $Q \succeq 0$.
2) $\check{A}W - \beta(\Gamma)\hat{B}Q\hat{B}^{\mathrm{T}} \in \mathbb{M}$.
3) $W\check{A}^{\mathrm{T}} + \check{A}W - 2\mathrm{Re}(\lambda_2(\Gamma))\hat{B}Q\hat{B}^{\mathrm{T}} \in \mathbb{M}$.
4) $W\hat{A}^{\mathrm{T}} + \hat{A}W - 2\mathrm{Re}(\lambda_2(\Gamma))\check{B}Q\check{B}^{\mathrm{T}} < 0$.

Proof: As $B \succeq 0$, conditions 1), 2) and 3) imply that $\forall A \in [\check{A}, \hat{A}]$, $\forall B \in [\check{B}, \hat{B}]$, $BQB^{\mathrm{T}} \succeq 0$, $AW - \beta(\Gamma)BQB^{\mathrm{T}} \in \mathbb{M}$, and $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(\Gamma))BQB^{\mathrm{T}} \in \mathbb{M}$, respectively. It is easy to see $W\hat{A}^{\mathrm{T}} + \hat{A}W - 2\mathrm{Re}(\lambda_2(\Gamma))\check{B}Q\check{B}^{\mathrm{T}} \succeq WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(\Gamma))BQB^{\mathrm{T}}$, then by *Lemma 3*, we have $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2)BQB^{\mathrm{T}} < 0$. The proof is completed. □

Remark 1: Notice that, Corollaries 1 and 2 only use $\beta(\Gamma)$, as well as the second largest Laplacian eigenvalue of topology $\mathcal{G}$, that is, $\lambda_2(\Gamma)$, instead of all its Laplacian eigenvalues. From the proofs of Corollaries 1 and 2, one can see that the gain matrix K obtained for Laplacian matrix Γ will also lead to positive consensus for any other Laplacian matrix Γ' as long as $\lambda_2(\Gamma) \preceq \lambda_2(\Gamma')$ and $\beta(\Gamma) \succeq \beta(\Gamma')$. More precisely, for a given topology $\mathcal{G}$ with $\lambda_2(\Gamma)$ and $\beta(\Gamma)$, a controller K is obtained by Corollaries 1 and 2, then *Problem PCDMAS* is also solvable for agents on a set of graphs: $\{(\mathcal{G}', \Gamma') \mid \lambda_2(\Gamma) \preceq \lambda_2(\Gamma'), \beta(\Gamma) \succeq \beta(\Gamma')\}$. Therefore, the results in Corollaries 1 and 2 indicate that the obtained controller gain K can robustly stabilize a set of positive multiagent systems with different topologies, although they remain as sufficient conditions for the solvability of *Problem PCDMAS*.

Based on *Theorem 1*, we can further derive the following equivalent results in *Theorem 2*.

Theorem 2: Problem PCDMAS is solved by gain matrix K if and only if all the following conditions hold:

1) $BK \succeq 0$.
2) $A - \beta(\Gamma)BK \in \mathbb{M}$.
3) For $k = 2, 3, \ldots, N$, $\exists P_k > 0, G_k, H_k$ such that one of the following conditions is satisfied:

$\Psi_k(P_k, G_k, H_k, K) :=$

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A} - \tilde{K}^{\mathrm{T}}\tilde{K} & * & * \\ P_k - G_k + H_k\tilde{A} & -H_k - H_k^{\mathrm{T}} & * \\ \tilde{B}_k^{\mathrm{T}}G_k - \tilde{K} & \tilde{B}_k^{\mathrm{T}}H_k^{\mathrm{T}} & -\mathbb{I} \end{bmatrix} < 0 \tag{11}$$

$\Xi_k(P_k, G_k, H_k, K) :=$

$$\begin{bmatrix} \tilde{A}G_k + G_k^{\mathrm{T}}\tilde{A}^{\mathrm{T}} - \tilde{B}_k\tilde{K}\tilde{K}^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} & * & * \\ P_k - G_k + H_k\tilde{A}^{\mathrm{T}} & -H_k - H_k^{\mathrm{T}} & * \\ \tilde{K}^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} - G_k & -H_k^{\mathrm{T}} & -\mathbb{I} \end{bmatrix} < 0. \tag{12}$$

where

$$\tilde{A} := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \tilde{K} := \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$$

$$\tilde{B}_k := \begin{bmatrix} \mathrm{Re}(\lambda_k)B & \mathrm{Im}(\lambda_k)B \\ -\mathrm{Im}(\lambda_k)B & \mathrm{Re}(\lambda_k)B \end{bmatrix}.$$

Proof: It suffices to show that condition 3) in *Theorem 2* is equivalent to condition 3) in *Theorem 1*. Notice that (11) is the dual of (12). Here, we only give the proof for (11) as the proof for (12) follows similarly.

If (11) holds, that is, $\tilde{A}_k \in \mathbb{H}$, $k = 2, 3, \ldots, N$, define $S_k := \begin{bmatrix} \mathbb{I} & \tilde{A}_k^{\mathrm{T}} & -\tilde{K}^{\mathrm{T}} \end{bmatrix}$ and notice that

$$S_k\Psi_kS_k^{\mathrm{T}} = \tilde{A}_k^{\mathrm{T}}P_k + P_k\tilde{A}_k < 0, k = 2, 3, \ldots, N \tag{13}$$

since $\Psi_k < 0$, $k = 2, \ldots, N$. By *Lemma 2*, we have that $\tilde{A}_k$, $k = 2, 3, \ldots, N$, are Hurwitz matrices.

On the other hand, if $\tilde{A}_k \in \mathbb{H}$, $k = 2, 3, \ldots, N$, by *Lemma 2* there must exist matrices $\tilde{P}_k > 0$ and $\tilde{H}_k > 0$ such that

$$\tilde{A}_k^{\mathrm{T}}\tilde{P}_k + \tilde{P}_k^{\mathrm{T}}\tilde{A}_k + \frac{1}{2}\tilde{A}_k^{\mathrm{T}}\tilde{H}_k\tilde{A}_k < 0 \tag{14}$$

which is equivalent to that

$$\Phi_k := \begin{bmatrix} \tilde{A}_k^{\mathrm{T}}\tilde{P}_k + \tilde{P}_k^{\mathrm{T}}\tilde{A}_k & * \\ \tilde{H}_k\tilde{A}_k & -\tilde{H}_k - \tilde{H}_k^{\mathrm{T}} \end{bmatrix} < 0. \tag{15}$$

Then, we define a positive semidefinite matrix

$$\Upsilon_k = \begin{bmatrix} \tilde{P}_k^{\mathrm{T}}\tilde{B}_k\tilde{B}_k^{\mathrm{T}}\tilde{P}_k & * \\ \tilde{H}_k^{\mathrm{T}}\tilde{B}_k\tilde{B}_k^{\mathrm{T}}\tilde{P}_k & \tilde{H}_k^{\mathrm{T}}\tilde{B}_k\tilde{B}_k^{\mathrm{T}}\tilde{H}_k \end{bmatrix} \geq 0. \tag{16}$$

There must exist $\epsilon_k > 0$ such that $\Phi_k + \epsilon_k\Upsilon_k < 0$, which is equivalent to $\epsilon_k\Phi_k + \epsilon_k^2\Upsilon_k < 0$, that is

$$\begin{bmatrix} \tilde{A}_k^{\mathrm{T}}\tilde{P}_k\epsilon_k + \epsilon_k\tilde{P}_k^{\mathrm{T}}\tilde{A}_k + \epsilon_k\tilde{P}_k^{\mathrm{T}}\tilde{B}_k\tilde{B}_k^{\mathrm{T}}\tilde{P}_k\epsilon_k & * \\ \epsilon_k\tilde{H}_k\tilde{A}_k + \epsilon_k\tilde{H}_k^{\mathrm{T}}\tilde{B}_k\tilde{B}_k^{\mathrm{T}}\tilde{P}_k\epsilon_k & -\epsilon_k\tilde{H}_k - \epsilon_k\tilde{H}_k^{\mathrm{T}} + \epsilon_k\tilde{H}_k^{\mathrm{T}}\tilde{B}_k\tilde{B}_k^{\mathrm{T}}\tilde{H}_k\epsilon_k \end{bmatrix} < 0. \tag{17}$$

Notice that, the matrix in (17) is a Schur complement to the following matrix:

$$\begin{bmatrix} \tilde{A}_k^{\mathrm{T}}\tilde{P}_k\epsilon_k + \epsilon_k\tilde{P}_k^{\mathrm{T}}\tilde{A}_k & * & * \\ \epsilon_k\tilde{H}_k\tilde{A}_k & -\epsilon_k\tilde{H}_k - \epsilon_k\tilde{H}_k^{\mathrm{T}} & * \\ \tilde{B}_k^{\mathrm{T}}\tilde{P}_k\epsilon_k & \tilde{B}_k^{\mathrm{T}}\tilde{H}_k\epsilon_k & -\mathbb{I} \end{bmatrix} < 0. \tag{18}$$

Taking $G_k = P_k = \epsilon_k\tilde{P}_k$ and $H_k = \epsilon_k\tilde{H}_k$, the above result implies the fact that

$$\Theta_k := \begin{bmatrix} \tilde{A}_k^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A}_k & * & * \\ P_k - G_k + H_k\tilde{A}_k & -H_k - H_k^{\mathrm{T}} & * \\ \tilde{B}_k^{\mathrm{T}}G_k & \tilde{B}_k^{\mathrm{T}}H_k & -\mathbb{I} \end{bmatrix} < 0. \tag{19}$$

Then define the following matrix:

$$T = \begin{bmatrix} \mathbb{I} & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ -\tilde{K} & 0 & \mathbb{I} \end{bmatrix} \tag{20}$$

and it is easy to see that $T\Theta_k T^{\mathrm{T}} = \Psi_k$, which implies that $\Psi_k < 0$, $k = 2, 3, \ldots, N$. The whole proof is completed. □

IV. Positive Consensus Design

In this section, the design of positive consensus is investigated, and a numerical algorithm is developed to solve *Problem PCDMAS* through using linear matrix inequalities.

Theorem 3: Problem PCDMAS is solved by gain matrix K if and only if all the following conditions are satisfied:

1) $BK \succeq 0$.

2) $A - \beta(\Gamma)BK \in \mathbb{M}$.

3) For $k = 2, 3, \ldots, N$, $\exists P_k > 0, G_k, H_k, U$ such that one of the following conditions is satisfied:

$\tilde{\Psi}_k(P_k, G_k, H_k, U, K) :=$

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A} - \tilde{K}^{\mathrm{T}}U - U^{\mathrm{T}}\tilde{K} + U^{\mathrm{T}}U & * & * \\ P_k - G_k + H_k\tilde{A} & -H_k - H_k^{\mathrm{T}} & * \\ \tilde{B}_k^{\mathrm{T}}G_k - \tilde{K} & \tilde{B}_k^{\mathrm{T}}H_k^{\mathrm{T}} & -\mathbb{I} \end{bmatrix} < 0 \tag{21}$$

,

$\tilde{\Xi}_k(P_k, G_k, H_k, U, K) :=$

$$\begin{bmatrix} \tilde{A}G_k + G_k^{\mathrm{T}}\tilde{A}^{\mathrm{T}} - \tilde{B}_k\tilde{K}U^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} - \tilde{B}_kU\tilde{K}^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} + \tilde{B}_kUU^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} & * & * \\ P_k - G_k + H_k\tilde{A}^{\mathrm{T}} & -H_k - H_k^{\mathrm{T}} & * \\ \tilde{K}^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} - G_k & -H_k^{\mathrm{T}} & -\mathbb{I} \end{bmatrix} < 0. \tag{22}$$

Proof: It suffices to show that condition 3) in *Theorem 2* is equivalent to condition 3) in *Theorem 3*. Since (21) is dual to (22), here we only give the proof for (21). If (11) in *Theorem 2* holds, obviously there exists a matrix $U := \tilde{K}$ such that $\tilde{A}^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A} - \tilde{K}^{\mathrm{T}}U - U^{\mathrm{T}}\tilde{K} + U^{\mathrm{T}}U = \tilde{A}^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A} - \tilde{K}^{\mathrm{T}}\tilde{K}$, so (21) in *Theorem 3* also holds. On the other hand, if (21) in *Theorem 3* holds, for $k = 2, 3, \ldots, N$, $\tilde{A}^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A} - \tilde{K}^{\mathrm{T}}U - U^{\mathrm{T}}\tilde{K} + U^{\mathrm{T}}U = \tilde{A}^{\mathrm{T}}G_k + G_k^{\mathrm{T}}\tilde{A} - \tilde{K}^{\mathrm{T}}\tilde{K} + (\tilde{K} - U)^{\mathrm{T}}(\tilde{K} - U)$, and so there exists a matrix

$$R := \begin{bmatrix} (\tilde{K} - U)^{\mathrm{T}}(\tilde{K} - U) & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} \geq 0 \tag{23}$$

such that $\Psi_k + R = \tilde{\Psi}_k < 0$, which implies $\Psi_k < 0$ and thus (11) in *Theorem 2* holds. Notice that $\tilde{A}G_k + G_k^{\mathrm{T}}\tilde{A}^{\mathrm{T}} - \tilde{B}_k\tilde{K}U^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} - \tilde{B}_kU\tilde{K}^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} + \tilde{B}_kUU^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} = \tilde{A}G_k + G_k^{\mathrm{T}}\tilde{A}^{\mathrm{T}} - \tilde{B}_k\tilde{K}\tilde{K}^{\mathrm{T}}\tilde{B}_k^{\mathrm{T}} + (\tilde{B}_k\tilde{K} - \tilde{B}_kU)(\tilde{B}_k\tilde{K} - \tilde{B}_kU)^{\mathrm{T}}$. The proof for (22) follows similarly. The whole proof is completed. □

Two numerical schemes are developed in *Algorithms 1* and *2* based on *Theorem 3*, respectively, to obtain the appropriate controller gain matrix. Define $\sigma(K)$ as the minimal element of matrices BK and $A - \beta(\Gamma)BK + vI$, where v is a free variable. Notice that, throughout the iterations of *Algorithm 1*, we have that $\xi^{(r+1)} \preceq \xi^{(r)}$ for $r \succeq 2$, since there always exist $K = K^{(r)}$ and $P_k > 0, G_k, H_k$, $2 \leq k \leq N$ such that $-\xi^{(r-1)} \preceq \sigma(K^{(r-1)}) \preceq -\xi^{(r)} \preceq \sigma(K^{(r)})$, $2 \leq k \leq N$. Similarly, one can show that *Algorithm 2* has an identical monotonic property. Combining *Algorithms 1* and *2* together, a primal-dual iterative algorithm is developed in *Algorithm 3* to solve *Problem PCDMAS*. The initial value of K, denoted by $\mathcal{K}$, is determined through solving a linear matrix inequality $WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2(\Gamma))BQB^{\mathrm{T}} < 0$ with respect to $W > 0$ and $Q > 0$, and then $\mathcal{K} = QB^{\mathrm{T}}W^{-1}$. $\mathcal{K}$ is a feasible solution for consensus of the multiagent system without positivity.

V. Numerical Simulation

In this section, a numerical example is provided to demonstrate the effectiveness of our theoretical results in Sections III and IV. Consider a four-agent system in (3) with a single topology, where the system matrices are

$$A = \begin{bmatrix} a & 3 & 2 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

and $a \in [-0.5, 0.5]$. Notice that this system is unstable for any $a \in [-0.5, 0.5]$. The communication topology is described by a directed graph $\mathcal{G}$ whose Laplacian matrix is

$$\Gamma := \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Algorithm 1: Primal Positive Consensus Design.

procedure PPCD $A, B, \Gamma, \mathcal{K}, \mathrm{Tol}$
  Initialize $\xi^{(1)} \leftarrow +\infty$, $r \leftarrow 1$ and $K^{(1)} \leftarrow \mathcal{K}$.
  while $\xi^{(r)} \succ 0$ **do**
    Update $U^{(r)} \leftarrow \tilde{K}^{(r)}$.
    Minimize ξ
  w.r.t. v, $P_k > 0, G_k, H_k, K$, $2 \le k \le N$.

$$\text{s.t.} \begin{cases} BK \succeq -\xi J_p \\ A - \beta(\Gamma)BK + vI \succeq -\xi J_p \\ \tilde{\Psi}_k(P_k, G_k, H_k, U^{(r)}, K) < 0,\ 2 \le k \le N \end{cases}$$

    Update $r \leftarrow r + 1$.
    Update $\xi^{(r)} \leftarrow \xi$ and $K^{(r)} \leftarrow K$.
    if $|\xi^{(r)} - \xi^{(r-1)}|/\xi^{(r)} \prec \mathrm{Tol}$ **then**
      stop
    end if
  end while

  returnξ, K.
end procedure

Algorithm 2: Dual Positive Consensus Design.

procedure DPCD $A, B, \Gamma, \mathcal{K}, \mathrm{Tol}$
  Initialize $\xi^{(1)} \leftarrow +\infty$, $r \leftarrow 1$ and $K^{(1)} \leftarrow \mathcal{K}$.
  while $\xi^{(r)} \succ 0$ **do**
    Update $U^{(r)} \leftarrow \tilde{K}^{(r)}$.
    Minimize ξ
  w.r.t. v, $P_k > 0, G_k, H_k, K$, $2 \le k \le N$.

$$\text{s.t.} \begin{cases} BK \succeq -\xi J_p \\ A - \beta(\Gamma)BK + vI \succeq -\xi J_p \\ \tilde{\Xi}_k(P_k, G_k, H_k, U^{(r)}, K) < 0,\ 2 \le k \le N \end{cases}$$

    Update $r \leftarrow r + 1$.
    Update $\xi^{(r)} \leftarrow \xi$ and $K^{(r)} \leftarrow K$.
    if $|\xi^{(r)} - \xi^{(r-1)}|/\xi^{(r)} \prec \mathrm{Tol}$ **then**
      stop
    end if
  end while

  returnξ, K.
end procedure

By the above model settings, it is easy to see that $\beta(\Gamma) = 2$. The Laplacian eigenvalues are, respectively, $\lambda_1(\Gamma) = 0$, $\lambda_2(\Gamma) = 1.5 + 0.866j$, $\lambda_3(\Gamma) = 1.5 - 0.866j$ and $\lambda_4(\Gamma) = 2$.

To compare the performance of *Algorithms 1 to 3* in solving *Problem PCDMAS*, we generate a set of 500 non-Hurwitz system matrices by selecting $a := -1/2 + (1/500)n$ $(n = 1, 2, \ldots, 500)$ for simulation. The simulation results are summarized in Table I, and we can see that *Algorithm 3* showed the best performance for solution as its success rate is 100%. Besides, Algorithm 2 had the lowest average iteration number in success. Therefore, the success rate of Algorithm 3 was improved significantly due to the combination of Algorithms 1 and 2, though the average iteration number increased slightly. In particular, neither *Algorithm 1* nor *Algorithm 2* was able to provide a feasible solution for

Algorithm 3: Primal-Dual Positive Consensus Design.

procedure PDPCD $A, B, \Gamma, \mathrm{Tol}$
  Initialize $\xi^{(1)} \leftarrow +\infty$, $r \leftarrow 1$ and $l \leftarrow 1$.
  Initialize $K^{(l)} \leftarrow QB^{\mathrm{T}}W^{-1}$ by finding W, Q:

$$\text{s.t.} \begin{cases} WA^{\mathrm{T}} + AW - 2\mathrm{Re}(\lambda_2)BQB^{\mathrm{T}} < 0 \\ W > 0,\ Q > 0 \end{cases}$$

  while $\xi^{(r)} \succ 0$ **do**
    Update $\mathcal{K} \leftarrow K^{(l)}$.
    Update $\xi, K \leftarrow \mathrm{PPCD}(A, B, \Gamma, \mathcal{K}, \mathrm{Tol})$.
    if $\xi \le 0$ **then**
      break
    else
      Update $r \leftarrow r + 1$ and $l \leftarrow l + 1$.
      Update $\xi^{(r)} \leftarrow \xi$ and $K^{(l)} \leftarrow K$.
    end if
    Update $\mathcal{K} \leftarrow K^{(l)}$.
    Update $\xi, K \leftarrow \mathrm{DPCD}(A, B, \Gamma, \mathcal{K}, \mathrm{Tol})$.
    if $\xi \le 0$ **then**
      break
    else
      Update $r \leftarrow r + 1$ and $l \leftarrow l + 1$.
      Update $\xi^{(r)} \leftarrow \xi$ and $K^{(l)} \leftarrow K$.
    end if
    if $|\xi^{(r)} - \xi^{(r-1)}|/\xi^{(r)} \prec \mathrm{Tol}$ **then**
      stop
    end if
  end while

  returnξ, K.
end procedure

TABLE I
SIMULATION RESULTS OF ALGORITHMS 1 TO 3

Algorithm	Success Rate	Average Iteration Number
1	90.2%	3.11
2	80.4%	3.03
3	100%	3.39

$a = 0.5$. However, *Algorithm 3* successfully gave a feasible solution as

$$K = \begin{bmatrix} 0.1579 & 0.7369 & 0.2575 \\ 0.1579 & 0.1184 & 1.2815 \end{bmatrix}. \tag{24}$$

It can be verified that

$$BK = \begin{bmatrix} 0.3158 & 1.4737 & 0.5150 \\ 0.4737 & 2.2106 & 0.7725 \\ 0.4737 & 0.9737 & 2.8205 \end{bmatrix} \succ 0$$

$$A - \beta(\Gamma)BK = \begin{bmatrix} -0.1316 & 0.0526 & 0.9700 \\ 0.0526 & -7.4211 & 0.4550 \\ 0.0526 & 0.0526 & -8.6409 \end{bmatrix} \in \mathbb{M}.$$

Also, the eigenvalues of $\tilde{A}_2$, $\tilde{A}_3$, and $\tilde{A}_4$ are $\{-0.0757 \pm 0.4569j,\ -5.9955 \pm 2.5966j,\ -7.4490 \pm 1.5768j,$ $\{-0.0757 \pm 0.4569j,\ -5.9955 \pm 2.5966j,\ -7.4490 \pm 1.5768j\}$, and $\{-0.1252,\ -0.1252,\ -8.6658,\ -8.6658,\ -7.4027,\ -7.4027\}$, respectively. Therefore, the conditions in Theorem 1 are all satisfied.

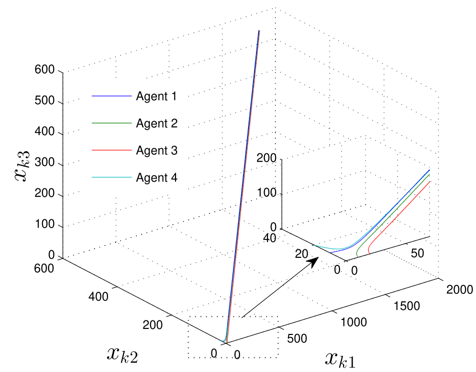

Fig. 1. Consensus of the multiagent system with controller (24).

The state trajectories of the multiagent system with controller (24) are shown in Fig. 1.

VI. Conclusion

This technical article has investigated the positive consensus issue for directed multiagent systems. The objective of positive consensus is to deign a controller such that the overall system can reach consensus and meanwhile the states of all the agents remain nonnegative throughout the evolutionary process. Using the spectral graph theory and positive linear systems theory, several necessary and sufficient conditions on PCDMASs have been derived. A primal-dual iterative algorithm was developed for computation, and finally numerical simulations were given to verify the theoretical results and algorithm.

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