Stability Analysis of Discrete-Time Cone-Preserving Systems with Time-Varying Delays

Bohao Zhu, *Member, IEEE,* James Lam, *Fellow, IEEE,* Xiujuan Lu, and Ka-Wai Kwok, *Senior Member, IEEE*

***Abstract*—This paper investigates stability issues related to linear discrete-time cone-preserving systems with time-varying delays. The study begins by examining the monotonicity of discrete-time systems with cone invariance. Equivalent asymptotic stability conditions for discrete-time cone-preserving systems with time-varying delays are then provided based on the monotonicity of cone-preserving systems and the comparison principle. The results indicate that time delays do not affect asymptotic stability. Furthermore, the α-exponential stability is analyzed to characterize the decay rate of the system. Finally, a numerical example is presented to illustrate the theoretical findings. These results contribute to the understanding of stability analysis of cone-preserving systems with time-varying delays.**

***Index Terms*—Cone invariance, Exponential stability, Stability analysis, Time-delay systems.**

I. Introduction

CONE-PRESERVING system, whose states are confined in a cone located in the linear space, has garnered much attention from researchers over the past decade. Examples of cone-preserving systems include those with states in nonnegative orthants, polyhedral cones [1], and ice-cream cones [2]. Due to their cone-invariant property, positive systems can be considered a special case of cone-preserving systems. As a popular research topic, there has been a significant amount of work conducted on positive systems. Positive systems under different conditions, like positive systems with time delays [3]–[8], switched positive systems [9]–[11] and 2-D positive systems [12], [13], have been taken into consideration. In [14], Frobenius eigenvalues was introduced to prove the asymptotic stability of positive systems, and several equivalent stability conditions were presented. In [15], Haddad, Chellaboina and Rajpurohit proved that positive systems with constant time delays are asymptotically stable if and only if the sum of the state matrices is Hurwitz. In [5], [16], Liu, Yu and Wang introduced the comparison principle to analyze the asymptotic stability of positive systems with time-varying delays. It is shown that the necessary and sufficient condition for positive systems with time-varying delays is the same as the one in [15]. In [17], Ngoc considered a general linear differential positive system and pointed out that the asymptotic stability of such a system is independent of magnitude bound of the time delays. In addition, exponential stability for positive systems was also investigated. Based on the Perron-Frobenius theorem, exponential stability was discussed [18], [19]. In [20]. Ilchmann and Ngoc pointed out that the exponential stability condition for positive linear integro-differential systems is independent of magnitude of the constant time delays. In [21], [22], Zhu, Li, *et al.* further discussed the decay rate for continuous-time and discrete-time positive systems with time delays by introducing the definition of α-exponential stability. Furthermore, in [6] Liu and Lam analyzed the relationships between asymptotic stability and exponential stability of positive delay systems. The results showed that, unlike general dynamic systems, asymptotic and exponential stability of positive systems are equivalent under bounded time-delay condition.

All of the aforementioned research showed that the stability of systems under nonnegative constraint is not affected by time delays. However, the above research is only focused on a special kind of cone-preserving systems whose states are confined to a cone formed by nonnegative linear combination of bases of vector space. The real-world applications for cone-preserving systems could be found in traffic control systems [23], viral escape mechanisms [24], rendezvous problems of multi-agent systems [25], and chemical reaction networks [26]. Take the rendezvous problem as an example. The state of each agent is constrained within the specified proper cones to ensure the relationship between each state component. As a result, analyzing consensus among the agents becomes the stability analysis for the multi-agent system. Till now, only a few considered the monotonicity and stability of general cone-preserving systems [27], [28]. In [29]–[31], Shen, Tanaka, *et al.* discussed monotonicity conditions for systems with cone invariance. Necessary and sufficient asymptotic stability criteria for linear systems with cone invariance and time delays were established. The results showed that stability of continuous-time cone-preserving systems with constant time delays and time-varying delays are not affected by the magnitude of time delays. These conclusions also indicated that delay robustness in positive systems is not due to system's positivity but cone invariance property.

Motivated by above works, this paper proposes a method for analyzing the stability of discrete-time systems with cone invariance. Different from the one in [31], multiple time delays are considered and the comparison method is proposed to build the relation between the constant time delays systems and time-varying delay systems. The paper is outlined as follows: First, a necessary and sufficient condition to characterize a cone-preserving system with time delays is introduced. Then, based on the partial ordering relation of state in discrete-

This work was supported in part by the Research Grants Council of Hong Kong under Grant 17201820 and in part by the HKSAR Government under the InnoHK initiative , Hong Kong, via Centre for Garment Production Limited. (Corresponding authors: Xiujuan Lu and James Lam).

B. Zhu, J. Lam, X. Lu, and K.-W. Kwok are with the Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong (Email: zhubh@hku.hk; james.lam@hku.hk; u3007423@connect.hku.hk; kwokkw@hku.hk).

time cone-preserving systems, asymptotic stability conditions of linear discrete-time cone-preserving systems with constant delays and time-varying delays are analyzed. Furthermore, the definition of α-exponential stability for cone-preserving systems is introduced, and a necessary and sufficient exponential stability condition for discrete-time systems with cone invariance and time delays is established. In this paper some major contributions to the analysis of discrete-time cone-preserving systems will be carried out, including: 1) Comparison methods are proposed to analyze the discrete-time systems, highlighting the monotonicity and relationships between systems with constant time delays and those with time-varying delays; 2) The derived stability conditions indicate the delay robustness inherent in discrete-time cone-preserving systems; 3) An equivalent condition to characterize the convergent speed of the state in the system is proposed.

II. Preliminaries

In this section, the notions and definitions on cones will be introduced. First, some mathematical notions are given as follows. $\mathbb{R}^n$ denotes the set of n-dimensional real vectors. $\mathbb{R}^{n\times m}$ denotes the set of real matrices of $n\times m$ dimension. A^T denotes the transpose of matrix A. $\mathbb{N}=\{1,2,\dots\}$ and $\mathbb{N}_0=\{0\}\cup\mathbb{N}$. $\|v\|_2=\sqrt{\sum_{i=1}^n v_i^2}$ denotes the 2-norm of the vector v on $\mathbb{R}^n$.

Some basic definitions about cones in [14] are recalled. Give a set $S\subseteq\mathbb{R}^n$, and S^G denotes a set consisting all nonnegative linear combinations of elements in set S. The boundary of a set and its interior are denoted by ∂S and $\operatorname{int}(S)$ accordingly. If there exists a set K that satisfies $K=K^G$, then K is called a closed cone. A cone K is said to be pointed if $K\cap(-K)=\{0\}$ and solid if the interior of K is not empty. If a cone is closed, pointed, solid and convex, then it is called a proper cone. K^* denotes the dual of the cone K satisfying $K^*=\left\{y\in\mathbb{R}^n | y^Tx\geq 0,\forall x\in K\right\}$. $x\prec_K y\ \ (x\succ_K y)$ denotes $y-x\in\operatorname{int}(K)\ \ (x-y\in\operatorname{int}(K))$ and $x\preceq_K y\ \ (x\succeq_K y)$ denotes $y-x\in K\ \ (x-y\in K)$. Furthermore, for a proper cone K and a matrix A, if for all $x\in K$, $Ax\in K$, then matrix A is called a K-nonnegative matrix.

In the following, some lemmas and definitions about the K-nonnegative matrix and cone-induced vector norms are given.

Lemma 1. [32] *For a K-nonnegative matrix A, there exists a vector $\lambda\in\operatorname{int}(K)$ letting $(I-A)\lambda\in\operatorname{int}(K)$ if and only if matrix A is a Schur matrix.*

Definition 1. *Let $K\subset\mathbb{R}^n$ be a proper cone and let $u\in\operatorname{int}(K)$, an order interval is formed as*

$$B_u=\{x\in\mathbb{R}^n \mid -u\preceq_K x\preceq_K u\}.$$

Then for any vector $y\in\mathbb{R}^n$, a cone-induced vector norm on $\mathbb{R}^n$ is defined as

$$\|y\|_{u,\infty}=\inf\{t\geq 0 \mid y\in tB_u\}.$$

Note that the cone-induced vector norm $\|x\|_{u,\infty}=0$ if and only if $x=0$. The cone-induced norm $\|x\|_{u,\infty}$ is monotonic with respect to cone K, that is, if $x\succeq_K y$, then $\|x\|_{u,\infty}\geq\|y\|_{u,\infty}$ holds. Furthermore, it should be mentioned that Definition 1 and Lemma 1 are given for proper cones. In the subsequent sections of this paper, all cones referred to are considered to be proper cones.

III. Main Results

The system taken into consideration is as follows:

$$\begin{aligned} x(k+1)&=A_0x(k)+\sum_{i=1}^{N}A_ix(k-d_i(k)),\quad k=0,1,\dots,\\ x(s)&=\varphi(s),\quad s=-\tau,-\tau+1,\dots,0, \end{aligned}\tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state vector, $A_i\in\mathbb{R}^{n\times n}$ is the system matrix, $\tau\in\mathbb{N}_0$, and $d_i(k)\in\mathbb{N}_0$ is the time delay satisfying $0\leq d_i(k)\leq\tau$ for all $i\in\{1,2,\dots,N\}$.

Lemma 2. *For any $\varphi(\cdot)\in K_x$ and $d_i(k)\in\{0,1,\dots,\tau\}$, $x(k)\in K_x$ for all $k\in\mathbb{N}_0$, if and only if A_i is K_x-nonnegative matrix for all $i\in\{0,1,2,\dots,N\}$.*

Proof. *Sufficiency*: Mathematical induction is applied to prove the sufficiency. Assume that A_i is K_x-nonnegative and $\varphi(k)\in K_x$. First, when $k=0$, the above state space equation can be written as $x(1)=A_0x(0)+\sum_{i=1}^N A_ix(-d_i(0))$. Since $-d_i(0)\in\{-\tau,-\tau+1,\dots,0\}$ and A_i is K_x-nonnegative matrix, $A_0x(0)$ and $A_ix(-d_i(0))\in K_x$ and their nonnegative linear combination $x(1)\in K_x$. Then we assume $x(k)\in K_x$ for all $k\leq p$. When $k=p$, the state space equation can be rewritten as, $x(p+1)=A_0x(p)+\sum_{i=1}^N A_ix(p-d_i(p))$. Similar to the proof of $x(1)\in K_x$, $x(p+1)\in K_x$ holds. By induction, $x(k)\in K_x$, for all $k\in\mathbb{N}_0$.

Necessity: Assume that there exists a number $i_s\in\mathbb{N}_0$ such that A_{i_s} is not a K_x-nonnegative matrix and $x(k)\in K_x$ for all $k\in\mathbb{N}$. There exists at least one $x_{out}\in K_x$ such that $A_{i_s}x_{out}\notin K_x$. Without loss of generality, we assume that $i_s=0$. Then $\varphi(k)$ can be chosen as follows, $\varphi(0)=x_{out}$, $\varphi(k)=0$ where $k\in\{-\tau,-\tau+1,\dots,-1\}$ and $d_i(0)\in\{0,1,\dots,\tau\}$. When $k=0$, $x(1)=A_0x_{out}\notin K_x$. This statement contradicts the assumption. The necessity of Lemma 2 is established. □

Lemma 2 provides an equivalent condition for the cone preserving property in system (1). In the following discussion, system (1) denotes the discrete-time cone-preserving system, where matrices A_i are K_x-nonnegative for all $i\in\{0,1,\dots,N\}$. Before analyzing the stability condition of system (1), a preliminary lemma that illustrates the relation of the states in system (1) under different initial conditions is presented. This lemma will assist in determining the stability of system (1) under arbitrary initial conditions.

Lemma 3. *Let x_I and x_{II} denote the states of system (1) with different initial conditions $\varphi_I(k)$ and $\varphi_{II}(k)$, respectively. If $\varphi_I(k)\preceq_{K_x}\varphi_{II}(k)$ holds for all $k\in\{-\tau,\tau+1,\dots,0\}$, the inequality $x_I(k)\preceq_{K_x}x_{II}(k)$ holds for all $k\in\mathbb{N}_0$.*

Proof. A new system is formed by defining $x(k)=x_{II}(k)-x_I(k)$ and $\varphi(k)=\varphi_{II}(k)-\varphi_I(k)$. Since $\varphi_I(k)\preceq_{K_x}\varphi_{II}(k)$, $\varphi(k)\in K_x$ holds. According to Lemma 2, $x(k)\in K_x$ for all $k\in\mathbb{N}_0$. In other words, $x_I(k)\preceq_{K_x}x_{II}(k)$ for all $k\in\mathbb{N}_0$. □

Cone-preserving systems with constant time delays can be viewed as a special kind of time-varying delay systems, as described in (1), in which the delays $d_i(\cdot)$ are constant functions. As a special case, the equivalent stability condition for systems with constant time delays serves as a necessary stability condition for system (1). Moreover, this condition assists in analyzing the sufficient condition using the comparison method. Therefore, rather than directly analyzing the stability condition of system (1), the linear cone-preserving systems with constant time delays are first introduced as follows:

$$\begin{aligned} y(k+1) &= A_0 y(k) + \sum_{i=1}^{N} A_i y(k-\tau_i), \quad k=0,1,\ldots, \\ y(s) &= \phi(s), \quad s=-\tau,-\tau+1,\ldots,0, \end{aligned} \tag{2}$$

where $\tau_i \in \mathbb{N}$, $\tau_i = \max_{\forall k \in \mathbb{N}_0} \{d_i(k)\}$, and A_i is K_x-nonnegative matrix for all $i \in \{0,1,\ldots,N\}$. In the subsequent discussion, we present two lemmas that demonstrate the monotonicity and stability conditions of system (2) when the initial condition $\phi(\cdot)$ is a constant vector function.

Lemma 4. *(**Monotonicity**) Consider system (2) with initial condition $\phi(\cdot) \equiv \lambda \in K_x$. If the vector λ satisfies the condition $\left(\sum_{i=0}^{N} A_i - I\right)\lambda \preceq_{K_x} 0$, the state of system (2) satisfies $y(k+1) \preceq_{K_x} y(k)$ for all $k \in \{-\tau,-\tau+1,\ldots,0\} \cup \mathbb{N}$.*

Proof. First, when $k=0$, system (2) can be written as $y(1) = A_0 y(0) + \sum_{i=1}^{N} A_i y(0-\tau_i)$. Since $\phi(\cdot) \equiv \lambda$, the equation can be written as $y(1) = y(0) + \left(\sum_{i=0}^{N} A_i - I\right)\lambda$. Since the inequality $\left(\sum_{i=0}^{N} A_i - I\right)\lambda \preceq_{K_x} 0$ holds, inequality $y(1) \preceq_{K_x} y(0)$ is obtained.

Then assume that inequality $y(k+1) \preceq_{K_x} y(k)$ holds for all $k \in \{-\tau,-\tau+1,\ldots,p\}$. For $k=p+1$,

$$y(p+2) = A_0 y(p+1) + \sum_{i=1}^{N} A_i y(p+1-\tau_i).$$

Since $y(p) - y(p+1) \succeq_{K_x} 0$, $y(p-\tau_i) - y(p+1-\tau_i) \succeq_{K_x} 0$ and A_i is K_x-nonnegative, inequalities $A_0 y(p) \succeq_{K_x} A_0 y(p+1)$ and $A_i y(p-\tau_i) \succeq_{K_x} A_i y(p+1-\tau_i)$ hold. Moreover, equation $y(p+2) \preceq_{K_x} A_0 y(p) + \sum_{i=1}^{N} A_i y(p-\tau_i) = y(p+1)$ holds. By induction, the above system satisfies $y(k+1) \preceq_{K_x} y(k)$ for all $k \in \{-\tau,-\tau+1,\ldots,0\} \cup \mathbb{N}$. □

Lemma 5. *(**Asymptotic Stability of System (2)**) System (2) with initial condition $\phi(\cdot) \equiv \lambda_1 \in \operatorname{int}(K_x)$ is asymptotically stable if and only if there exists a vector $\lambda \succ_{K_x} 0$ satisfying that $(\sum_{i=0}^{N} A_i - I)\lambda \prec_{K_x} 0$.*

Proof. *Sufficiency*: Since $\lambda \succ_{K_x} 0$, without loss of generality, one could assume that there exist a positive scalar l such that $l\lambda \succ_{K_x} \lambda_1$. First, we consider the system (2) with initial condition $\phi. \equiv l\lambda$, where the state is denoted by y^*. According to Lemma 4, for any $k \in \{-\tau,-\tau+1,\ldots,0\} \cup \mathbb{N}$, inequality $y^*(k+1) \preceq_{K_x} y^*(k)$ holds. Lemma 2 indicates that when A_i is K_x-nonnegative, $y^*(k) \succeq_{K_x} 0$ holds. Therefore, the limit of $y^*(k)$ exists and it can be defined as $c = \lim_{k\to\infty} y^*(k)$. Then following equations are obtained:

$$\begin{aligned} \lim_{k\to\infty} y^*(k+1) &= \lim_{k\to\infty} A_0 y^*(k) + \sum_{i=1}^{N} \lim_{k\to\infty} A_i y^*(k-\tau_i), \\ c &= \sum_{i=0}^{N} A_i c. \end{aligned}$$

By Lemma 1, eigenvalues of $\sum_{i=0}^{N} A_i$ are inside the unit circle. Matrix $\sum_{i=0}^{N} A_i - I$ is a full rank matrix. Equation $c = \sum_{i=0}^{N} A_i c$ holds if and only if $c=0$. Based on Lemma 3, the state $y(k)$ with the initial condition $\phi(\cdot) \equiv \lambda_1$ satisfies $y(k) \preceq_{K_x} y^*(k)$. When $k\to\infty$, $y(k)\to 0$. System (2) with initial condition $\phi(\cdot) \equiv \lambda_1$ is asymptotically stable.

Necessity: According to system (2), for a positive integer p, following p equations can be obtained,

$$\begin{aligned} y(1) &= A_0 y(0) + \sum_{i=1}^{N} A_i y(0-\tau_i), \\ y(2) &= A_0 y(1) + \sum_{i=1}^{N} A_i y(1-\tau_i), \\ &\vdots \\ y(p) &= A_0 y(p-1) + \sum_{i=1}^{N} A_i y(p-1-\tau_i). \end{aligned}$$

Sum the above equations, one has $y(p) + \sum_{i=1}^{p-1} y(i) = A_0 \sum_{i=0}^{p-1} y(i) + \sum_{j=1}^{N} A_j \sum_{i=0}^{p-1} y(i-\tau_j)$. Rewrite the equation, one can obtain

$$\begin{aligned} y(p) - y(0) &= (A_0 - I)\sum_{i=0}^{p-1} y(i) + \sum_{j=1}^{N} A_j \sum_{i=0}^{p-1} y(i-\tau_j), \\ y(p) - y(0) - \sum_{j=1}^{N} A_j \sum_{i=0}^{\tau_j-1} y(i-\tau_j) &= -\sum_{i=0}^{p-1} y(i) + \sum_{j=0}^{N} A_j \sum_{i=0}^{p-\tau_j-1} y(i), \end{aligned}$$

where $\tau_0 = 0$. Since the system is asymptotically stable, one has $y(p)\to 0$ when $p\to\infty$. When $p\to\infty$, the above equation can be written as

$$\begin{aligned} -y(0) &= \lim_{p\to\infty}\left[\sum_{j=1}^{N} A_j \sum_{i=0}^{\tau_j-1} y(i-\tau_j) - \sum_{i=0}^{p-1} y(i) + \sum_{j=0}^{N} A_j \sum_{i=0}^{p-\tau_j-1} y(i)\right] \\ &= \lim_{p\to\infty}\left[\sum_{j=1}^{N} A_j \sum_{i=0}^{\tau_j-1} y(i-\tau_j) - \sum_{i=0}^{p-1} y(i) + \sum_{j=0}^{N} A_j \sum_{i=\tau_j}^{p-1} y(i-\tau_j)\right] \\ &= \lim_{p\to\infty}\left[\sum_{j=1}^{N} A_j \sum_{i=0}^{p-1} y(i-\tau_j) + A_0 \sum_{i=0}^{p-1} y(i) - \sum_{i=0}^{p-1} y(i)\right] \\ &= \lim_{p\to\infty}\left[\sum_{j=0}^{N} A_j \sum_{i=0}^{p-1} y(i-\tau_j) - \sum_{i=0}^{p-1} y(i)\right] \\ &\succeq_{K_x} \lim_{p\to\infty}\left[\sum_{j=0}^{N} A_j \sum_{i=0}^{p-1-\tau_j} y(i) - \sum_{i=0}^{p-1} y(i)\right] = \left(\sum_{i=0}^{N} A_i - I\right)\sum_{i=0}^{\infty} y(i). \end{aligned}$$

Since $y(0) = \lambda_1 \in \operatorname{int}(K_x)$, there exists a $\lambda = \sum_{i=0}^{\infty} y(i) \in \operatorname{int}(K_x)$ such that $\left(\sum_{i=0}^{N} A_i - I\right)\lambda \prec_{K_x} 0$. □

To establish the relationship between the stability conditions of systems (2) and (1), Lemma 6 is presented. It illustrates the connection between the states in systems (1) and (2) when subjected to the same initial condition $\phi(\cdot)$.

Lemma 6. *Consider system (1) and system (2) with initial condition $\varphi(\cdot) \equiv \lambda$ and $\phi(\cdot) \equiv \lambda$, respectively, If there exists a vector λ such that $\left(\sum_{i=0}^{N} A_i - I\right)\lambda \preceq_{K_x} 0$, the inequality $x(k) \preceq_{K_x} y(k)$ holds for all $k \in \mathbb{N}_0$, where $x(k)$ and $y(k)$ are states of system (1) and system (2), respectively.*

Proof. For $k=0$, $x(1)$ and $y(1)$ can be written as

$$x(1)=A_0x(0)+\sum_{i=1}^{N}A_ix\left(0-d_i(0)\right),$$
$$y(1)=A_0y(0)+\sum_{i=1}^{N}A_iy(0-\tau_i).$$

Since $\phi(\cdot)=\lambda$, $y(1)-x(1)=A_0(\lambda-\lambda)+\sum_{i=1}^{N}A_i(\lambda-\lambda)=0$, which satisfies $x(1)\preceq_{K_x}y(1)$. Then we assume $x(k)\preceq_{K_x}y(k)$ when $k\le p$. For $x(p+1)$ and $y(p+1)$, following equations hold:

$$x(p+1)=A_0x(p)+\sum_{i=1}^{N}A_ix\left(p-d_i(p)\right),$$
$$y(p+1)=A_0y(p)+\sum_{i=1}^{N}A_iy(p-\tau_i).$$

By considering the difference of the above two equations, we can obtain $y(p+1)-x(p+1)=A_0\left[y(p)-x(p)\right]+\sum_{i=1}^{N}A_i\left[y(p-\tau_i)-x\left(p-d_i(p)\right)\right]$.

Based on Lemma 4, the state $y(k)$ monotonically decreases under the partial order defined by the cone K_x and the inequality $y(p-\tau_i)\succeq_{K_x}y\left(p-d_i(p)\right)$ holds. Combining the inequality with the assumption, inequality $y(p-\tau_i)\succeq_{K_x}y\left(p-d_i(p)\right)\succeq_{K_x}x\left(p-d_i(p)\right)$ holds. It is obviously that $y(k)\succeq_{K_x}x(k)$ for all $k\in\mathbb{N}_0$. □

Based on the stability condition of system (2) and the relation between states outlined in Lemma 3 and Lemma 6, Theorem 1 is presented to demonstrate several equivalent asymptotic stability conditions for system (1).

Theorem 1. ***(Asymptotic Stability of System (1))*** *For a discrete-time cone-preserving system (1) with time-varying delays, the following statements are equivalent:*
i) System (1) is asymptotically stable;
ii) There exists a vector $\lambda\in\operatorname{int}(K_x)$ that satisfies condition $\left(\sum_{i=0}^{N}A_i-I\right)\lambda\prec_{K_x}0$;
iii) Matrix $\sum_{i=0}^{N}A_i$ is a Schur matrix.

Proof. Based on Lemma 1, condition ii) and condition iii) are equivalent. Consequently, to establish the equivalence for Theorem 1, it is only necessary to demonstrate the equivalence between conditions i) and ii).
Sufficiency: Assume that there is a vector λ which satisfies equation $\left(\sum_{i=0}^{N}A_i-I\right)\lambda\preceq_{K_x}0$. Inequality $\left(\sum_{i=0}^{N}A_i-I\right)m\lambda\prec_{K_x}0$ holds for all $m>0$. Denote that $\varphi_{max}=\max_k\{\|\varphi(k)\|_2\}$. For a λ_1 satisfies conditions $\left(\sum_{i=0}^{N}A_i-I\right)\lambda_1\prec_{K_x}0$ and $\|\lambda_1\|_2=1$, there always exists a positive scalar ε letting $\lambda_1-\varepsilon x\in K_x$ for any $\|x\|_2=1, x\in\mathbb{R}^n$. Letting vector $\lambda=m\lambda_1$, where $m=\varphi_{max}/\varepsilon$. For any $\varphi(k)$, it holds that $\lambda-\varphi(k)\succeq_{K_x}0$. By Lemma 4 and Lemma 6, inequality $0\preceq_{K_x}x(k)\preceq_{K_x}y_\lambda(k)$ holds, where $y_\lambda(k)$ is the solution of system (2) with the initial condition $\phi(k)\equiv\lambda$. By the Squeeze Theorem, when $k\to\infty$, there is $0\preceq_{K_x}\lim_{t\to\infty}x(k)\preceq_{K_x}\lim_{k\to\infty}y_\lambda(k)$. In other words, $\lim_{k\to\infty}x(k)=0$. System (1) is asymptotically stable for any $\varphi(k)\in K_x$.

Necessity: Assume $\varphi(k)\equiv\lambda\in\operatorname{int}(K_x)$ and $d_i(\cdot)\equiv\tau_i$. According to Lemma 1, there exists a $\lambda\in\operatorname{int}(K_x)$ letting $\left(\sum_{i=0}^{N}A_i-I\right)\preceq_{K_x}0$ is a necessary and sufficient asymptotic stability condition for linear discrete-time cone-preserving systems with fixed initial condition and constant time delays. Therefore, the condition that there exists a vector $\lambda\in K_x$ satisfying $\left(\sum_{i=0}^{N}A_i-I\right)\prec_{K_x}0$ is a necessary condition for system (1). □

Remark 1. *When letting the proper cone K_x be the nonnegative orthant constructed by unit basis vectors, Theorem 1 could be used to characterize the stability of discrete-time positive systems with constant time delays [33], [34] or time-varying delays.*

Definition 2. *(**α-exponential stability**) For a given scalar $0<\alpha<1$, system (1) is called α-exponentially stable if there exist a scalar $\Gamma>0$ and a vector $\lambda\in\operatorname{int}(K_x)$ letting the state of system (1) satisfy the equation $\|x(k)\|_{u,\infty}\le\Gamma\|\lambda\|_{u,\infty}\alpha^k$ for all $k\in\mathbb{N}_0$.*

Note that in the definition of α-exponential stability of the cone-preserving system, convergent rate α is prescribed. We can always find an $\varepsilon>0$ letting $\alpha+\varepsilon<1$. With the definition of α-exponential stability, we derive Theorem 2 to analyse whether a cone-preserving system is α-exponentially stable.

Theorem 2. *System (1) is α-exponentially stable if and only if $A_0\alpha^{-1}+\sum_{i=1}^{N}A_i\alpha^{-\tau_i-1}$ is a Schur matrix.*

Proof. *Sufficiency*: First consider system (2) with initial condition $\phi(s)\equiv\alpha^s\lambda$. Define $\bar{y}(k)=\alpha^{-k}y(k)$. Then system (2) with the initial condition $\phi(s)$ can be written as following form,

$$\bar{y}(k+1)=A_0\alpha^{-1}\bar{y}(k)+\sum_{i=1}^{N}A_i\alpha^{-\tau_i-1}\bar{y}(k-\tau_i),\quad k=0,1,\ldots,$$
$$\bar{y}(k)=\alpha^{-k}\phi(k)\equiv\lambda,\quad k=-\tau,-\tau+1,\ldots,0.$$

Since $\alpha>0$, $A_0\alpha^{-1}$, $A_i\alpha^{-\tau_i-1}$ are K_x-nonnegative matrices and equation $\left(A_0\alpha^{-1}+\sum_{i=1}^{N}A_i\alpha^{-d_i-1}\right)\lambda\prec_{k_x}0$ holds, $\bar{y}(k+1)\preceq_{K_x}\bar{y}(k)$ for any $k\in\mathbb{N}$ can be proved by Lemma 4. The following inequality holds,

$$\|\bar{y}(k)\|_{u,\infty}=\left\|\alpha^{-k}y(k)\right\|_{u,\infty}\le\left\|\alpha^{-k+1}y(k-1)\right\|_{u,\infty}.$$

Then inequality $\|y(k)\|_{u,\infty}\le\alpha^k\|\lambda\|_{u,\infty}$ can be obtained. Letting $\bar{x}(k)=\alpha^{-k}x(k)$, where $x(k)$ is the state of system (1) with initial condition $\varphi(\cdot)\in K_x$. Similar to the proof in Theorem 1, there always exists a scalar named Γ letting $\Gamma\lambda\succeq_{K_x}\alpha^{-s}\varphi(s)$, for all $s\in\{-\tau,-\tau+1,\ldots,0\}$. Equation for $\bar{x}$ is as follows:

$$\bar{x}(k+1)=A_0\alpha^{-1}\bar{x}(k)+\sum_{i=1}^{N}A_i\alpha^{-d_i(k)-1}\bar{x}\left(k-d_i(k)\right).$$

Let the error $\bar{e}(k)\triangleq\bar{y}(k)-\bar{x}(k)$, the error system can be written in the following form:

$$\begin{aligned}\bar{e}(k+1)=&A_0\alpha^{-1}\bar{e}(k)+\sum_{i=1}^{N}A_i\alpha^{-d_i(k)-1}\bar{e}\left(k-d_i(k)\right)\\&+\sum_{i=1}^{N}\Big\{A_i\left(\alpha^{-\tau_i-1}-\alpha^{-d_i(k)-1}\right)\bar{y}(k-\tau_i)\\&+A_i\alpha^{-d_i(k)-1}\left[\bar{y}(k-\tau_i)-\bar{y}\left(k-d_i(k)\right)\right]\Big\}.\end{aligned}$$

Since $\alpha^{-s}\phi(s) \preceq_{K_x} \Gamma\lambda$ and the fact that $\sum_{i=1}^{N}\left\{A_i\left(\alpha^{-\tau_i-1}-\alpha^{-d_i(k)-1}\right)\bar{y}(k-\tau_i)+A_i\alpha^{-d_i(k)-1}[\bar{y}(k-\tau_i)-\bar{y}(k-d_i(k))]\right\}$ is in cone K_x, we can derive that $\bar{e}\in K_x$ by Lemma 2. Combining with the above conclusion, inequality $\|x(k)\|_{u,\infty} \le \Gamma\|\lambda\|_{u,\infty}\alpha^k$ holds.

Necessity: First, the following assumption is made that there exist a set of matrices A_i such that $A_0\alpha^{-1}+\sum_{i=1}^{N}A_i\alpha^{-\tau_i-1}$ is not a Schur matrix, and system (1) is α-exponentially stable. Since $0<\alpha<1$, there exists a scalar $\varepsilon>0$ letting $\alpha+\varepsilon<1$. Define $\tilde{x}(k)=\tilde{\alpha}^{-k}x(k)$, where $\tilde{\alpha}=\alpha+\varepsilon$. Then system (1) can be transformed into the following forms:

$$\tilde{x}(k+1)=A_0\tilde{\alpha}^{-1}\tilde{x}(k)+\sum_{i=1}^{N}A_i\tilde{\alpha}^{-1-d_i(k)}\tilde{x}(k-d_i(k)),\ k\ge 0,$$
$$\tilde{x}(k)=\tilde{\alpha}^{-k}\varphi(k), \quad k=-\tau,-\tau+1,\ldots,0.$$

Since $A_0\tilde{\alpha}^{-1}$ and $A_i\tilde{\alpha}^{-1-d_i(k)}$ are K_x-nonnegative matrices and $\tilde{\alpha}^{-k}\varphi(\cdot)$ is in cone K_x, $\tilde{x}(k)$ is always in cone K_x. In other words, the above system is a cone-preserving system. Furthermore, according to the above assumption, system (1) is α-exponentially stable and inequality $\|x(k)\|_{u,\infty}\le\Gamma\|\lambda\|_{u,\infty}\alpha^k$ holds. For $\tilde{x}$, there is

$$\|\tilde{x}(k)\|_{u,\infty}=\left\|\tilde{\alpha}^{-k}x(k)\right\|_{u,\infty}\le\left(\frac{\alpha+\varepsilon}{\alpha}\right)^{-k}\Gamma\|\lambda\|_{u,\infty}.$$

When $k\to\infty$, $\|\tilde{x}(k)\|_{u,\infty}\to 0$. The above system is asymptotically stable. According to Theorem 1, when time delay $d_i(k)$ is set to be τ_i, the statement that $A_0\tilde{\alpha}^{-1}+\sum_{i=1}^{N}A_i\tilde{\alpha}^{-1-\tau_i}$ is Schur matrix is true for all $\frac{\alpha+\varepsilon}{\alpha}>1$. It contradicts the assumption, so the necessity of Theorem 2 is proved. □

Remark 2. *The necessity of Theorem 2 means that for a given α, $\rho\left(A_0\alpha^{-1}+\sum_{i=1}^{N}A_i\alpha^{-\tau_i-1}\right)<1$ is a necessary condition for α-exponentially stable. It does not mean that for any $0<\alpha<1$ system (1) is α-exponentially stable only if $A_0\alpha^{-1}+\sum_{i=1}^{N}A_i\alpha^{-\tau_i-1}$ is a Schur matrix. Furthermore, when choosing different vectors u, the values of the induced vector norm $\|x(k)\|_{u,\infty}$ are different. However, α-exponential stability of system (1) is not affected by vector u.*

IV. Illustrative Examples

To illustrate the theoretical results, system (1) with the time delay $d_1(k)$ and following system matrices

$$A_0=\begin{bmatrix}0.033 & 0.227 & 0.24\\ 0.2 & 0.113 & 0.167\\ 0.087 & 0.12 & 0.013\end{bmatrix},\ A_1=\begin{bmatrix}0.287 & 0.033 & 0.02\\ 0.06 & 0.207 & 0.093\\ 0.173 & 0.14 & 0.307\end{bmatrix},$$

are given. Matrices A_0 and A_1 are K_x-nonnegative, with cone K_x formed by three edges

$$e_1=\begin{bmatrix}2 & 2 & -1\end{bmatrix}^T,\ e_2=\begin{bmatrix}2 & -1 & 2\end{bmatrix}^T,\ e_3=\begin{bmatrix}-1 & 2 & 2\end{bmatrix}^T.$$

It can be observed that the edges remain within the cone K_x even after applying linear transformations by matrices A_0 and A_1. According to the definition of cone-preserving systems, this confirms the K_x-nonnegative property of matrices A_0 and A_1.

In this illustrative example, two types of time delays are considered for $d_1(k)$. The first is a constant time delay, where

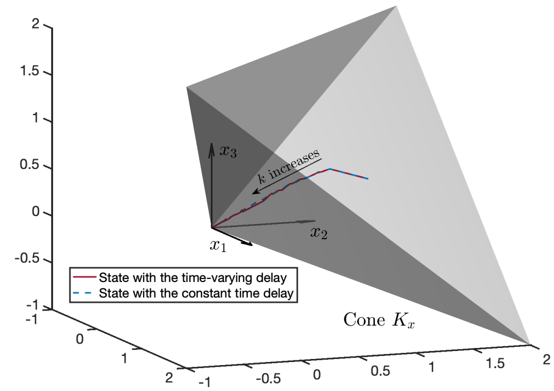

Fig. 1. Trajectory of states for system (1) with the time-varying delay and the constant time delay

α_i	eigenvalues of matrix $A_0\alpha_i^{-1}+\sum_{i=1}^{N}A_i\alpha_i^{-d_i-1}$
$\alpha_1=0.90$	1.0699, 0.1736 and 0.1387
$\alpha_2=0.93$	0.9879, 0.1325 and 0.1090

TABLE I
Eigenvalues of matrix $A_0\alpha_i^{-1}+\sum_{i=1}^{N}A_i\alpha_i^{-d_i-1}$ for different α_i

$d_1(k)=4$ for all $k\in\mathbb{N}_0$. The second is a time-varying delay, in which $d_1(k)$ is an integer randomly selected from the range 0 to 4, with equal probability for each value. The initial condition $\varphi(s)$ in system (1) is set to be

$$\varphi(s)=\begin{bmatrix}1 & 0.95+0.05\sin\frac{\pi(5+s)}{6} & 0.6+0.1\sin\frac{\pi(5+s)}{6}\end{bmatrix}^T,$$

where $s=-4,-3,\ldots,0$.

The state trajectories for different time delays are depicted in Fig. 1. Arrows illustrate the evolution of the states, demonstrating that the states ultimately converge to zero in both cases. Furthermore, α-exponential stability of system (1) is also discussed. By introducing the vector norm $\|\cdot\|_{u,\infty}$ where $u=\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T$ and applying $\alpha_1=0.9$ and $\alpha_2=0.93$ to α in matrix $A_0\alpha^{-1}+\sum_{i=1}^{N}A_i\alpha^{-d_i-1}$, the eigenvalues of the matrix with two different α are shown in Table I. It can be found that the spectral radius of the matrix with α_1 is more than 1 and the one with α_2 is less than 1. It indicates that the decay rate of the system is faster than 0.93 but slower than 0.90. Since $\max\{\varphi(\cdot)\}=1.1$, $\Gamma\|\lambda\|_{u,\infty}$ can be chosen as 1.1. Fig. 2 shows the variation of the function $\ln\|x(k)\|_{u,\infty}$, where $x(k)$ is the state of the system with the time-varying delay. It can be found that the convergent rate is in the range of $[0.90, 0.93]$, which verifies the theoretical results.

V. Conclusion

In this paper, a necessary and sufficient asymptotic stability condition for discrete-time linear cone-preserving systems with bounded time-varying delays is proposed based on comparison principle. Furthermore, by introducing α-exponential stability of systems with cone invariance, a necessary and sufficient condition is established for exponential stability of discrete-time systems with cone invariance and time-varying

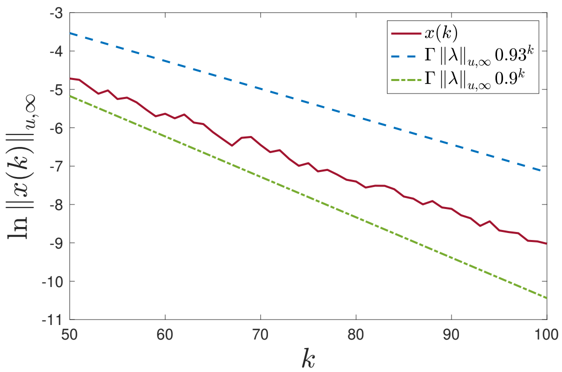

Fig. 2. Variation of $\ln\|x(k)\|_{u,\infty}$ for the time-varying delay system

delays. Finally, some numerical examples are presented to illustrate the theoretical results.

References

[1] Ying Chen, Paolo Bolzern, Patrizio Colaneri, Yuming Bo, and Baozhu Du. Stability and stabilization for markov jump linear systems in polyhedral cones. In *Proceedings of the 2018 IEEE Conference on Decision and Control*, pages 4779–4784. IEEE, 2018.

[2] Ying Chen, Paolo Bolzern, and Patrizio Colaneri. Stability, $\mathscr{L}_1$ performance and state feedback design for linear systems in ice-cream cones. *International Journal of Control*, 94(3):784–792, 2021.

[3] Meng-Jie Hu, Ju H Park, and Yan-Wu Wang. Stabilization of positive systems with time delay via the Takagi–Sugeno fuzzy impulsive control. *IEEE Transactions on Cybernetics*, 52(6):4275–4285, 2020.

[4] Bohao Zhu, James Lam, Xiaochen Xie, Xiaoqi Song, and Ka-Wai Kwok. Stability and L_∞-gain analysis of periodic piecewise positive systems with constant time delay. *IEEE Transactions on Automatic Control*, 67(5):2655–2662, 2021.

[5] Xingwen Liu, Wensheng Yu, and Long Wang. Stability analysis for continuous-time positive systems with time-varying delays. *IEEE Transactions on Automatic Control*, 55(4):1024–1028, 2010.

[6] Xingwen Liu and James Lam. Relationships between asymptotic stability and exponential stability of positive delay systems. *International Journal of General Systems*, 42(2):224–238, 2013.

[7] Ligang Wu, James Lam, Zhan Shu, and Baozhu Du. On stability and stabilizability of positive delay systems. *Asian Journal of Control*, 11(2):226–234, 2009.

[8] Shuo Li, Zhengrong Xiang, and Junfeng Zhang. Dwell-time conditions for exponential stability and standard L_1-gain performance of discrete-time singular switched positive systems with time-varying delays. *Nonlinear Analysis: Hybrid Systems*, 38:100939, 2020.

[9] Juyong Zhang, Zhu Han, and Hai Wu. Robust finite-time stability and stabilisation of switched positive systems. *IET Control Theory & Applications*, 8(1):67–75, 2014.

[10] Peng Wang and Jun Zhao. Dissipativity of positive switched systems using multiple linear supply rates. *Nonlinear Analysis: Hybrid Systems*, 32:37–53, 2019.

[11] Zhongyang Fei, Weizhong Chen, and Xudong Zhao. Interval estimation for asynchronously switched positive systems. *Automatica*, 143:110427, 2022.

[12] JE Kurek. Stability of positive 2-D system described by the Roesser model. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(4):531–533, 2002.

[13] Jinna Fu, Zhaoxia Duan, and Zhengrong Xiang. On mixed ℓ_1/ℓ_- fault detection observer design for positive 2D Roesser systems: Necessary and sufficient conditions. *Journal of the Franklin Institute*, 359(1):160–177, 2022.

[14] Lorenzo Farina and Sergio Rinaldi. *Positive Linear Systems: Theory and Applications*. New York: Wiley-Interscience, 2000.

[15] Wassim M Haddad, VijaySekhar Chellaboina, and Tanmay Rajpurohit. Dissipativity theory for nonnegative and compartmental dynamical systems with time delay. *IEEE Transactions on Automatic Control*, 49(5):747–751, 2004.

[16] Xingwen Liu, Wensheng Yu, and Long Wang. Stability analysis of positive systems with bounded time-varying delays. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 56(7):600–604, 2009.

[17] Pham Huu Anh Ngoc. Stability of positive differential systems with delay. *IEEE Transactions on Automatic Control*, 58(1):203–209, 2013.

[18] Jason JR Liu, James Lam, Bohao Zhu, Xiaomei Wang, Zhan Shu, and Ka-Wai Kwok. Nonnegative consensus tracking of networked systems with convergence rate optimization. *IEEE Transactions on Neural Networks and Learning Systems*, 33(12):7534–7544, 2021.

[19] Jun-e Feng, James Lam, Ping Li, and Zhan Shu. Decay rate constrained stabilization of positive systems using static output feedback. *International Journal of Robust and Nonlinear Control*, 21(1):44–54, 2011.

[20] Achim Ilchmann and Pham Huu Anh Ngoc. Stability and robust stability of positive Volterra systems. *International Journal of Robust and Nonlinear Control*, 22(6):604–629, 2012.

[21] Shuyuan Zhu, Zuyi Li, and Chenghui Zhang. Exponential stability analysis for positive systems with delays. *IET Control Theory & Applications*, 6(6):761–767, 2012.

[22] Shuqian Zhu, Min Meng, and Chenghui Zhang. Exponential stability for positive systems with bounded time-varying delays and static output feedback stabilization. *Journal of the Franklin Institute*, 350(3):617–636, 2013.

[23] Ping Li and James Lam. Decentralized control of compartmental networks with H_∞ tracking performance. *IEEE Transactions on Industrial Electronics*, 60(2):546–553, 2012.

[24] Esteban Hernandez-Vargas, Patrizio Colaneri, Richard Middleton, and Franco Blanchini. Discrete-time control for switched positive systems with application to mitigating viral escape. *International journal of robust and nonlinear control*, 21(10):1093–1111, 2011.

[25] Raktim Bhattacharya, Jimmy Fung, Abhishek Tiwari, and Richard M Murray. Ellipsoidal cones and rendezvous of multiple agents. In *Proceedings of IEEE Conference on Decision and Control*, volume 1, pages 171–176. IEEE, 2004.

[26] Murad Banaji. Monotonicity in chemical reaction systems. *Dynamical Systems*, 24(1):1–30, 2009.

[27] Jun Shen and James Lam. Input–output gain analysis for linear systems on cones. *Automatica*, 77:44–50, 2017.

[28] Jun Shen and James Lam. Some extensions on the bounded real lemma for positive systems. *IEEE Transactions on Automatic Control*, 62(6):3034–3038, 2016.

[29] Jun Shen and Wei Xing Zheng. Stability analysis of linear delay systems with cone invariance. *Automatica*, 53:30–36, 2015.

[30] Takashi Tanaka, Cédric Langbort, and Valeri Ugrinovskii. DC-dominant property of cone-preserving transfer functions. *Systems & Control Letters*, 62(8):699–707, 2013.

[31] Jun Shen and James Lam. On the decay rate of discrete-time linear delay systems with cone invariance. *IEEE Transactions on Automatic Control*, 62(7):3442–3447, 2016.

[32] Abraham Berman and Robert J Plemmons. *Nonnegative Matrices*. Philadephia, PA: SIAM, 1994.

[33] M Busłowicz. Simple stability conditions for linear positive discrete-time systems with delays. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, pages 325–328, 2008.

[34] Xingwen Liu, Long Wang, Wensheng Yu, and Shouming Zhong. Constrained control of positive discrete-time systems with delays. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 55(2):193–197, 2008.