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Output Reachable Set-Based Leader-Following Consensus of Positive Agents Over Switching Networks

Chenchen Fan, James Lam, *Fellow, IEEE*, Kai-Fung Chu, *Member, IEEE*, Xiujuan Lu, and Ka-Wai Kwok, *Senior Member, IEEE*

***Abstract*—This work addresses the output reachable set-based leader-following consensus problem, focusing on a group of positive agents over directed dwell-time switching networks. Two types of non-negative disturbances, namely, 1) L_1-norm bounded disturbances and 2) $L_{\infty,1}$-norm bounded disturbances are studied. Meanwhile, a class of directed dwell-time switching networks for modeling the communication protocol of positive agents is investigated. To deal with the presence of disturbances, an output-feedback control protocol is developed to achieve a robust consensus with positivity preserved based on the output reachable set. By exploiting the positive characteristics, switched linear copositive Lyapunov functions are adopted to establish output reachable set-based consensus conditions. These conditions can facilitate the control protocol design by solving a bilinear programming problem, and also generate hyperpyramidal regions to confine the output consensus error. A particle swarm optimization-based (PSO-based) algorithm is applied to compute the controller gains and optimize the volume of the hyperpyramids. The proposed methods are verified by the presented numerical case studies.**

***Index Terms*—Directed switching networks, output reachable set-based consensus, particle swarm optimization-based (PSO-based) algorithm, positive linear systems.**

I. Introduction

COOPERATIVE coordination is ubiquitous in natural populations and social networks. Among the studies on cooperative control for multiagent systems, consensus is an important topic that receives great attention [1], [2]. The main task of consensus is to develop control laws by using neighboring information to make all agents reach an agreement. Leader-following consensus for multiagent systems has attracted particular attention due to its vast practical applications in quadrotor formation flight [3], vehicle platooning [4], multirobot coordination [5], and distributed voltage regulation of power networks [6]. Leader-following consensus is investigated from different perspectives. In terms of the communication topology, two types of distributed adaptive control protocols have been developed to achieve leader-following consensus for networked Lagrangian systems under both directed and undirected graphs [7]. Apart from fixed topology, switching topologies and time-varying topologies have been considered in dealing with leader-following consensus problems [8], [9]. From the perspective of agent dynamics, the leader-following consensus problem for linear and Lipschitz nonlinear agents has been addressed in [10], which employs distributed event-triggered control protocol with quantized information. Moreover, second-order and high-order agents have been widely studied to compensate for the modeling limitation of first-order agents, achieving leader-following consensus [11], [12]. Recently, positive agents have received significant attention due to their modeling capabilities for real non-negative group behaviors [13], [14]. Several studies have reported leader-following consensus for a class of positive agents under undirected fixed topology [15] and switching topologies [16]. However, the leader-following consensus with positivity preservation under directed switching topologies has not been fully explored.

The ubiquity of disturbances in practical systems has motivated many studies on the robustness of control [17], [18], [19]. For multiagent systems affected by disturbances, various robust consensus results based on different performance indexes have been developed. Robust H_∞ consensus and leader-following H_∞ consensus problems have been widely investigated for multiagent systems subject to energy-bounded disturbances [20], [21], [22]. Dissipativity theory in terms of input-output energy is the generalization of H_∞ performance and passivity performance. Based on this, dissipativity-based consensus has been conducted for fuzzy multiagent systems [23] and time-varying delayed nonlinear multiagent systems [24] to guarantee the strict dissipativity performance of consensus error systems under energy-bounded disturbances. However, both H_∞ consensus

Manuscript received 24 August 2022; revised 31 December 2022 and 19 May 2023; accepted 7 June 2023. This work was supported in part by the Research Grants Council (RGC) under Grant 17205721, Grant 17209021, and Grant 17207020, and in part by the Innovation and Technology Commission of the HKSAR Government under the InnoHK initiative, Centre for Garment Production Ltd. This article was recommended by Associate Editor F. Wu. *(Corresponding authors: Ka-Wai Kwok; James Lam.)*

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Color versions of one or more figures in this article are available at https://doi.org/10.1109/TCYB.2023.3286416.

Digital Object Identifier 10.1109/TCYB.2023.3286416

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and dissipativity-based consensus are limited to handling energy-bounded disturbances. The aforementioned disturbances using H_∞ and dissipativity performance measures are based on L_2 signal space, namely, energy-bounded signals. To characterize the performance of positive dynamics, 1-norm of signals which measures the size of signals by summing the component values is more appropriate [25]. This article investigates two types of non-negative disturbances based on the 1-norm, namely, one with a bounded L_1-norm and the other with a bounded $L_{\infty,1}$-norm bounded. The L_1-norm measures the accumulation of 1-norm over all time, while the $L_{\infty,1}$-norm measures the maximum magnitude of 1-norm at a given time. Therefore, both the cumulative and instantaneous effects of disturbances are studied. Instead of using commonly employed quadratic Lyapunov functions, linear copositive Lyapunov functions are more suitable for exploiting the considered positive scenarios.

Furthermore, in the consensus problem under disturbance effects, rather than just obtaining a disturbance attenuation level, what the consensus error can reach under disturbances is often more meaningful. Therefore, the concept of reachable set is introduced. Reachable set refers to the region of the system state or output that is reachable under some specific classes of disturbances [25], [26], [27]. The problems of estimating reachable sets and synthesizing controllers that determine bounding regions of system dynamic variables are classical topics in control theory [28], [29], [30], [31]. Regarding the consensus problem under disturbance effects, consensus error may not converge to zero [17], but a bounded consensus can often be achieved under control protocols. It is worth studying to quantify such consensus boundedness by using the concept of reachable set.

Although the results focusing on consensus of positive agents under switching topologies can be found in [16] and [32], the output reachable set-based leader-following consensus problem for positive agents over directed dwell-time switching networks is studied for the first time by this work. The primary contribution can be summarized as follows.

1) Different from [16] without considering disturbances and [32] with only considering L_1-norm bounded disturbances, our work studies the effect of both L_1-norm bounded disturbances and $L_{\infty,1}$-norm bounded disturbances. In the presence of such kinds of disturbances, a novel scheme of output reachable set-based leader-following consensus that can also maintain the positivity property is established.
2) Sufficient conditions based on a switched linear copositive Lyapunov function are developed to guarantee that the output reachable set of the consensus error is enclosed by a hyperpyramid bounding region. The adopted switched linear copositive Lyapunov functions can reduce conservatism as the value of M increases, whereas the multiple co-positive Lyapunov function used in [16] does not provide explicit guidance for conservatism reduction.
3) Controller design and reachable set bounding are accomplished simultaneously by solving a bilinear programming problem, for which a particle swarm optimization-based (PSO-based) algorithm is proposed. In comparison to iterative algorithms for solving bilinear programming problems [33], [34], [35], the proposed PSO-based algorithm can avoid being stuck in local optima and reduce the dependence on initial values.

An outline of the remainder is given as follows. Some mathematical preliminaries and problem descriptions are presented in Section II. In Section III, the main results for deriving output reachable set-based consensus conditions are provided. Section IV develops numerical simulations to validate the proposed methods. Finally, the work is concluded in Section V.

Notation: The symbols $\mathbb{R}^n$, $\mathbb{R}^n_{0,+}$ and $\mathbb{R}^n_+$ represent the sets of n-dimensional real vectors with real entries, non-negative entries, and positive entries, respectively. The real matrix with dimension $m \times n$ is denoted by $\mathbb{R}^{m\times n}$. The set of natural numbers is represented by $\mathbb{N}_0$. The superscript "T" is used to denote the matrix transposition. The identity matrix with appropriate dimensions is I. The column vector in appropriate size with all elements being 1 is denoted by $\mathbf{1}$. Vector $x \succeq 0$ and matrix $M \succeq 0$ denote, respectively, that every entry of x and M is non-negative. For vector $\omega \in \mathbb{R}^{n_\omega}$, the 1-norm $\|\omega\|_1$, L_1-norm $\|\omega\|_{1,1}$, and $L_{\infty,1}$-norm $\|\omega\|_{\infty,1}$ are defined as $\sum_{i=1}^{n_\omega} |\omega_i|$, $\int_0^\infty \|\omega(\tau)\|_1 d\tau$, and $\sup_{t\geq 0} \|\omega(t)\|_1$, respectively. Matrix $M \in \mathbb{R}^{n\times n}$ is Metzler if all its off-diagonal entries are non-negative. The Kronecker product is represented by $\otimes$.

II. Preliminaries and Problem Formulation

A. Preliminaries of Positive Systems

Given a linear system as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D\omega(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $u(t) \in \mathbb{R}^{n_u}$, $\omega(t) \in \mathbb{R}^{n_\omega}$ and $y(t) \in \mathbb{R}^{n_y}$ denote the system state, the control input, the disturbance input, and the system output, respectively. Some statements regarding positivity are given below.

Definition 1 [36]: System (1) is called positive if for all initial conditions $x(0) \succeq 0$, input $u(t) \succeq 0$ and disturbance $\omega(t) \succeq 0$, its state $x(t)$ and output $y(t)$ are in the non-negative orthant for all $t \geq 0$.

Lemma 1 [36]: System (1) is positive if and only if matrix A is Metzler, and B, C, and D are non-negative matrices.

B. Graph Theory

A directed graph $\bar{\mathcal{G}}$ containing N followers and a leader labeled as v_0 is considered. Among the N followers, a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with node set as $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and edge set as $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is employed to represent the connection. A directed edge described by pair (v_i, v_j) implies that there is an information flow from node v_j to node v_i. The neighbors of node v_i are denoted by $\mathcal{N}_i := \{v_j \mid (v_j, v_i) \in \mathcal{E}\}$. The adjacency matrix is given as $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ where $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The Laplacian matrix of a directed graph is defined as $L = (l_{ij})_{N\times N}$ with $l_{ii} = \sum_{j\in\mathcal{N}_i} a_{ij}$ and $l_{ij} = -a_{ij}$, for $i \neq j$. A directed path is a sequence of connected edges in the graph. The leader adjacency matrix is denoted by a diagonal matrix $G = \mathrm{diag}\{g_1, g_2, \ldots, g_N\}$, where

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$g_i \geq 0$ for any i. For an unweighed graph, if the leader is a neighbor of node v_i, then $g_i = 1$; otherwise, $g_i = 0$. The graph $\bar{G}$ is said to contain a directed spanning tree if at least one node is found to have a directed path to all other nodes.

C. Problem Formulation

We consider a class of multiagent systems consisting of N followers and a leader with positive dynamics. Each follower is described as

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t) + D\omega_i(t), \\ y_i(t) = Cx_i(t) \end{cases} i = 1, 2, \ldots, N \quad (2)$$

where $x_i(t) \in \mathbb{R}^{n_x}$, $u_i(t) \in \mathbb{R}^{n_u}$, $\omega_i(t) \in \mathbb{R}^{n_\omega}$ and $y_i(t) \in \mathbb{R}^{n_y}$ represent the state, the control input, the disturbance input and the output of agent i, respectively. Matrix A is Metzler, and matrices B, C, and D are non-negative.

The leader is represented as

$$\begin{cases} \dot{x}_0(t) = Ax_0(t) \\ y_0(t) = Cx_0(t) \end{cases} \quad (3)$$

where $x_0 \in \mathbb{R}^{n_x}$ and $y_0 \in \mathbb{R}^{n_y}$ are the state and output of the leader agent, respectively.

Assumption 1: The pair (A, B) is stabilizable, and (A, C) is detectable.

In practical scenarios, communication networks are often subject to variations that render them neither fixed nor static. Switching communication networks are preferable to approximate some time-varying or complex communication networks than a fixed communication network. In this work, the following dwell-time switching communication networks are employed to describe the communication of agents.

Dwell-Time Switching Networks: Assume that the follower and leader communicate with their neighbors through directed switching networks $\bar{G}_{\sigma(t)}$, where $\sigma(t)$ is a switching signal as $\sigma(t) : [0, +\infty) \rightarrow \mathcal{S} \triangleq \{1, 2, \ldots, S\}$. Suppose there is a strictly increasing switching sequence $\{t_k\}$, $k = 0, 1, 2, \ldots$, and $t_0 = 0$. $\bar{G}_{\sigma(t)}$ is unchanged for $t \in [t_k, t_{k+1})$, and obviously, $\bar{G}_{\sigma(t)} \in \{\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_S\}$. Corresponding to switching networks $\bar{G}_{\sigma(t)}$, the Laplacian matrices of followers are denoted by $L_{\sigma(t)} = (l_{ij\sigma(t)})_{N\times N}$, and the leader adjacency matrix is given by $G_{\sigma(t)} = \text{diag}\{g_{1\sigma(t)}, g_{2\sigma(t)}, \ldots, g_{N\sigma(t)}\}$. The switching law of the communication networks is assumed to obey the dwell-time switching. The relevant definition and assumption are given as follows.

Definition 2: For switching signal $\sigma(t)$ with the switching sequence $\{t_k\}$, $k = 0, 1, \ldots, \tau_d = \inf_{k\in\mathbb{N}_0}\{t_{k+1} - t_k\}$ is known as the dwell time of $\sigma(t)$, and $\sigma(t) \in \mathcal{D}_{\tau_d} \triangleq \{\sigma(t) \mid \{t_k\} : t_{k+1} - t_k \geq \tau_d,\ \tau_d > 0\}$ denotes all switching strategies with dwell time not less than τ_d.

Remark 1: From Definition 2, it is known that the dwell time τ_d is the minimum time duration of any switching mode. This consideration is natural in practice because when switching from one component to another, the system may need to perform certain operations with non-negligible duration leading to minimal dwell time requirements on each mode. Under the dwell-time switching setting, the dwell time τ_d is often prespecified or assumed to be known [26].

Assumption 2 [37]: The directed graph $\bar{G}_{\sigma(t)}$ contains a spanning tree, whose leader node is the root.

Given a switching communication network, the following output-feedback control protocol is developed:

$$u_i(t) = K_{\sigma(t)}\left(\sum_{j\in\mathcal{N}_i} a_{ij\sigma(t)}\big(y_j(t) - y_i(t)\big) + g_{i\sigma(t)}(y_0(t) - y_i(t))\right) \quad (4)$$

where $K_{\sigma(t)}$ is the control gain to be designed.

Remark 2: It is noted that the designed control protocol (4) with $K_{\sigma(t)}$ is based on an assumption that switching signal $\sigma(t)$ of the communication network is known to all agents. This corresponds to situations where the switching behavior can be broadcast globally. However, if $\sigma(t)$ is unknown globally, a distributed control protocol can be developed by replacing $K_{\sigma(t)}$ with K. The design principle of the control gain remains the same.

Under control protocol (4), with $x(t) = [x_1^{\mathrm{T}}(t)\ x_2^{\mathrm{T}}(t) \ldots x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $y(t) = \big[y_1^{\mathrm{T}}(t)\ y_2^{\mathrm{T}}(t) \ldots y_N^{\mathrm{T}}(t)\big]^{\mathrm{T}}$, $\omega(t) = \big[\omega_1^{\mathrm{T}}(t)\ \omega_2^{\mathrm{T}}(t) \ldots \omega_N^{\mathrm{T}}(t)\big]^{\mathrm{T}}$, $\bar{x}_0(t) = \mathbf{1} \otimes x_0(t)$, the closed-loop dynamics of the followers can be written in the following compact form:

$$\begin{cases} \dot{x}(t) = \big(I \otimes A - \big(L_{\sigma(t)} + G_{\sigma(t)}\big) \otimes BK_{\sigma(t)}C\big)x(t) \\ \qquad\quad + \big(G_{\sigma(t)} \otimes BK_{\sigma(t)}C\big)\bar{x}_0(t) + (I \otimes D)\omega(t) \\ y(t) = (I \otimes C)x(t). \end{cases} \quad (5)$$

Within the framework of positive multiagent systems, the closed-loop system (5) under the control protocol should maintain positivity. Denoting $X_{\sigma(t)} = L_{\sigma(t)} + G_{\sigma(t)}$, it follows that:

$$X_{\sigma(t)} = \begin{bmatrix} l_{11\sigma(t)} + g_{1\sigma(t)} & \cdots & l_{1N\sigma(t)} \\ \vdots & \ddots & \vdots \\ l_{N1\sigma(t)} & \cdots & l_{NN\sigma(t)} + g_{N\sigma(t)} \end{bmatrix} \triangleq \begin{bmatrix} \xi_{11,\sigma(t)} & \cdots & \xi_{1N,\sigma(t)} \\ \vdots & \ddots & \vdots \\ \xi_{N1,\sigma(t)} & \cdots & \xi_{NN,\sigma(t)} \end{bmatrix}. \quad (6)$$

Then, with $\sigma(t) \in \mathcal{S} = \{1, 2, \ldots, S\}$, the following proposition gives the conditions to preserve the positivity property of the closed-loop system.

Proposition 1 [16]: The closed-loop system (5) is positive if and only if $((I \otimes A) - (L_{\sigma(t)} + G_{\sigma(t)}) \otimes BK_{\sigma(t)}C)$ is Metzler, $(G_{\sigma(t)} \otimes BK_{\sigma(t)}C) \succeq 0$, $(I \otimes D) \succeq 0$, and $(I \otimes C) \succeq 0$, which give the following equivalent conditions:

$$A - \xi_{\max}^{\sigma(t)} BK_{\sigma(t)}C \quad \text{is Metzler} \quad (7)$$

$$BK_{\sigma(t)}C \succeq 0 \quad (8)$$

where $\xi_{\max}^{\sigma(t)} = \max_{i=1,2,\ldots,N} \xi_{ii,\sigma(t)}$, $\sigma(t) \in \mathcal{S}$.

To investigate the leader-following consensus problem under external disturbances, the consensus errors of state and output are denoted as $\eta_{xi}(t) = x_i(t) - x_0(t)$ and $\eta_{yi}(t) = y_i(t) - y_0(t)$, respectively. With $\eta_x(t) = [\eta_{x1}^{\mathrm{T}}(t)\ \eta_{x2}^{\mathrm{T}}(t) \ldots \eta_{xN}^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\eta_y(t) = [\eta_{y1}^{\mathrm{T}}(t)\ \eta_{y2}^{\mathrm{T}}(t) \ldots \eta_{yN}^{\mathrm{T}}(t)]^{\mathrm{T}}$, the overall consensus error dynamic system is represented by

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$$\begin{cases} \dot{\eta}_x(t) = \big(I \otimes A - \big(L_{\sigma(t)} + G_{\sigma(t)}\big) \otimes BK_{\sigma(t)}C\big)\eta_x(t) \\ +(I \otimes D)\omega(t) \\ \eta_y(t) = (I \otimes C)\eta_x(t). \end{cases} \tag{9}$$

When the closed-loop system (5) is positive, it holds that $((I \otimes A) - (L_{\sigma(t)} + G_{\sigma(t)}) \otimes BK_{\sigma(t)}C)$ is Metzler, $(I \otimes D) \succeq 0$, and $(I \otimes C) \succeq 0$. In this work, we consider a class of cases where the initial condition of error dynamic system (9) satisfies $\eta_x(0) \succeq 0$. It can be obtained that error dynamic system (9) is also positive.

To facilitate the output reachable set-based consensus, with the initial consensus error $\eta_x(0) \succeq 0$, one can always find $\bar{r}$ such that

$$\bar{r}^{\mathrm{T}}\eta_x(0) \leq 1 \tag{10}$$

where $\bar{r} = \big[r_1^{\mathrm{T}}\ r_2^{\mathrm{T}}\ \ldots\ r_N^{\mathrm{T}}\big]^{\mathrm{T}}$, and $r_i \in \mathbb{R}_+^{n_x}$, $i = 1, 2, \ldots, N$, are positive vectors.

Given a group of leader-following positive agents under external disturbances, the output reachable set-based consensus as a kind of robust consensus is investigated. A hyperpyramid $\Theta(\check{q})$ as the following form:

$$\Theta(\check{q}) \triangleq \Big\{\zeta \in \mathbb{R}_{0,+}^{n_y} \mid \check{q}^{\mathrm{T}}\zeta \leq 1,\ \check{q} \in \mathbb{R}_+^{n_y}\Big\} \tag{11}$$

will be employed for bounding the reachable set of the output consensus error $\eta_{yi}(t)$, $i = 1, 2, \ldots, N$, generated by system (9) under the initial condition (10). Then, the definition of the output reachable set-based leader-following consensus is given as follows.

Definition 3: The output reachable set-based leader-following consensus of positive multiagent system (2)–(3) under control protocol (4) is said to be achieved, if the closed-loop system (5) is positive, and the reachable set of the output consensus error $\eta_{yi}(t)$, $i = 1, 2, \ldots, N$, generated by system (9) under the initial condition (10) is enclosed by the hyperpyramid $\Theta(\check{q})$ in (11).

Remark 3: Regarding the positivity property, motivated by [25] and [28], hyperpyramids are appropriate for characterizing the reachable set of the output consensus error in this work. It is noted that the reachable set of the output consensus error $\eta_{yi}(t)$, $i = 1, 2, \ldots, N$, from the initial condition (10) is also known as all the trajectories of $\eta_{yi}(t)$ generated by error dynamic system (9) under the initial condition (10). For $\eta_{yi}(t) \in \mathbb{R}_{0,+}^{n_y}$ from system (9) under the initial condition (10), if one can find a positive vector $\check{q} \in \mathbb{R}_+^{n_y}$ such that $\check{q}^{\mathrm{T}}\eta_{yi}(t) \leq 1$, $t \geq 0$, then it can be said that the reachable set of the output consensus error $\eta_{yi}(t)$ is enclosed by the hyperpyramid $\Theta(\check{q})$ in (11).

III. Main Results

In this section, two typical classes of non-negative disturbances subject to L_1-norm bounded and $L_{\infty,1}$-norm bounded are considered. Due to the effect of disturbances, the main objective is to find control protocols to ensure the output reachable set-based consensus. Regarding the positivity property of the considered systems, the copositive Lyapunov function method is more appropriate and effective for dealing with the 1-norm based disturbances. Therefore, pyramidal bounding regions will be generated by the following results.

A. L_1-Norm Bounded Disturbances

For the ith follower, $i = 1, 2, \ldots, N$, the non-negative disturbance ω_i is assumed to be L_1-norm bounded, satisfying

$$\|\omega_i\|_{1,1} \triangleq \int_0^{\infty} \|\omega_i(\tau)\|_1 d(\tau) \leq \bar{\omega}_{1,1,i} \tag{12}$$

where $\bar{\omega}_{1,1,i}$ is a positive scalar. For the error dynamic system (9) with $\omega(t) = [\omega_1^{\mathrm{T}}(t), \omega_2^{\mathrm{T}}(t), \ldots, \omega_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, since $\|\omega\|_{1,1} = \int_0^{\infty} \|\omega(\tau)\|_1 d\tau = \int_0^{\infty} \sum_{i=1}^{N} \|\omega_i(\tau)\|_1 d\tau$, it can be deduced that

$$\|\omega\|_{1,1} \leq \sum_{i=1}^{N} \bar{\omega}_{1,1,i}. \tag{13}$$

Then, under the L_1-norm bounded disturbances, the following theorem regarding the defined output reachable set-based positive consensus is developed.

Theorem 1: Consider the leader-following system (2) and (3) under disturbance (12) and dwell-time switching networks $\bar{G}_{\sigma(t)}$. If there exist vectors $p_{i,n,m} \in \mathbb{R}_+^{n_x}$, $q_i \in \mathbb{R}_+^{n_y}$, $\check{q} \in \mathbb{R}_+^{n_y}$, and matrices K_n, $i = 1, 2, \ldots, N$, $n \in \mathcal{S}$, $m = 0, 1, 2, \ldots, M$, such that (7)–(8), and

$$\frac{M}{\tau_d}\Big(p_{i,n,m'}^{\mathrm{T}} - p_{i,n,m'-1}^{\mathrm{T}}\Big) + p_{i,n,m'-1}^{\mathrm{T}}A - \Bigg(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n,m'-1}^{\mathrm{T}}\Bigg) BK_nC \preceq 0,\ \ m' = 1, 2, \ldots, M \tag{14}$$

$$\frac{M}{\tau_d}\Big(p_{i,n,m'}^{\mathrm{T}} - p_{i,n,m'-1}^{\mathrm{T}}\Big) + p_{i,n,m'}^{\mathrm{T}}A - \Bigg(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n,m'}^{\mathrm{T}}\Bigg) BK_nC \preceq 0,\ \ m' = 1, 2, \ldots, M \tag{15}$$

$$p_{i,n,M}^{\mathrm{T}}A - \Bigg(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n,M}^{\mathrm{T}}\Bigg) BK_nC \preceq 0 \tag{16}$$

$$p_{i,n,m}^{\mathrm{T}}D - \frac{1}{2\sum_{i=1}^{N} \bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}} \preceq 0,\ \ m = 0, 1, 2, \ldots, M \tag{17}$$

$$q_i^{\mathrm{T}}C - p_{i,n,m}^{\mathrm{T}} \preceq 0,\quad m = 0, 1, 2, \ldots, M \tag{18}$$

$$p_{i,n,0} - p_{i,l,M} \preceq 0,\quad l \neq n,\quad l \in \mathcal{S} \tag{19}$$

$$2p_{i,n,0} - r_i \preceq 0 \tag{20}$$

$$\check{q} - q_i \preceq 0 \tag{21}$$

hold, then the designed control protocol (4) can ensure that the closed-loop system (5) is positive, and the reachable set of the output consensus error $\eta_{yi}(t)$, $i = 1, 2, \ldots, N$, under L_1-norm bounded disturbances is enclosed by the hyperpyramid $\Theta(\check{q})$ in (11).

Proof: First, based on Proposition 1, conditions (7)–(8) can ensure that the closed-loop system (5) is positive.

Then, consider the dwell time interval $[t_k, t_k + \tau_d)$ and divide it into M segments, subintervals $\Omega_{k,m'} = [t_k + (m' - 1)(\tau_d/M), t_k + m'[\tau_d/M])$, $m' = 1, 2, \ldots, M$, can be

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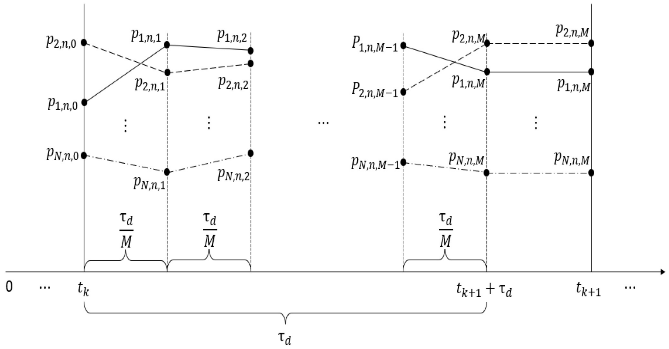

Fig. 1. Sketch of vector functions $p_{i,n}(t)$, $i = 1, 2, \ldots, N$ for mode n.

obtained. Corresponding to the dwell-time switching topology, the following switched linear copositive Lyapunov function is constructed:

$$V_n(\eta_x(t)) = \bar{p}_n^{\mathrm{T}}(t)\eta_x(t) \quad \forall \sigma(t) = n \in \mathcal{S} \tag{22}$$

where $\bar{p}_n(t) = \left[p_{1,n}^{\mathrm{T}}(t)\ p_{2,n}^{\mathrm{T}}(t) \ldots p_{N,n}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and $p_{i,n}(t) \in \mathbb{R}_+^{n_x}$, $i = 1, 2, \ldots, N$, $n \in \mathcal{S}$. Each $p_{i,n}(t)$ is given in the following form:

$$p_{i,n}(t) = \begin{cases} \beta_{k,m'}(t)p_{i,n,m'-1} + \left(1 - \beta_{k,m'}(t)\right)p_{i,n,m'}, & t \in \Omega_{k,m'} \\ p_{i,n,M}, & t \in \left[t_k + \tau_d, t_{k+1}\right) \end{cases} \tag{23}$$

where $\beta_{k,m'}(t) = (M/\tau_d)(t_k + (m'\tau_d/M) - t)$, and $p_{i,n,m} \in \mathbb{R}_+^{n_x}$, $m = 0, 1, 2, \ldots, M$. For mode n of switching networks, a sketch illustrating the construction of positive vector functions $p_{i,n}(t)$, $i = 1, 2, \ldots, N$ is presented in Fig. 1.

Along the trajectory of error dynamic system (9), when $t \in \Omega_{k,m'}$, one can get that

$$\begin{aligned} &\dot{V}_n(\eta_x(t)) \\ &= \dot{\bar{p}}_n^{\mathrm{T}}(t)\eta_x(t) + \bar{p}_n^{\mathrm{T}}(t)\dot{\eta}_x(t) \\ &= \Bigg(\left[\dot{p}_{1,n}^{\mathrm{T}}(t)\ \dot{p}_{2,n}^{\mathrm{T}}(t) \ldots \dot{p}_{M,n}^{\mathrm{T}}(t)\right] + \left[p_{1,n}^{\mathrm{T}}(t)A\ p_{2,n}^{\mathrm{T}}(t)A \right.\\ &\quad \left. \ldots p_{N,n}^{\mathrm{T}}(t)A\right] - \Bigg[\left(\sum_{j=1}^{N}\xi_{j1,n}p_{j,n}^{\mathrm{T}}(t)\right)BK_nC\left(\sum_{j=1}^{N}\right. \\ &\quad \left.\xi_{j2,n}p_{j,n}^{\mathrm{T}}(t)\right)BK_nC \ldots \left(\sum_{j=1}^{N}\xi_{jN,n}p_{j,n}^{\mathrm{T}}(t)\right)BK_nC\Bigg]\Bigg) \\ &\quad \eta_x(t) + \left[p_{1,n}^{\mathrm{T}}(t)D\ p_{2,n}^{\mathrm{T}}(t)D \ldots p_{N,n}^{\mathrm{T}}(t)D\right]\omega(t). \end{aligned}$$

To obtain $\dot{V}_n(\eta_x(t)) - (1/2\sum_{i=1}^{N}\bar{\omega}_{1,1,i})\mathbf{1}^{\mathrm{T}}\omega(t) \le 0$, $t \in \Omega_{k,m'}$, it follows that:

$$\dot{p}_{i,n}^{\mathrm{T}}(t) + p_{i,n}^{\mathrm{T}}(t)A - \left(\sum_{j=1}^{N}\xi_{ji,n}p_{j,n}^{\mathrm{T}}(t)\right)BK_nC \preceq 0 \tag{24}$$

$$p_{i,n}^{\mathrm{T}}(t)D - \frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}} \preceq 0. \tag{25}$$

With $p_{i,n}(t) = \beta_{k,m'}(t)p_{i,n,m'-1} + (1-\beta_{k,m'}(t))p_{i,n,m'}$, $t \in \Omega_{k,m'}$, $\dot{p}_{i,n}(t) = (M/\tau_d)(p_{i,n,m'} - p_{i,n,m'-1})$ can be obtained. Therefore, it holds that

$$\begin{aligned} &\frac{M}{\tau_d}\left(p_{i,n,m'}^{\mathrm{T}} - p_{i,n,m'-1}^{\mathrm{T}}\right) + \beta_{k,m'}(t)p_{i,n,m'-1}^{\mathrm{T}}A + \big(1 \\ &\quad - \beta_{k,m'}(t)\big)p_{i,n,m'}^{\mathrm{T}}A - \Bigg(\sum_{j=1}^{N}\xi_{ji,n}\big(\beta_{k,m'}(t)p_{j,n,m'-1}^{\mathrm{T}} \\ &\quad + \big(1 - \beta_{k,m'}(t)\big)p_{j,n,m'}^{\mathrm{T}}\big)\Bigg)BK_nC \preceq 0 \end{aligned} \tag{26}$$

$$\begin{aligned} &\beta_{k,m'}(t)p_{i,n,m'-1}^{\mathrm{T}}D + \big(1 - \beta_{k,m'}(t)\big)p_{i,n,m'}^{\mathrm{T}}D \\ &\quad - \frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}} \preceq 0. \end{aligned} \tag{27}$$

With $0 < \beta_{i,m'}(t) \le 1$, the above conditions (26)–(27) are equivalent to the conditions (14)–(15) and (17).

When $t \in [t_k + \tau_d, t_{k+1})$, $p_{i,n}(t) = p_{i,n,M}$, alone the trajectory of error dynamic system (9), it holds that

$$\begin{aligned} &\dot{V}_n(\eta_x(t)) \\ &= \bar{p}_n^{\mathrm{T}}(t)\dot{\eta}_x(t) \\ &= \Bigg(\left[p_{1,n,M}^{\mathrm{T}}A\ p_{2,n,M}^{\mathrm{T}}A \ldots p_{N,n,M}^{\mathrm{T}}A\right] - \Bigg[\Bigg(\sum_{j=1}^{N}\xi_{j1,n} \\ &\quad p_{j,n,M}^{\mathrm{T}}\Bigg)BK_nC\left(\sum_{j=1}^{N}\xi_{j2,n}p_{j,n,M}^{\mathrm{T}}\right)BK_nC \ldots \\ &\quad \left(\sum_{j=1}^{N}\xi_{jN,n}p_{j,n,M}^{\mathrm{T}}\right)BK_nC\Bigg]\Bigg)\eta_x(t) \\ &\quad + \left[p_{1,n,M}^{\mathrm{T}}D\ p_{2,n,M}^{\mathrm{T}}D \ldots p_{N,n,M}^{\mathrm{T}}D\right]\omega(t). \end{aligned}$$

To get $\dot{V}_n(\eta_x(t)) - (1/2\sum_{i=1}^{N}\bar{\omega}_{1,1,i})\mathbf{1}^{\mathrm{T}}\omega(t) \le 0$, $t \in [t_k + \tau_d, t_{k+1})$, it follows that:

$$p_{i,n,M}^{\mathrm{T}}A - \left(\sum_{j=1}^{N}\xi_{ji,n}p_{j,n,M}^{\mathrm{T}}\right)BK_nC \preceq 0 \tag{28}$$

$$p_{i,n,M}^{\mathrm{T}}D - \frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}} \preceq 0. \tag{29}$$

Therefore, it can be concluded that

$$\dot{V}_n(\eta_x(t)) - \frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}}\omega(t) \le 0,\ \ t \in \left[t_k, t_{k+1}\right). \tag{30}$$

Introduce a Lyapunov candidate $V(t) = \sum_{n\in\mathcal{S}}\lambda_n(t)V_n(\eta_x(t)) = \sum_{n\in\mathcal{S}}\lambda_n(t)\bar{p}_n^{\mathrm{T}}(t)\eta_x(t)$, where $\lambda_n(t) \in \{0, 1\}$ and $\sum_{n\in\mathcal{S}}\lambda_n(t) = 1$. λ_n can be regarded as the indicator function to describe the active mode at time instant t. For any $t \in [t_k, t_{k+1})$, (30) gives that $\dot{V}(t) - (1/2\sum_{i=1}^{N}\bar{\omega}_{1,1,i})\mathbf{1}^{\mathrm{T}}\omega(t) \le 0$. Integrating this from t_k to t, one has

$$V(t) - V\left(t_k^+\right) \le \int_{t_k}^{t}\frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}}\omega(\tau)d\tau. \tag{31}$$

Moreover, integrating $\dot{V}(t) - (1/2\sum_{i=1}^{N}\bar{\omega}_{1,1,i})\mathbf{1}^{\mathrm{T}}\omega(t) \le 0$ over each subsystem time interval, it follows that:

$$V\left(t_k^-\right) - V\left(t_{k-1}^+\right) \le \int_{t_{k-1}}^{t_k}\frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}}\omega(\tau)d\tau \tag{32}$$

$$\vdots$$

$$V\left(t_1^-\right) - V(t_0) \le \int_{t_0}^{t_1}\frac{1}{2\sum_{i=1}^{N}\bar{\omega}_{1,1,i}}\mathbf{1}^{\mathrm{T}}\omega(\tau)d\tau. \tag{33}$$

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At the switching instants, it is observed that $p_{i,n,0} \preceq p_{i,l,M}$, $l \neq n$. This condition results in $V(t_k^+) \leq V(t_k^-)$, $k = 1, 2, \ldots$ Therefore, summing up (31)–(33) and $V(t_k^+) - V(t_k^-) \leq 0$, $k = 1, 2, \ldots$, one can get that

$$V(t) - V(t_0) \leq \int_{t_0}^{t} \frac{1}{2\sum_{i=1}^{N} \bar{\omega}_{1,1,i}} \mathbf{1}^{\mathrm{T}} \omega(\tau) d\tau. \quad (34)$$

With $2p_{i,n,0} - r_i \preceq 0$ as constraint (20), and $t_0 = 0$, it holds that

$$V(0) \leq \frac{1}{2}\bar{r}^{\mathrm{T}} \eta_x(0) = \frac{1}{2}\left[r_1^{\mathrm{T}}\ r_2^{\mathrm{T}}\ \ldots\ r_N^{\mathrm{T}}\right] \eta_x(0) \leq \frac{1}{2}. \quad (35)$$

Since $\|\omega\|_{1,1} = \int_0^\infty \|\omega(\tau)\|_1 d\tau = \int_0^\infty \mathbf{1}^{\mathrm{T}} \omega(\tau) d\tau \leq \sum_{i=1}^{N} \bar{\omega}_{1,1,i}$, based on (34) and (35), for any $t \in [t_k, t_{k+1})$, one can get that

$$\begin{aligned} V(t) &\leq V(0) + \int_0^t \frac{1}{2\sum_{i=1}^{N} \bar{\omega}_{1,1,i}} \mathbf{1}^{\mathrm{T}} \omega(\tau) d\tau \\ &\leq \frac{1}{2} + \frac{1}{2\sum_{i=1}^{N} \bar{\omega}_{1,1,i}} \sum_{i=1}^{N} \bar{\omega}_{1,1,i} = 1. \end{aligned} \quad (36)$$

Thus, it holds that $V(t) \leq 1$, $t \geq 0$. With $V(t) \leq 1$, it is obtained that $\bar{p}_n^{\mathrm{T}}(t)\eta_x(t) \leq 1$, $n \in \mathcal{S}$.

To investigate the reachable set of the output consensus error, denoting $\bar{q} = \left[q_1^{\mathrm{T}}\ q_2^{\mathrm{T}}\ \ldots\ q_N^{\mathrm{T}}\right]^{\mathrm{T}}$, one has

$$\begin{aligned} &\bar{q}^{\mathrm{T}} \eta_y(t) - \bar{p}_n^{\mathrm{T}}(t)\eta_x(t) \\ &= \left[q_1^{\mathrm{T}}\ q_2^{\mathrm{T}}\ \ldots\ q_N^{\mathrm{T}}\right](I \otimes C)\eta_x(t) - \left[p_{1,n}^{\mathrm{T}}(t)\ p_{2,n}^{\mathrm{T}}(t)\right. \\ &\qquad\qquad \left.\ldots p_{N,n}^{\mathrm{T}}(t)\right]\eta_x(t) \\ &= \left[q_1^{\mathrm{T}}C - p_{1,n}^{\mathrm{T}}(t)\ q_2^{\mathrm{T}}C - p_{2,n}^{\mathrm{T}}(t)\ \ldots\ q_N^{\mathrm{T}}C - p_{N,n}^{\mathrm{T}}(t)\right] \times \eta_x(t). \end{aligned} \quad (37)$$

To obtain $\bar{q}^{\mathrm{T}} \eta_y(t) - \bar{p}_n^{\mathrm{T}}(t)\eta_x(t) \leq 0$, based on (37), one has

$$\begin{aligned} &q_i^{\mathrm{T}}C - p_{i,n}^{\mathrm{T}}(t) \\ &= q_i^{\mathrm{T}}C - \beta_{k,m'}(t)p_{i,n,m'-1}^{\mathrm{T}} - \left(1 - \beta_{k,m'}(t)\right)p_{i,n,m'}^{\mathrm{T}} \preceq 0. \end{aligned} \quad (38)$$

Moreover, condition (18) is equivalent to (38). Therefore, condition (18) implies that $\bar{q}^{\mathrm{T}}\eta_y(t) - \bar{p}_n^{\mathrm{T}}(t)\eta_x(t) \leq 0$. Consequently, $\bar{q}^{\mathrm{T}}\eta_y(t) \leq 1$ is obtained from $\bar{p}_n^{\mathrm{T}}(t)\eta_x(t) \leq 1$.

With $\bar{q}^{\mathrm{T}}\eta_y(t) \leq 1$, where $\bar{q} = \left[q_1^{\mathrm{T}}\ q_2^{\mathrm{T}}\ \ldots\ q_N^{\mathrm{T}}\right]^{\mathrm{T}}$, and $\eta_y(t) = \left[\eta_{y1}^{\mathrm{T}}(t)\ \eta_{y2}^{\mathrm{T}}(t)\ \ldots\ \eta_{yN}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, it arrives that $q_i^{\mathrm{T}}\eta_{yi}(t) \leq 1$, $i = 1, 2, \ldots, N$. From condition (21), $\check{q}^{\mathrm{T}}\eta_{yi}(t) \leq q_i^{\mathrm{T}}\eta_{yi}(t) \leq 1$ can be obtained. According to Remark 3, it can be concluded that the reachable set of output consensus error $\eta_{yi} \in \mathbb{R}_{0,+}^{n_y}$, $i = 1, 2, \ldots, N$, is enclosed by the hyperpyramid $\Theta(\check{q})$ in (11). The proof is done. ■

B. $L_{\infty,1}$-Norm Bounded Disturbances

In this section, another class of non-negative disturbances is considered. For each follower i, $i = 1, 2, \ldots, N$, the non-negative disturbance ω_i is assumed to be $L_{\infty,1}$-norm bounded, satisfying

$$\|\omega_i\|_{\infty,1} \triangleq \sup \|\omega_i(t)\|_1 \leq \bar{\omega}_{\infty,1,i} \quad (39)$$

where $\bar{\omega}_{\infty,1,i}$ are positive scalars. Since $\|\omega\|_{\infty,1} = \sup \|\omega(t)\|_1 = \sup \sum_{i=1}^{N} \|\omega_i(t)\|_1$, it is obtained that

$$\|\omega\|_{\infty,1} \leq \sum_{i=1}^{N} \bar{\omega}_{\infty,1,i}. \quad (40)$$

Theorem 2: Consider the leader-following system (2) and (3) under disturbance (39) and dwell-time switching networks $\bar{G}_{\sigma(t)}$. Given a scalar $\alpha > 0$, if there exist vectors $p_{i,n,m} \in \mathbb{R}_+^{n_x}$, $q_i \in \mathbb{R}_+^{n_y}$, $\check{q} \in \mathbb{R}_+^{n_y}$, and matrices K_n, $i = 1, 2, \ldots, N$, $n \in \mathcal{S}$, $m = 0, 1, 2, \ldots, M$, such that (7)–(8), (18)–(19), (21) and

$$\begin{aligned} &\frac{M}{\tau_d}\left(p_{i,n,m'}^{\mathrm{T}} - p_{i,n,m'-1}^{\mathrm{T}}\right) + p_{i,n,m'-1}^{\mathrm{T}}A - \left(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n,m'-1}^{\mathrm{T}}\right) \\ &BK_nC + \alpha p_{i,n,m'-1}^{\mathrm{T}} \preceq 0,\ \ m' = 1, 2, \ldots, M \end{aligned} \quad (41)$$

$$\begin{aligned} &\frac{M}{\tau_d}\left(p_{i,n,m'}^{\mathrm{T}} - p_{i,n,m'-1}^{\mathrm{T}}\right) + p_{i,n,m'}^{\mathrm{T}}A - \left(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n,m'}^{\mathrm{T}}\right) \\ &BK_nC + \alpha p_{i,n,m'}^{\mathrm{T}} \preceq 0,\ \ m' = 1, 2, \ldots, M \end{aligned} \quad (42)$$

$$p_{i,n,M}^{\mathrm{T}}A - \left(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n,M}^{\mathrm{T}}\right) BK_nC + \alpha p_{i,n,M}^{\mathrm{T}} \preceq 0 \quad (43)$$

$$p_{i,n,m}^{\mathrm{T}}D - \frac{\alpha}{\sum_{i=1}^{N} \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}} \preceq 0,\ \ m = 0, 1, 2, \ldots, M \quad (44)$$

$$p_{i,n,0} - r_i \preceq 0 \quad (45)$$

hold, then the designed control protocol (4) can ensure that the closed-loop system (5) is positive, and the reachable set of the output consensus error $\eta_{yi}(t)$, $i = 1, 2, \ldots, N$, under $L_{\infty,1}$-norm bounded disturbances is enclosed by the hyperpyramid $\Theta(\check{q})$ in (11).

Proof: First, conditions (7)–(8) can guarantee that the closed-loop system (5) is positive. Then, consider the switched linear copositive Lyapunov function $V_n(\eta_x(t)) = \bar{p}_n^{\mathrm{T}}(t)\eta_x(t)$, $n \in \mathcal{S}$, where $\bar{p}_n(t) = \left[p_{1,n}^{\mathrm{T}}(t)\ p_{2,n}^{\mathrm{T}}(t)\ \ldots\ p_{N,n}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$, and $p_{i,n}(t) \in \mathbb{R}_+^{n_x}$, $i = 1, 2, \ldots, N$, with $p_{i,n}(t)$ defined in (23).

When $t \in \Omega_{k,m'}$, according to the evolution of the error dynamic system (9), one has

$$\begin{aligned} &\dot{V}_n(\eta_x(t)) + \alpha V_n(\eta_x(t)) - \frac{\alpha}{\sum_{i=1}^{N} \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}} \omega(t) \\ &\quad = \left(\left[\dot{p}_{1,n}^{\mathrm{T}}(t)\ \dot{p}_{2,n}^{\mathrm{T}}(t)\ \ldots\ \dot{p}_{N,n}^{\mathrm{T}}(t)\right] + \left[p_{1,n}^{\mathrm{T}}(t)A\ p_{2,n}^{\mathrm{T}}(t)A\right.\right. \\ &\qquad \left.\ldots\ p_{N,n}^{\mathrm{T}}(t)A\right] - \left[\left(\sum_{j=1}^{N} \xi_{j1,n} p_{j,n}^{\mathrm{T}}(t)\right)BK_nC\ \left(\sum_{j=1}^{N}\right.\right. \\ &\qquad \left.\left.\xi_{j2,n} p_{j,n}^{\mathrm{T}}(t)\right)BK_nC\ \ldots\ \left(\sum_{j=1}^{N} \xi_{jN,n} p_{j,n}^{\mathrm{T}}(t)\right)BK_nC\right] \\ &\qquad \left. + \left[\alpha p_{1,n}^{\mathrm{T}}(t)\ \alpha p_{2,n}^{\mathrm{T}}(t)\ \ldots\ \alpha p_{N,n}^{\mathrm{T}}(t)\right]\right)\eta_x(t) \\ &\qquad + \left[p_{1,n}^{\mathrm{T}}(t)D - \tfrac{\alpha}{\sum_{i=1}^{N} \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\ \ p_{2,n}^{\mathrm{T}}(t)D - \tfrac{\alpha}{\sum_{i=1}^{N} \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\right. \\ &\qquad \left.\ldots\ p_{N,n}^{\mathrm{T}}(t)D - \tfrac{\alpha}{\sum_{i=1}^{N} \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\right]\omega(t). \end{aligned}$$

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To guarantee $\dot{V}_n(\eta_x(t)) + \alpha V_n(\eta_x(t)) - (\alpha/\sum_{i=1}^N \bar{\omega}_{\infty,1,i}) \mathbf{1}^{\mathrm{T}}\omega(t) \leq 0$, it holds that

$$\dot{p}_{i,n}(t) + p_{i,n}^{\mathrm{T}}(t)A - \left(\sum_{j=1}^{N} \xi_{ji,n} p_{j,n}^{\mathrm{T}}(t)\right) BK_n C + \alpha p_{i,n}^{\mathrm{T}}(t) \preceq 0 \tag{46}$$

$$p_{i,n}^{\mathrm{T}}(t)D - \frac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}} \preceq 0. \tag{47}$$

With $0 < \beta_{k,m'}(t) \leq 1$, (46) and (47) are equivalent to (41)–(42) and (44).

When $t \in [t_k + \tau_d, t_{k+1})$, $p_{i,n}(t) = p_{i,n,M}$, it holds that

$$\begin{aligned}
&\dot{V}_n(\eta_x(t)) + \alpha V_n(\eta_x(t)) - \frac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\omega(t) \\
&= \Bigg(\left[p_{1,n,M}^{\mathrm{T}}A \; p_{2,n,M}^{\mathrm{T}}A \; \dots \; p_{N,n,M}^{\mathrm{T}}A\right] - \Bigg[\Big(\sum_{j=1}^{N} \xi_{j1,n} p_{j,n,M}^{\mathrm{T}}\Big) \\
&\quad BK_nC \; \Big(\sum_{j=1}^{N} \xi_{j2,n} p_{j,n,M}^{\mathrm{T}}\Big) BK_nC \dots \Big(\sum_{j=1}^{N} \xi_{jN,n} p_{j,n,M}^{\mathrm{T}}\Big) BK_nC\Bigg] \\
&\quad + \left[\alpha p_{1,n,M}^{\mathrm{T}} \; \alpha p_{2,n,M}^{\mathrm{T}} \; \dots \alpha p_{N,n,M}^{\mathrm{T}}\right]\Bigg)\eta_x(t) \\
&\quad + \Bigg[p_{1,n,M}^{\mathrm{T}}D - \tfrac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}} \; p_{2,n,M}^{\mathrm{T}}D \\
&\quad - \tfrac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}} \dots p_{N,n,M}^{\mathrm{T}}D - \tfrac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\Bigg]\omega(t).
\end{aligned}$$

To guarantee $\dot{V}_n(\eta_x(t)) + \alpha V_n(\eta_x(t)) - (\alpha/\sum_{i=1}^N \bar{\omega}_{\infty,1,i}) \mathbf{1}^{\mathrm{T}}\omega(t) \leq 0$, conditions (43)–(44) should be held.

With $V_n(\eta_x(t))$ being continuous in $[t_k, t_{k+1})$, it is obtained that $\dot{V}_n(\eta_x(t)) + \alpha V_n(\eta_x(t)) - (\alpha/\sum_{i=1}^N \bar{\omega}_{\infty,1,i})\mathbf{1}^{\mathrm{T}}\omega(t) \leq 0$, $t \in [t_k, t_{k+1})$. By considering $V(t) = \sum_{n\in\mathcal{S}} \lambda_n(t) V_n(\eta_x(t))$, for any $t \in [t_k, t_{k+1})$, one has

$$\dot{V}(t) + \alpha V(t) - \frac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\omega(t) \leq 0. \tag{48}$$

Integrating (48) from t_k to t, it yields that

$$V(t) \leq e^{-\alpha(t-t_k)} V(t_k^+) + \int_{t_k}^{t} e^{-\alpha(t-\tau)} \left(\frac{\alpha}{\sum_{i=1}^N \bar{\omega}_{\infty,1,i}} \mathbf{1}^{\mathrm{T}}\omega(\tau)\right) d\tau. \tag{49}$$

With $\mathbf{1}^{\mathrm{T}}\omega(t) = \|\omega(t)\|_1 \leq \sum_{i=1}^N \bar{\omega}_{\infty,1,i}$, one has

$$V(t) \leq e^{-\alpha(t-t_k)}\left(V(t_k^+) - 1\right) + 1.$$

Then, with $p_{i,n,0} \preceq p_{i,l,M}$, $l \neq n$, generating $V(t_k^+) \leq V(t_k^-)$, $k = 1, 2, \dots$, it follows that:

$$\begin{aligned}
V(t) &\leq e^{-\alpha(t-t_k)}\left(V(t_k^+) - 1\right) + 1 \\
&\leq e^{-\alpha(t-t_k)}\left(V(t_k^-) - 1\right) + 1 \\
&\leq e^{-\alpha(t-t_k)}\left(e^{-\alpha(t_k - t_{k-1})}\left(V(t_{k-1}^+) - 1\right)\right) + 1 \\
&\;\;\vdots \\
&\leq e^{-\alpha(t-t_0)}(V(t_0) - 1) + 1.
\end{aligned} \tag{50}$$

Based on condition (45), $V(t_0) = \bar{p}_{n,0}^{\mathrm{T}}\eta_x(0) \leq \bar{r}^{\mathrm{T}}\eta_x(0) \leq 1$ holds with $\bar{p}_{n,0} = \left[p_{1,n,0}^{\mathrm{T}}\, p_{2,n,0}^{\mathrm{T}} \cdots p_{N,n,0}^{\mathrm{T}}\right]^{\mathrm{T}}$. Therefore, from (50), $V(t) \leq 1$ is obtained. By following the similar proof of Theorem 1, $q_i^{\mathrm{T}}C - p_{i,n}^{\mathrm{T}} \preceq 0$ can guarantee that $\bar{q}^{\mathrm{T}}\eta_y(t) \leq 1$ holds, and condition (21) gives that $\check{q}^{\mathrm{T}}\eta_{yi}(t) \leq q_i^{\mathrm{T}}\eta_{yi}(t) \leq 1$. Therefore, the reachable set of output consensus error $\eta_{yi} \in \mathbb{R}_{0,+}^{n_y}$, $i = 1, 2, \dots, N$, can be enclosed by the hyperpyramid $\Theta(\check{q})$ in (11). The proof is completed. ∎

Remark 4: The derivations of Theorems 1 and 2 are based on elements rather than matrices with the utilization of the switched linear copositive Lyapunov function. Meanwhile, under the dwell-time switching communication networks, the adopted switched linear copositive Lyapunov candidate can lead to a tractable conservatism reduction with the increasing value of M similar to the idea in [26] and [38].

Remark 5: It is noted that the output reachable set-based consensus problem of this work is essentially an output-feedback control synthesis problem. Assumption 1 is a basic assumption to ensure the existence of the control matrices K_n, $n \in \mathcal{S}$ in Theorems 1 and 2. About Assumption 2, if the directed graph $\bar{G}_{\sigma(t)}$ does not contain a spanning tree at some time, $X_{\sigma(t)}$ may contain all-zero columns, that is, $\xi_{ji,n} = 0$, $j = 1, 2, \dots, N$. Then, constraint (16) in Theorem 1 and constraint (43) in Theorem 2 will result in a restriction that A is stable. As it is more desired to achieve an output reachable set-based consensus of possible unstable agents, the directed graph $\bar{G}_{\sigma(t)}$ is assumed to contain a spanning tree in this work.

C. Optimization

Based on the obtained theorems, the reachable set of the output consensus error $\eta_{yi} \in \mathbb{R}_{0,+}^{n_y}$, $i = 1, 2, \dots, N$ can be enclosed by the hyperpyramid $\Theta(\check{q})$ in (11). With vector $\check{q} = \left[\check{q}_1\; \check{q}_2 \cdots \check{q}_{n_y}\right]^{\mathrm{T}}$, the volume of the hyperpyramid $\Theta(\check{q})$ in (11) is proportional to $\prod_{i=1}^{n_y}(1/\check{q}_i)$. To find a hyperpyramid with the volume as small as possible that encloses the reachable set of the output consensus error, the optimization problem is formulated as follows:

$$\text{minimize } J(\check{q}) \triangleq -\sum_{i=1}^{n_y} \log \check{q}_i, \quad \text{subject to}$$
$$\begin{cases} (7)\text{–}(8),\ (14)\text{–}(20),\ (21), \text{for Theorem 1} \\ (7)\text{–}(8),\ (18)\text{–}(19),\ (21),\ (41)\text{–}(45), \text{for Theorem 2.} \end{cases} \tag{51}$$

Remark 6: The optimization problem is formulated to minimize the reachable set of the output consensus error. In this work, due to the effect of exogenous disturbances, the robust consensus is investigated. As a kind of robust consensus problem, the reachable set-based consensus intuitively characterizes the bounding region of the set of all consensus errors reached from the initial condition. Minimizing the size of the output consensus error bounding region can achieve a robust controller design with disturbance attenuation.

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Algorithm 1 PSO-Based Algorithm

Input: ranges of K_n, $n \in \mathcal{S}$, based on (7)–(8), w_0, w_l, w_g
Output: best solution K_n^*, $n \in \mathcal{S}$, $\check{q}^*$

for each particle j **do**
  Initialization: K_n^j and ΔK_n^j based on ranges of K_n, $n \in \mathcal{S}$, and uniformly random distribution
  Initialization: best $K_n^{j*} \leftarrow K_n^j$, and best $K_n^* \leftarrow K_n^j$, $n \in \mathcal{S}$, of all problem instances
end for
while maximum number of iteration is not reached **do**
  for each particle j **do**
    Random generate r_l and r_g in uniformly (0,1) distribution
    Update $\Delta K_n^j \leftarrow w_0 \Delta K_n^j + w_l r_l (K_n^{j*} - K_n^j) + w_g r_g (K_n^* K_n^j)$, $n \in \mathcal{S}$
    Update $K_n^j \leftarrow K_n^j + \Delta K_n^j$, $n \in \mathcal{S}$
    $\check{q}^j \leftarrow \arg\min \sum_{i=1}^{n_y} \log \check{q}_i | \{K_1^j, K_2^j, \ldots, K_S^j\}$
    Calculate objective function value $J(\check{q}^j) = -\sum_{i=1}^{n_y} \log \check{q}_i^j$ subject to (51)
    if $J(\check{q}^j) | \{K_1^j, K_2^j, \ldots, K_S^j\} < J(\check{q}^{j*}) | \{K_1^{j*}, K_2^{j*}, \ldots, K_S^{j*}\}$ **then**
      $K_n^{j*} \leftarrow K_n^j$, $n \in \mathcal{S}$
      if $J(\check{q}^{j*}) | \{K_1^{j*}, K_2^{j*}, \ldots, K_S^{j*}\} < J(\check{q}^*) | \{K_1^*, K_2^*, \ldots, K_S^*\}$ **then**
        $K_n^* \leftarrow K_n^{j*}$, $n \in \mathcal{S}$
      end if
    end if
  end for
end while
return best solution K_n^*, $n \in \mathcal{S}$, $\check{q}^*$

D. PSO-Based Algorithm

Notice that the obtained conditions of Theorems 1 and 2 contain the coupling items of variables K_n and $p_{i,n,m}$, which lead the formulated optimization problem to a bilinear programming problem. It can be found that iterative approaches are commonly used to solve bilinear problems [33], [34], [35], and often yield a local optimum because of the selection of initial values. Motivated by [39] and [40], in our work, a heuristic algorithm based on PSO is proposed to solve the involved bilinear programming problem.

PSO [39] is an evolutionary algorithm inspired by social behavior and swarm intelligence. It simulates a set of particles in the search space where each particle represents the candidate solution containing the location and velocity attributes. The location and velocity represent the solution and movement to another location, respectively. In each iteration, the velocity of the particle will be updated based on the best location among all particles, and the particle itself will move to a new location. After some iterations, the best location discovered by the particles will be considered the optimized solution of the algorithm.

In our problem, determining the search space of $p_{i,n,m}$, $i = 1, 2, \ldots, N$, $n \in \mathcal{S}$, $m = 0, 1, 2, \ldots, M$, is challenging as the number of vectors $p_{i,n,m}$ changes with the value of M. However, identifying the ranges of K_n, $n \in \mathcal{S}$, is relatively easy. Therefore, K_n, $n \in \mathcal{S}$, will be regarded as the location of particles in our problem. For each particle, once the value of K_n is set, the remaining problem becomes a linear programming problem that can be solved by a standard solver. The procedure of our proposed PSO-based algorithm can be summarized in Algorithm 1.

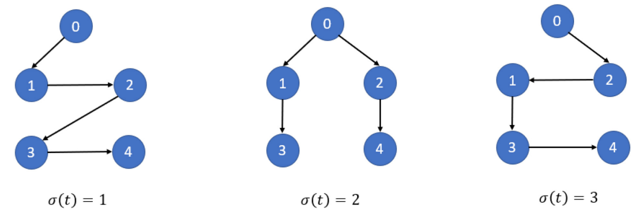

Fig. 2. Three modes of switching communication networks.

TABLE I
J^* WITH INCREASING VALUE OF M FOR CASE 1

M	1	2	3	4	5
J^*	0.7613	0.7600	0.7586	0.7581	0.7580

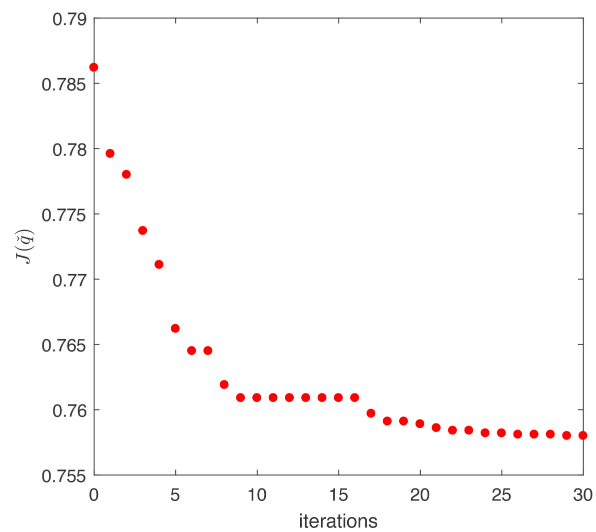

Fig. 3. Value of objective function $J(\check{q})$ via iterations ($M = 5$) for case 1.

IV. EXAMPLE

Consider a positive leader-following multiagent system under switching communication networks in the multi-input–multi-output (MIMO) form. One leader and four followers are contained, and the relevant parameters are presented below

$$A = \begin{bmatrix} -5.1 & 3 & 4 \\ 2 & -4.5 & 1 \\ 2.5 & 3 & -4.8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0.5 & 1.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0.3 & 0.6 & 0.5 \end{bmatrix}^{\mathrm{T}}.$$

It can be checked that A is Metzler, and $B, C,$ and D are non-negative. The eigenvalues of A are 0.1973, −7.8988, and −6.6865, thus the system matrix A is unstable.

The communication networks are described as directed switching graphs in Fig. 2 with $\sigma(t) \in \mathcal{S} = \{1, 2, 3\}$.

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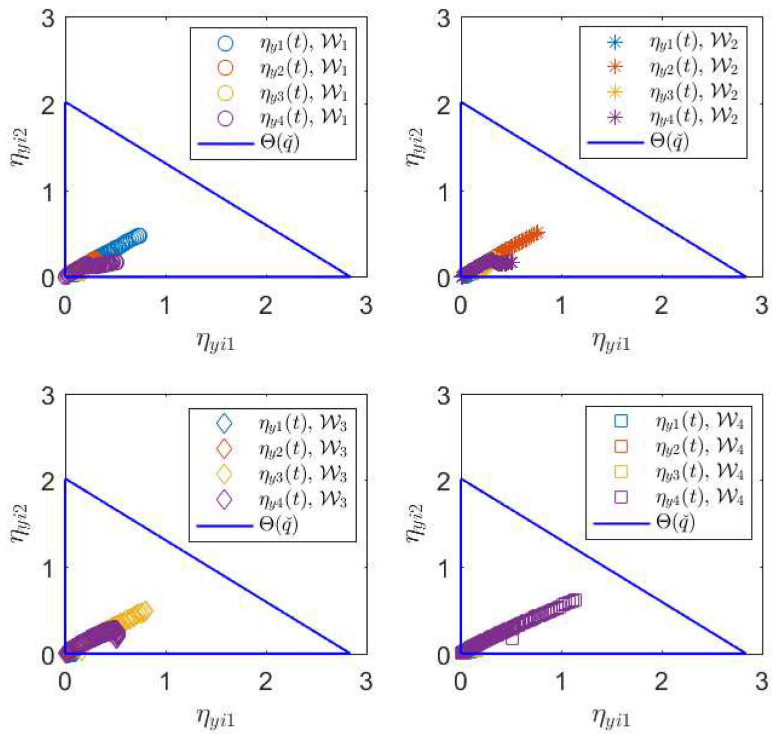

Fig. 4. Reachable set of output consensus error and bounding hyperpyramid for case 1.

From Fig. 2, with $X_{\sigma(t)} = L_{\sigma(t)} + G_{\sigma(t)}$ $\forall\sigma(t) = n \in \{1, 2, 3\}$, it can be obtained that

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$
$$X_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The communication networks obey a switching law with dwell time $\tau_d = 4.5$.

Given the initial condition of the leader is $x_0(0) = [0.1, 0.1, 0.1]^{\mathrm{T}}$. The initial conditions of followers are assumedThen, we can select $r_i = [2, 2, 2]^{\mathrm{T}}$, $i = 1, 2, 3, 4$. The following cases consider two types of non-negative disturbances.

Case 1 (L_1-Norm Bounded Disturbances): Consider four sets of L_1-norm bounded disturbances as follows:

$$\mathcal{W}_1 \triangleq \begin{cases} \omega_1 = 0.4ae^{-at}, \\ \omega_2 = 0, \\ \omega_3 = 0, \\ \omega_4 = 0, \end{cases} \quad \mathcal{W}_2 \triangleq \begin{cases} \omega_1 = 0 \\ \omega_2 = 0.4ae^{-at} \\ \omega_3 = 0 \\ \omega_4 = 0 \end{cases}$$
$$\mathcal{W}_3 \triangleq \begin{cases} \omega_1 = 0, \\ \omega_2 = 0, \\ \omega_3 = 0.4ae^{-at}, \\ \omega_4 = 0, \end{cases} \quad \mathcal{W}_4 \triangleq \begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_3 = 0 \\ \omega_4 = 0.4ae^{-at}. \end{cases}$$

Large a can make ae^{-at} approximate a Dirac delta function. In our simulation, a is taken as 1000. Since $\int_0^\infty ae^{-at} = 1$, it is obtained that all four admissible L_1-norm bounded disturbances give $\sum_{i=1}^{4} \bar{\omega}_{1,1,i} = 0.4$.

Based on Theorem 1, after conducting the PSO-based algorithm by setting the number of particles to 30 and the maximum number of iterations to 30, the optimal values J^* of $J(\check{q})$ with scenarios $M = \{1, 2, 3, 4, 5\}$ are collected in Table I.

It can be observed from Table I that a larger M generates a smaller value J^*. Therefore, conservatism can be reduced by increasing M.

The simulation results with $M = 5$ are displayed for illustration. When $M = 5$, after the implementation of the PSO-based algorithm, the variation of $J(\check{q})$ via iterations is exhibited in Fig. 3, and the obtained controller gain matrices K_n, $n \in \{1, 2, 3\}$, are $K_1 = \begin{bmatrix} 0.8333 & 2.6667 \\ 1.3324 & 0.4462 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.7384 & 2.5621 \\ 0.0376 & 0.0742 \end{bmatrix}$, $K_3 = \begin{bmatrix} 0.8340 & 2.6640 \\ 1.3334 & 0.6667 \end{bmatrix}$, and $\check{q} = \begin{bmatrix} 0.3520 \\ 0.4960 \end{bmatrix}$.

The initial conditions of followers are given as $x_1(0) = [0.1, 0.15, 0.1]^{\mathrm{T}}$, $x_2(0) = [0.1, 0.16, 0.12]^{\mathrm{T}}$, $x_3(0) = [0.1, 0.18, 0.1]^{\mathrm{T}}$, $x_4(0) = [0.11, 0.35, 0.13]^{\mathrm{T}}$. Then, under the obtained control protocol, it can be observed that in Fig. 4, the reachable set of the output consensus error η_{yi}, $i = 1, 2, 3, 4$, under disturbances $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, and $\mathcal{W}_4$, can be enclosed by the hyperpyramid $\Theta(\check{q})$. In addition, the output trajectories of a leader and four followers under all four admissible disturbances are shown in Fig. 5. As shown in Figs. 4 and 5, under L_1-norm bounded disturbances, the output reachable set-based consensus with positivity preserved has been achieved.

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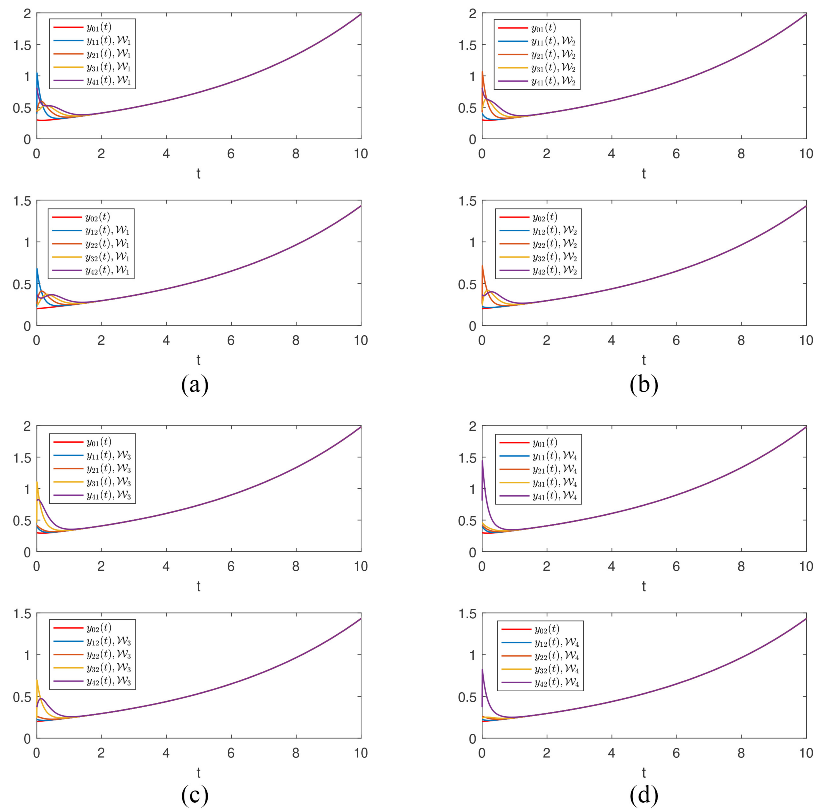

Fig. 5. Outputs of leader and followers for case 1.

TABLE II
J^* WITH INCREASING VALUE OF M FOR CASE 2

M	1	2	3	4	5
J^*	0.4116	0.4040	0.3962	0.3941	0.3932

Case 2 ($L_{\infty,1}$-Norm Bounded Disturbances): Consider four sets of $L_{\infty,1}$-norm bounded disturbances as follows:

$$\mathcal{W}_1 \triangleq \begin{cases} \omega_1 = 0.125, \\ \omega_2 = 0.125, \\ \omega_3 = 0.125, \\ \omega_4 = 0.125, \end{cases} \quad \mathcal{W}_2 \triangleq \begin{cases} \omega_1 = 0 \\ \omega_2 = 0.5 \\ \omega_3 = 0 \\ \omega_4 = 0 \end{cases}$$

$$\mathcal{W}_3 \triangleq \begin{cases} \omega_1 = 0, \\ \omega_2 = 0, \\ \omega_3 = 0.5, \\ \omega_4 = 0, \end{cases} \quad \mathcal{W}_4 \triangleq \begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_3 = 0 \\ \omega_4 = 0.5 \end{cases}$$

which all give $\sum_{i=1}^{4} \bar{\omega}_{\infty,1,i} = 0.5$. Based on Theorem 2 by given $\alpha = 0.7$, the PSO-based algorithm is applied by setting the number of particles to 30 and the maximum number of iterations to 30. As a result, the optimal value J^* of $J(\check{q})$ with different values of M is presented in Table II. It appears that the optimal value J^* will be smaller with increasing M to achieve conservatism reduction.

The simulation results with $M = 5$ are displayed for illustration. When $M = 5$, the variation of $J(\check{q})$ via iterations is demonstrated in Fig. 6, and the obtained controller

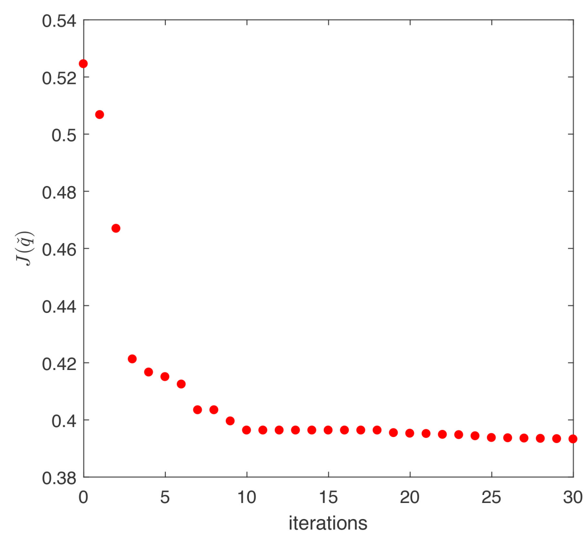

Fig. 6. Value of objective function $J(\check{q})$ via iterations ($M = 5$) for case 2.

gain matrices K_n, $n \in \{1, 2, 3\}$, are $K_1 = \begin{bmatrix} 0.8301 & 2.6667 \\ 1.3365 & 0.6667 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.4058 & 0.7297 \\ 1.9739 & 0.6510 \end{bmatrix}$, $K_3 = \begin{bmatrix} 0.8324 & 2.6667 \\ 1.3343 & 0.6667 \end{bmatrix}$, and $\check{q} = \begin{bmatrix} 0.6171 \\ 0.6553 \end{bmatrix}$.

The initial conditions of followers are given as: $\mathcal{I}_1 \triangleq \{x_1(0) = [0.1, 0.3, 0.15]^{\mathrm{T}}, x_2(0) = [0.1, 0.15, 0.3]^{\mathrm{T}}, x_3(0) = [0.1, 0.1, 0.1]^{\mathrm{T}}, x_4(0) = [0.1, 0.1, 0.1]^{\mathrm{T}}\}$, $\mathcal{I}_2 \triangleq$

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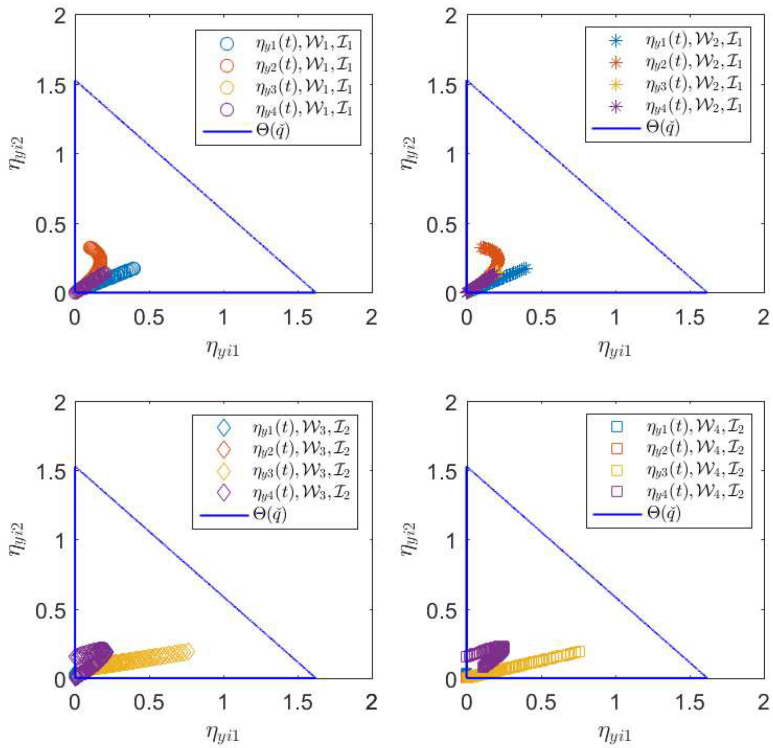

Fig. 7. Reachable set of output consensus error and bounding hyperpyramid for case 2.

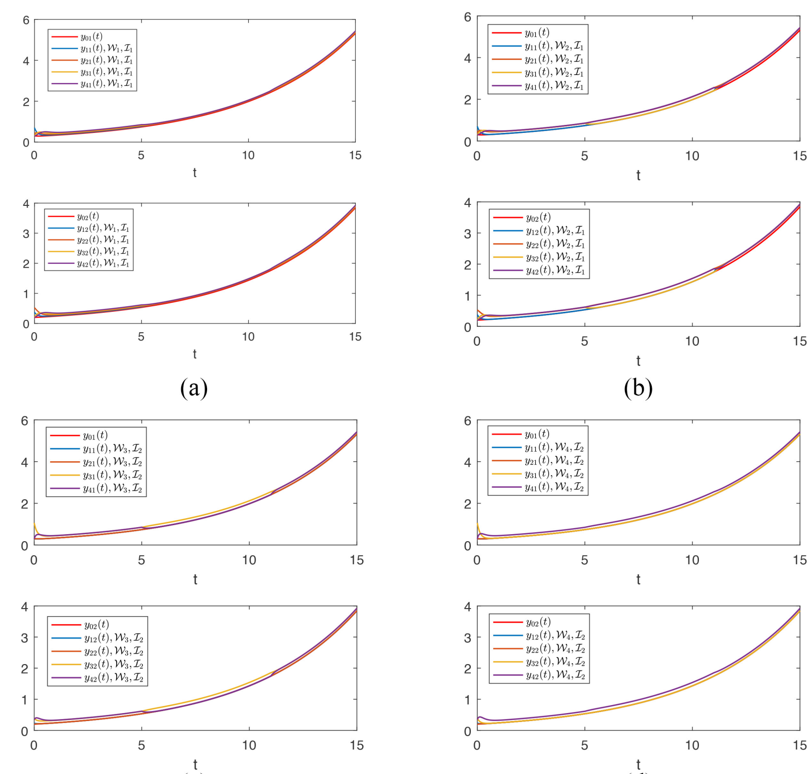

Fig. 8. Outputs of leader and followers for case 2.

$\{x_1(0) = [0.1, 0.1, 0.12]^{\mathrm{T}}, x_2(0) = [0.1, 0.1, 0.1]^{\mathrm{T}}, x_3(0) = [0.1, 0.48, 0.1]^{\mathrm{T}}, x_4(0) = [0.1, 0.1, 0.2]^{\mathrm{T}}\}$.

With the obtained control protocol and hyperpyramid $\Theta(\check{q})$, the reachable set of the output consensus error η_{yi} under different admissible disturbances and initial conditions, and the bounding hyperpyramid $\Theta(\check{q})$ are depicted in Fig. 7. Moreover, the corresponding output trajectories of the leader and four followers are presented in Fig. 8.

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It can be concluded that under $L_{\infty,1}$-norm bounded disturbances, the output reachable set-based consensus of the closed-loop system with positivity preserved has been achieved.

V. Conclusion

In this work, the output reachable set-based leader-following consensus has been investigated for positive agents subject to two classes of non-negative disturbances over switching communication networks. By employing switched linear copositive Lyapunov functions, the conditions to guarantee the positivity, and the reachable set of output consensus errors contained by hyperpyramids have been established. In terms of the obtained conditions, a PSO-based algorithm has been proposed to achieve the control protocol design by solving a bilinear programming problem. Numerical studies have been conducted to verify the effectiveness of the developed output reachable set-based consensus results. Future research can extend to positive agents over jointly connected switching networks to achieve reachable set-based consensus and explore new conditions for characterizing the reachable set of consensus errors with other types of bounding regions.

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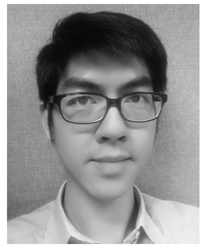

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