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Brief paper

A polynomial blossoming approach to stabilization of periodic time-varying systems☆

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ABSTRACT

This paper proposes a novel polynomial blossoming approach to designing stabilizing controller for a class of periodic time-varying systems. Utilizing multi-convexity of a non-homogeneous symmetric matrix polynomial, the approach can provide a series of convex optimization conditions to guarantee the negativity/positivity of matrix polynomial. Special cases of the proposed approach are also discussed, giving the conclusion that our approach generalizes two existing matrix polynomial approaches. For periodic systems formulated with [0, 1]-bounded time-varying coefficients, the designed stabilizing controller not only involves user-selectable varying gains over time intervals that are possibly non-identical to the system fundamental period, but can also guarantee the exponential stability of the closed-loop system. The effectiveness of our approach is validated and illustrated through two application-oriented simulation examples.

1. Introduction

Periodic systems, usually described by ordinary differential equations with parametric periodicity, arise in many practical applications such as helicopter rotor blades, mechanical oscillators and ecosystem regulation (Bittanti & Colaneri, 2008). Analytical studies on periodic systems in continuous time domain are admittedly more challenging than discrete-time cases, since such systems could not be readily reformulated into time-invariant forms by lifting techniques (Zhou & Duan, 2011). To deal with linear continuous-time periodic system involving fixed fundamental periods, approaches tackling their stability and control issues can be roughly categorized into two major types: **(i)** numerical approaches based on the Floquet theory (Bittanti & Colaneri, 2008; Vrabel, 2019); **(ii)** system averaging and approximation approaches amenable to convex optimization based on periodic or looped Lyapunov functions (Briat, 2016; Briat & Seuret, 2015; Zhou & Qian, 2017). Stability criteria in the Floquet theory are determined through the monodromy matrix eigenvalues associated with given parameters (Bittanti & Colaneri, 2008). Averaged or approximated models of periodic systems such as periodic time-varying systems that are piecewise in time (Li, Lam, & Cheung, 2015) also require a prescribed partition of system dynamics over each fundamental period, which may be represented by a number of time-invariant, time-delay or time-varying subsystems (Li, Lam, Kwok, & Lu, 2018; Li, Lam, Lu, & Kwok, 2019; Xie & Lam, 2018).

For periodic time-varying systems that can be exactly parameterized with time-periodic coefficients, the system and controller are usually assumed to share the same set of time-varying parameters (Sakai, Asai, Ariizumi, & Azuma, 2020a, 2020b). However, accurate identification and modelling of periodic time-varying systems may be inaccessible in practice due to the system nonlinearity and uncertainty. Previous studies employed periodic time-varying Lyapunov matrices to solve control and filtering problems with parametric uncertainty (Xie, Lam, & Fan, 2018) and polytopic uncertainty (Fan, Lam, & Xie, 2018) via linear matrix inequality (LMI) conditions. Moreover, matrix polynomials were applied to representing the time-varying dynamics in controller gains (Li et al., 2018) and systems over each period (Xie, Fan, Kwok, & Lam, 2021). In most of the existing results, periodic piecewise time-varying/constant controllers and filters are constructed over time intervals that share the same widths as subsystem dwell times under a fixed periodic switching sequence. An exception is our previous work (Xie, Lam, & Kwok, 2020), where the

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time interpolation-based controller enables more flexible time segmentation, resulting in the coexistence of two non-identical linear time-varying (LTV) coefficients in controller design criteria. However, the method in Xie et al. (2020) only allows at most two non-identical time-varying coefficients over each time segmentation. It will be challenging if the system contains more nonlinear time-varying coefficients that cannot be precisely characterized. To this end, we utilize the polynomial blossoming theory to tackle this problem.

Polynomial blossoming (also known as "polar form" or "polarization") was first proposed by Ramshaw in 1987, indicating that a univariate polynomial of degree n is essentially equivalent to a symmetric polynomial in n variables that is linear in each variable separately (Ramshaw, 1987, 1989). The Blossoming Principle provides a connection between Bézier simplices and symmetric multi-affine maps, which can be intuitively extended to tensor forms facilitating computer graphics (DeRose, Goldman, Hagen, & Mann, 1993), and linear programming-based synthesis for polynomial dynamical systems (Sassi & Girard, 2012). A univariate nth degree Bernstein polynomial (Rokne, 1979) can be regarded as a special case of its blossoming. In this paper, we propose a non-homogeneous symmetric matrix polynomial approach to stabilizing controller design for a class of continuous periodic time-varying systems with known system dynamic boundedness. The novelty is focused on its tractability for tackling multiple time-varying coefficients defined over non-identical time intervals. The contributions are threefold:

- A lemma inspired by polynomial blossoming is proposed, focusing on the negativity/positivity of a class of symmetric matrix polynomials with multiple non-identical time-varying coefficients.
- The polynomial blossoming approach generalizes the existing matrix polynomial approaches in Li et al. (2019), Xie, Lam, Fan, Wang, and Kwok (2022) and Xie et al. (2021), providing a new perspective for decoupling time-varying coefficients through LMI manipulation in periodic stabilizing controller design.
- The proposed stabilizing controller can guarantee the global uniformly exponential stability of periodic time-varying systems under a general framework.

The paper is organized as follows. Section 2 gives the problem formulation and preliminaries for stabilizing controller design and the existing methods. Section 3 proposes the polynomial blossoming approach, based on which some special cases are discussed, and sufficient conditions for controller design are established. Section 4 validates the proposed approach based on an equivalent mass–spring–damper system. Section 5 concludes the paper.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of natural numbers (including zero) and the set of positive integers, respectively. For $n \in \mathbb{N}^+$, I_n and 0_n denote the $n \times n$ identity matrix and the $n \times n$ zero matrix, respectively (dimensions are consistent with the context if subscripts omitted). P^T and P^{-1} are the transpose and inverse of matrix P, respectively. For real symmetric matrices P and Q, $P \geq Q$ (resp., $P > Q$) means that matrix $P - Q$ is positive semi-definite (resp., positive definite). $\underline{\lambda}(P)$, $\overline{\lambda}(P)$ denote the minimal and maximal eigenvalues of square matrix P, respectively, and $\mathbf{sym}(P) \triangleq P^T + P$. Integer set $\overline{i,j} \triangleq \{i, i+1, \ldots, j-1, j\}$ denotes an interval from integer i to j, $i < j$. Given a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$, and $\Upsilon(\mathcal{S}, k, q)$ denotes the product of all elements in the qth combination derived by selecting k distinct elements from set $\mathcal{S}$, $k = 1, 2, \ldots, |\mathcal{S}|$, $q = 1, 2, \ldots, \binom{|\mathcal{S}|}{k}$.

2. Problem formulation and preliminaries

Consider a class of continuous-time periodic systems with a known fundamental period $T_p > 0$:

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}^{n_u}$ are the state vector and the control input, respectively; $\mathcal{A} : [0, \infty) \to \mathbb{R}^{n_x \times n_x}$ and $\mathcal{B} : [0, \infty) \to \mathbb{R}^{n_x \times n_u}$ are continuously T_p-periodic time-varying matrix functions, that is, $\mathcal{A}(t) = \mathcal{A}(t + T_p)$ and $\mathcal{B}(t) = \mathcal{B}(t + T_p)$, $\forall t \geq 0$. Suppose $A_0 \in \mathbb{R}^{n_x \times n_x}$ is a constant matrix, based on Zhou and Huang (2020) we can represent periodic matrix function $\mathcal{A}(t)$ without any loss of generality as

$$\mathcal{A}(t) = A_0 - A_0 + \mathcal{A}(t) \triangleq A_0 + \tilde{\mathcal{A}}(t), \tag{2}$$

where $\tilde{\mathcal{A}}(t) = \tilde{\mathcal{A}}(t + T_p)$. The model in (2) is widely applied in practical studies on continuous-time periodic systems, such as the rain-wind induced vibration model for cable-stayed bridges (Hartono & van der Burgh, 2004). Hence, in this paper we consider the following approximation for system (1):

$$\begin{cases} \mathcal{A}(t) = A_0 + \tilde{\mathcal{A}}(t) = A_0 + \displaystyle\sum_{i=1}^{N} \alpha_i(t) A_i \\ \mathcal{B}(t) = B_0 + \tilde{\mathcal{B}}(t) = B_0 + \displaystyle\sum_{j=1}^{S} \beta_j(t) B_j \end{cases} \tag{3}$$

where $A_0, A_i \in \mathbb{R}^{n_x \times n_x}$ and $B_0, B_j \in \mathbb{R}^{n_x \times n_u}$ are known constant matrices; $\alpha_i(t) = \alpha_i(t + T_p)$, $i \in \overline{1, N}$, and $\beta_j(t) = \beta_j(t + T_p)$, $j \in \overline{1, S}$, are continuous T_p-periodic scalar functions, satisfying $\alpha_i(t) \in [0, 1]$, $\beta_i(t) \in [0, 1]$, $\forall i, j$.

Remark 1. The model in (3) uses time-varying coefficients $\alpha_i(t), \beta_j(t) \in [0, 1]$, $i \in \overline{1, N}$, $j \in \overline{1, S}$, to represent both periodic dynamics and uncertainty in measurement and/or approximation. These coefficients may be scaled to within [0, 1] during the modelling process (Schouten, Lou, & Weiland, 2019), or possibly be normalized in [0, 1] based on the known parameter bounds. Take the case $N = 2$ for example, consider $\mathcal{A}(t) = \mathscr{A}_0 + \epsilon_1(t)\mathscr{A}_1 + \epsilon_2(t)\mathscr{A}_2$, $\epsilon_i(t) \in [\underline{\epsilon}_i, \overline{\epsilon}_i]$, $\underline{\epsilon}_i < \overline{\epsilon}_i$, $i = 1, 2$, and we can obtain

$$\begin{aligned} \mathcal{A}(t) &= \mathscr{A}_0 + \epsilon_1(t)\mathscr{A}_1 + \epsilon_2(t)\mathscr{A}_2 \\ &= \mathscr{A}_0 + \left(\epsilon_1(t) - \underline{\epsilon}_1 + \underline{\epsilon}_1\right)\mathscr{A}_1 + \left(\epsilon_2(t) - \underline{\epsilon}_2 + \underline{\epsilon}_2\right)\mathscr{A}_2 \\ &= \left(\mathscr{A}_0 + \underline{\epsilon}_1\mathscr{A}_1 + \underline{\epsilon}_2\mathscr{A}_2\right) + \sum_{i=1}^{2} \frac{\epsilon_i(t) - \underline{\epsilon}_i}{\overline{\epsilon}_i - \underline{\epsilon}_i}(\overline{\epsilon}_i - \underline{\epsilon}_i)\mathscr{A}_i \\ &= A_0 + \sum_{i=1}^{2} \alpha_i(t) A_i, \end{aligned} \tag{4}$$

where $A_0 \triangleq \mathscr{A}_0 + \underline{\epsilon}_1\mathscr{A}_1 + \underline{\epsilon}_2\mathscr{A}_2$, $A_i \triangleq (\overline{\epsilon}_i - \underline{\epsilon}_i)\mathscr{A}_i$, $\alpha_i(t) \triangleq \frac{\epsilon_i(t) - \underline{\epsilon}_i}{\overline{\epsilon}_i - \underline{\epsilon}_i} \in [0, 1]$, $i = 1, 2$. Hence, formulation (3) can provide efficiency in modelling. It also enables more generality than periodic piecewise linear system models (Li et al., 2015; Xie et al., 2018), and covers the existing periodic piecewise LTV system models (Li et al., 2019; Xie et al., 2020) by setting $\alpha_i(t), \beta_j(t)$ as periodic piecewise LTV functions.

Note that periodic system (1) satisfying formulation (3) can also be regarded as a class of linear parameter varying (LPV) formulations. However, in most of the studies on periodic time-varying systems and LPV control systems, it is usually assumed that the varying parameters appeared in a quadratic Lyapunov function candidate

$$V(t) = x^T(t)\mathcal{P}(t)x(t) > 0,\ \ \mathcal{P}(t) > 0,\ \ t \geq 0, \tag{5}$$

and in the state feedback time-varying control law

$$u(t)=\mathcal{K}(t)x(t),\quad t\geq 0, \tag{6}$$

are identical to those in the system model (see Remark 2). In this paper, since the parameters in formulation (3) are time-periodic, our direct thought is to consider a controller in form of (6) with periodic time-varying gains $\mathcal{K}(t)=\mathcal{K}(t+T_p)$. We aim to find a direct and effective approach to stabilize periodic system (1) that satisfies (3), without imposing the time-varying parameters in the controller gains to be identical to those in the system model.

Remark 2. Existing studies on periodic time-varying control or LPV control tend to use the same set of parameters in both the system matrix and the controller. For instance, in Sakai et al. (2020a, 2020b), all the time-varying elements in system matrix $\mathcal{A}(t)$, input matrix $\mathcal{B}(t)$, Lyapunov matrix function $\mathcal{P}(t)$ and controller $\mathcal{K}(t)$ are assumed to share the same time-varying coefficients $\sin(r\omega t)$, $\cos(r\omega t)$, which are exactly parameterized before the controller design. However, in practice we may encounter the cases that such time-varying coefficients are not exactly known, while a bounding interval may be available. Thus, it is desirable to find a straightforward approach for periodic time-varying controller design, which allows using non-identical time-varying structures in the system and controller.

To construct a Lyapunov function in the form of (5) with a continuous symmetric matrix function $\mathcal{P}(t)>0$, we use the Dini derivative of $\mathcal{P}(t)$ to generalize the case when $\mathcal{P}(t)$ is not differentiable at $t\geq 0$. Considering a Dini-differentiable continuous periodic symmetric matrix function with a prescribed fundamental period $T_p>0$, that is, $\mathcal{P}(t)=\mathcal{P}(t+T_p)>0$, $\forall t\geq 0$, the upper right Dini derivative of $\mathcal{P}(t)$ is defined as

$$\mathcal{D}^+\mathcal{P}(t)=\limsup_{h\to 0^+}\frac{\mathcal{P}(t+h)-\mathcal{P}(t)}{h}. \tag{7}$$

When $\mathcal{P}(t)$ is differentiable for all $t\geq 0$, the derivative of $\mathcal{P}(t)$ is denoted as $\dot{\mathcal{P}}(t)$.

Definition 1 (*Definition of GUES*)**.** Periodic time-varying system (1) with $u(t)=0$ is said to be globally uniformly exponentially stable (GUES) if there exist two constants $\varphi>0$ and $\chi^*>0$, such that the solution of the system from any $x(0)\in\mathbb{R}^n_x$ satisfies $\|x(t)\|\leq\varphi e^{-\chi^* t}\|x(0)\|$, $\forall t>0$.

Definition 2 (*Definition of GUAS Zhou, 2016*)**.** A scalar system $\dot{y}(t)=\mu(t)y(t)$ is said to be globally uniformly asymptotically stable (GUAS) if there exists a function $\zeta\in\mathcal{KL}$ (Hespanha, 2004) such that $\|y(t)\|\leq\zeta(\|y(0)\|,t)$ for any $y(0)\in\mathbb{R}$.

Definition 3 (*Definition of USF Zhou, 2016*)**.** A piecewise continuous function $\mu(t)$ is said to be a uniformly stable function (USF) if $\dot{y}(t)=\mu(t)y(t)$ is GUAS.

We also revisit four useful lemmas on time-varying systems and matrix polynomials:

Lemma 1 (*Necessary and Sufficient Condition for USF Zhou, 2016*)**.** *A piecewise continuous scalar function $\mu(t)$ is a USF if and only if for two given constants $c_1>0$ and $c_2>0$, there exists a constant $T>0$ (maybe dependent on c_1) such that the following inequalities hold for any $t>0$:*

$$\int_t^{t+T}\mu(s)ds\leq -c_1, \tag{8}$$

$$\int_t^{t+\theta}\mu(s)ds\leq c_2,\ \forall\theta\in[0,T]. \tag{9}$$

Lemma 2 (*Sufficient Condition for GUES Time-varying System Zhou, 2016*)**.** *A time-varying system $\dot{x}(t)=\mathcal{A}(t)x(t)$ is GUES if there exist a Dini-differentiable continuous symmetric matrix function $\mathcal{P}(t)$, a USF $\mu(t)$, and constants $r_2\geq r_1>0$ such that*

$$\mathcal{A}^T(t)\mathcal{P}(t)+\mathcal{P}(t)\mathcal{A}(t)+\mathcal{D}^+\mathcal{P}(t)\leq\mu(t)\mathcal{P}(t), \tag{10}$$

$$r_1 I_{n_x}\leq\mathcal{P}(t)\leq r_2 I_{n_x}. \tag{11}$$

Remark 3. Lemma 2 is adapted from Zhou (2016) but we replace the ordinary derivative of $\mathcal{P}(t)$ by its upper right Dini derivative, generalizing the case when $\mathcal{P}(t)$ is composed of piecewise differentiable symmetric matrix functions.

Lemma 3 (*Negativity/Positivity Property for a Class of Matrix Polynomials Li et al., 2019; Xie et al., 2022*)**.** *Consider a symmetric matrix polynomial function $f:[0,1]^n\to\mathbb{R}^{d\times d}$, $n,d\in\mathbb{N}^+$, defined as*

$$f(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)=\Lambda_0+\varepsilon_1\Lambda_1+\varepsilon_1\varepsilon_2\Lambda_2+\cdots+\left(\prod_{k=1}^{n}\varepsilon_k\right)\Lambda_n, \tag{12}$$

with scalars $\varepsilon_k\in[0,1]$ and symmetric matrices $\Lambda_k\in\mathbb{R}^{d\times d}$, $k\in\overline{1,n}$. Matrix polynomial $f(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)<0$ (resp., >0) ***if and only if*** *the following inequalities hold:*

$$\sum_{\nu=0}^{k}\Lambda_\nu<0\ (\text{resp.}, >0),\quad k=0,1,\ldots,n. \tag{13}$$

Lemma 4 (*Negativity/Positivity Property for Another Class of Matrix Polynomials Xie et al., 2020*)**.** *Let $p:[0,1]^2\to\mathbb{R}^{d\times d}$ be a matrix polynomial function defined as*

$$p(\varepsilon_1,\varepsilon_2)=\Theta_0+\varepsilon_1\Theta_{1,1}+\varepsilon_2\Theta_{1,2}+\varepsilon_1\varepsilon_2\Theta_{2,1} \tag{14}$$

where $\varepsilon_1\in[0,1]$, $\varepsilon_2\in[0,1]$, and symmetric matrices $\Theta_0,\Theta_{1,1},\Theta_{1,2},\Theta_{2,1}\in\mathbb{R}^{d\times d}$. Symmetric matrix polynomial $p(\varepsilon_1,\varepsilon_2)<0$ (resp., >0) ***if and only if***

$$\Theta_0<0\ (\text{resp.}, >0), \tag{15}$$

$$\Theta_0+\Theta_{1,1}<0\ (\text{resp.}, >0), \tag{16}$$

$$\Theta_0+\Theta_{1,2}<0\ (\text{resp.}, >0), \tag{17}$$

$$\Theta_0+\Theta_{1,1}+\Theta_{1,2}+\Theta_{2,1}<0\ (\text{resp.}, >0). \tag{18}$$

From Lemma 3, we notice that matrix polynomial (12) can be written as:

$$f(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)=\Lambda_0+\varepsilon_1(\Lambda_1+\varepsilon_2(\Lambda_2+\cdots+\varepsilon_{n-1}(\Lambda_{n-1}+\varepsilon_n\Lambda_n))). \tag{19}$$

If we consider scalars $\varepsilon_k, k\in\overline{1,n}, n\in\mathbb{N}^+$, as variables that may not be identical but bounded in $[0,1]$, it can be found that the non-homogeneous symmetric matrix polynomials in (12) and (14) are multi-convex (or multi-affine) (Shen, Diamond, Udell, Gu, & Boyd, 2017). For $n=2$, matrix polynomial (12) becomes

$$f(\varepsilon_1,\varepsilon_2)=\Lambda_0+\varepsilon_1\Lambda_1+\varepsilon_1\varepsilon_2\Lambda_2, \tag{20}$$

with $\varepsilon_1\in[0,1]$, $\varepsilon_2\in[0,1]$. Comparing (20) with (14) by letting $\Lambda_0=\Theta_0$, $\Lambda_k=\Theta_{k,1}$, $k=1,2$, it can be seen that (14) not only covers (20), but also provides one more term $\varepsilon_2\Theta_{1,2}$. However, Lemma 4 only takes account of the case $n=2$. In the following section, the result in Lemma 4 will be extended to $\forall n\in\mathbb{N}^+$.

3. Main results

3.1. Polynomial blossoming approach

We first generalize the negativity/positivity property considered in Lemma 4 to a symmetric matrix polynomial blossoming form with $n \in \mathbb{N}^+$, based on which we can obtain some useful results for tackling matrix polynomials.

By extending $p(\varepsilon_1, \varepsilon_2)$ in (14) to the case of $n > 2$, we have a matrix polynomial in a form obtained by the product of n binomials inspired by polynomial blossoming based on a number of scalars $\varepsilon_k \in [0, 1]$ and symmetric matrices $\Theta_{k,q} \in \mathbb{R}^{d\times d}$, $k \in \overline{1, n}$, $q = 1, 2, \ldots, \binom{n}{k}$:

$$\begin{aligned} &p(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n) \\ &= \Theta_0 + \varepsilon_1\Theta_{1,1} + \varepsilon_2\Theta_{1,2} + \varepsilon_3\Theta_{1,3} + \cdots + \varepsilon_n\Theta_{1,\binom{n}{1}} \\ &\quad + \varepsilon_1\varepsilon_2\Theta_{2,1} + \varepsilon_1\varepsilon_3\Theta_{2,2} + \cdots + \varepsilon_1\varepsilon_n\Theta_{2,n-1} \\ &\quad + \varepsilon_2\varepsilon_3\Theta_{2,n} + \cdots + \varepsilon_{n-1}\varepsilon_n\Theta_{2,\binom{n}{2}} \\ &\quad + \varepsilon_1\varepsilon_2\varepsilon_3\Theta_{3,1} + \cdots + \varepsilon_{n-2}\varepsilon_{n-1}\varepsilon_n\Theta_{3,\binom{n}{3}} \\ &\quad + \cdots + \varepsilon_1\varepsilon_2\cdots\varepsilon_n\Theta_{n,\binom{n}{n}} \\ &= \Theta_0 + \sum_{k=1}^{n}\sum_{q=1}^{\binom{n}{k}}\Big(\prod_{\substack{\nu\in\mathcal{S}_k\\ \mathcal{S}_k\subseteq\overline{1,n}\\ |\mathcal{S}_k|=k}}\varepsilon_\nu\Big)\Theta_{k,q} \end{aligned} \tag{21}$$

where $\mathcal{S}_k$ denotes an integer set of the combinations obtained by selecting k distinct numbers from $1, 2, \ldots, n$. Matrix polynomial (21) can be regarded as a blossoming form varied by symmetric matrices $\Theta_{k,q}$. Here we obtain a generalized lemma concerning the negativity/positivity property for matrix polynomial (21).

Lemma 5 (*Negativity/Positivity Property Generalized by Polynomial Blossoming*). *Consider a non-homogeneous symmetric matrix polynomial function $p : [0, 1]^n \to \mathbb{R}^{d\times d}$ defined in (21), with scalars $\varepsilon_k \in [0, 1]$, $k \in \overline{1, n}$, and symmetric matrices $\Theta_{k,q} \in \mathbb{R}^{d\times d}$, $k \in \overline{1, n}$, $q = 1, 2, \ldots, \binom{n}{k}$, $n, d \in \mathbb{N}^+$. Symmetric matrix polynomial function $p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) < 0$ (resp., > 0)* ***if and only if*** *the following inequalities hold:*

$$\Theta_0 < 0 \ (resp., > 0), \tag{22}$$

$$\Theta_0 + \sum_{\nu=1}^{k}\sum_{\substack{q\in\mathcal{S}_\nu\\ \mathcal{S}_\nu\subseteq\overline{1,n}\\ |\mathcal{S}_\nu|=\binom{k}{\nu}}}\Theta_{\nu,q} < 0 \ (resp., > 0),\ k = 1, 2, \ldots, n. \tag{23}$$

Proof. For $n \in \mathbb{N}^+$, from (22) and (23) we can derive 2^n symmetric matrix inequalities. The necessary and sufficient condition is proved as follows.

Necessity: With $\varepsilon_k \in [0, 1]$, $k \in \overline{1, n}$, the necessity can be proved by substituting the endpoint values 0,1 of ε_k into (21). It can be seen that for $k = 1, 2, \ldots, n$, the endpoint values of ε_k provide a n-digit binary number combination, corresponding to the 2^n matrix inequalities. The same principle with further details can be found in the proof of Lemma 1 and Remark 3 in Xie et al. (2022).

Sufficiency: When $n \le 2$, the sufficiency has been proved by Lemma 4 (Xie et al., 2020). When $n = 3$, we can transform $p(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ into the following convex combination:

$$\begin{aligned} p(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \Theta_0 + \varepsilon_1\Theta_{1,1} + \varepsilon_2\Theta_{1,2} + \varepsilon_3\Theta_{1,3} \\ &\quad + \varepsilon_1\varepsilon_2\Theta_{2,1} + \varepsilon_1\varepsilon_3\Theta_{2,2} + \varepsilon_2\varepsilon_3\Theta_{2,3} \\ &\quad + \varepsilon_1\varepsilon_2\varepsilon_3\Theta_{3,1} \\ &= (1 - \varepsilon_3)\Delta_3^a(\varepsilon_1, \varepsilon_2) + \varepsilon_3\Delta_3^b(\varepsilon_1, \varepsilon_2), \end{aligned} \tag{24}$$

where $\Delta_3^a(\varepsilon_1, \varepsilon_2) \triangleq \Theta_0 + \varepsilon_1\Theta_{1,1} + \varepsilon_2\Theta_{1,2} + \varepsilon_1\varepsilon_2\Theta_{2,1}$ and

$$\begin{aligned} \Delta_3^b(\varepsilon_1, \varepsilon_2) &\triangleq (\Theta_0 + \Theta_{1,3}) + \varepsilon_1(\Theta_{1,1} + \Theta_{2,2}) \\ &\quad + \varepsilon_2(\Theta_{1,2} + \Theta_{2,3}) + \varepsilon_1\varepsilon_2(\Theta_{2,1} + \Theta_{3,1}). \end{aligned}$$

By (15)–(18) and $\varepsilon_k \in [0, 1]$, $k = 1, 2, 3$, we have

$$\Delta_3^a(\varepsilon_1, \varepsilon_2) = \Theta_0 + \varepsilon_1\Theta_{1,1} + \varepsilon_2\Theta_{1,2} + \varepsilon_1\varepsilon_2\Theta_{2,1} < 0,$$

and

$$\begin{aligned} \Delta_3^b(\varepsilon_1, \varepsilon_2) &= (\Theta_0 + \Theta_{1,3}) + \varepsilon_1(\Theta_{1,1} + \Theta_{2,2}) \\ &\quad + \varepsilon_2(\Theta_{1,2} + \Theta_{2,3}) + \varepsilon_1\varepsilon_2(\Theta_{2,1} + \Theta_{3,1}) < 0 \end{aligned}$$

$$\Leftarrow \begin{cases} \Theta_0 + \Theta_{1,3} < 0 \\ \Theta_0 + \Theta_{1,1} + \Theta_{1,3} + \Theta_{2,2} < 0 \\ \Theta_0 + \Theta_{1,2} + \Theta_{1,3} + \Theta_{2,3} < 0 \\ \Theta_0 + \sum_{j=1}^{3}\Theta_{1,j} + \sum_{j=1}^{3}\Theta_{2,j} + \Theta_{3,1} < 0 \end{cases} \tag{25}$$

based on the matrix inequalities derived from (22) and (23). Then we have $p(\varepsilon_1, \varepsilon_2, \varepsilon_3) < 0$ for $n = 3$. For $n \ge 3$, by repeating the similar procedures we can always obtain a convex combination in form of

$$\begin{aligned} p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) &\triangleq (1 - \varepsilon_n)\Delta_n^a(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) \\ &\quad + \varepsilon_n\Delta_n^b(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}), \end{aligned} \tag{26}$$

where $\Delta_n^a(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) < 0$ and $\Delta_n^b(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) < 0$ can be recursively proved by $p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) < 0$ and the 2^n matrix inequalities, similar to the case of $n = 3$. Hence, with $\varepsilon_k \in [0, 1]$, $k \in \overline{1, n}$, we have $p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) < 0$.

In addition, it is clear that $p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) > 0$ if and only if the matrix inequalities in (22) and (23) hold, with "<0" replaced by ">0". Therefore, both the necessity and sufficiency are proved. □

Note that conditions (22) and (23) in Lemma 5 provide a general formulation inspired by polynomial blossoming. To facilitate the understanding, we compare the inequality constraints to guarantee $p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) < 0$ under some different cases in Table 1, taking $n = 2$ and $n = 3$ for examples. It can be seen that Lemma 4 (Xie et al., 2020) is actually the $n = 2$ case of the proposed polynomial blossoming approach.

By Lemma 5, the resulting constraints for $n = 3$ can be obtained based on (24)–(25), with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in [0, 1]$ and symmetric matrices $\Theta_{k,q}$, $k \in \overline{1, 3}$, $q = 1, 2, \ldots, \binom{3}{k}$. Symmetric matrix polynomial $p(\varepsilon_1, \varepsilon_2, \varepsilon_3) < 0$ **if and only if** the inequality conditions listed in **Polynomial Blossoming Case,** Table 1 ($n = 3$) hold. Respectively, the same principle goes with $p(\varepsilon_1, \varepsilon_2, \varepsilon_3) > 0$ by changing "<0" to ">0" in the relevant inequalities.

3.2. Discussions on special cases

Based on Lemma 5, a special case arises when $\varepsilon_k \triangleq \epsilon \in [0, 1]$, $k \in \overline{1, n}$, and $p(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ will become a symmetric matrix polynomial $p(\epsilon)$, which can be expanded based on the Bernstein basis (Rokne, 1979):

$$\begin{aligned} p(\epsilon) &= \Theta_0 + \epsilon\sum_{q=1}^{\binom{n}{1}}\Theta_{1,q} + \epsilon^2\sum_{q=1}^{\binom{n}{2}}\Theta_{2,q} + \cdots + \epsilon^n\Theta_{n,1} \\ &= \sum_{k=0}^{n}\binom{n}{k}\epsilon^k(1-\epsilon)^{n-k}\Xi_k \end{aligned} \tag{27}$$

where $\Xi_0 \triangleq \Theta_0$, and

$$\Xi_k \triangleq \sum_{\nu=1}^{k}\frac{\binom{k}{\nu}}{\binom{n}{\nu}}\sum_{q=1}^{\binom{n}{\nu}}\Theta_{\nu,q},\ k = 1, 2, \ldots, n. \tag{28}$$

Table 1
Inequality constraints under different cases to guarantee $p(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) < 0$ $(n = 2, 3)$.

Degree	**Polynomial Blossoming Case** (Lemma 5)	**Bernstein Polynomial Case** $\varepsilon_k \triangleq \epsilon \in [0,1],\ k \in \overline{1,n}$ (Lemma 6 adapted based on Xie et al. (2021))	**Special Case** $\Theta_{k,q} = 0,\ k \in \overline{1,n},\ q = 2,3,\dots,\binom{n}{k}$ (Lemma 3 Li et al., 2019; Xie et al., 2022)
$n = 2$	$\Theta_0 < 0$ $\Theta_0 + \Theta_{1,1} < 0$ $\Theta_0 + \Theta_{1,2} < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{1,2} + \Theta_{2,1} < 0$	$\Theta_0 < 0$ $\Theta_0 + \frac{1}{2}(\Theta_{1,1} + \Theta_{1,2}) < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{1,2} + \Theta_{2,1} < 0$	$\Theta_0 < 0$ $\Theta_0 + \Theta_{1,1} < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{2,1} < 0$
$n = 3$	$\Theta_0 < 0$ $\Theta_0 + \Theta_{1,1} < 0$ $\Theta_0 + \Theta_{1,2} < 0$ $\Theta_0 + \Theta_{1,3} < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{1,2} + \Theta_{2,1} < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{1,3} + \Theta_{2,2} < 0$ $\Theta_0 + \Theta_{1,2} + \Theta_{1,3} + \Theta_{2,3} < 0$ $\Theta_0 + \sum_{q=1}^{3} \Theta_{1,q} + \sum_{q=1}^{3} \Theta_{2,q} + \Theta_{3,1} < 0$	$\Theta_0 < 0$ $\Theta_0 + \frac{1}{3}\sum_{q=1}^{3} \Theta_{1,q} < 0$ $\Theta_0 + \frac{2}{3}\sum_{q=1}^{3} \Theta_{1,q} + \frac{1}{3}\sum_{q=1}^{3} \Theta_{2,q} < 0$ $\Theta_0 + \sum_{q=1}^{3} \Theta_{1,q} + \sum_{q=1}^{3} \Theta_{2,q} + \Theta_{3,1} < 0$	$\Theta_0 < 0$ $\Theta_0 + \Theta_{1,1} < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{2,1} < 0$ $\Theta_0 + \Theta_{1,1} + \Theta_{2,1} + \Theta_{3,1} < 0$

According to the property of Bernstein polynomials discussed in Xie et al. (2021), we have the following lemma for the Bernstein Polynomial Case which is a special case when $\varepsilon_k \triangleq \epsilon \in [0,1]$, $k \in \overline{1,n}$, of its blossoming form (21).

Lemma 6 (*Negativity/Positivity Property for the Bernstein Polynomial Case*)**.** *Consider an nth degree symmetric matrix polynomial function $p : [0,1] \to \mathbb{R}^{d\times d}$ defined in (27) with a scalar $\epsilon \in [0,1]$. Given symmetric matrices $\Theta_{k,q} \in \mathbb{R}^{d\times d}$, $k \in \overline{1,n}$, $q = 1,2,\dots,\binom{n}{k}$, $n, d \in \mathbb{N}^+$, symmetric matrix function $p(\epsilon) < 0$ (resp., >0) if (22) and the following matrix inequalities hold:*

$$\Theta_0 + \Xi_k < 0 \ (\text{resp.}, >0),\ \ k = 1,2,\dots,n, \tag{29}$$

where matrices Ξ_k, $k = 1,2,\dots,n$, satisfy (28).

Proof. In the Bernstein Polynomial case, inequalities (22) and (29) constitute the constraints for a sufficient condition of symmetric matrix polynomial $p(\epsilon) < 0$ (resp., >0). Let $\Theta_k \triangleq \sum_{q=1}^{\binom{n}{k}} \Theta_{k,q}$, $k = 1,2,\dots,n$. When (22), (28) and (29) hold, we have $\Theta_0 < 0$ (resp., >0), and for $k = 1,2,\dots,n$, it holds that

$$\Theta_0 + \Xi_k = \Theta_0 + \sum_{\nu=1}^{k} \frac{\binom{k}{\nu}}{\binom{n}{\nu}} \Theta_\nu < 0 \ (\text{resp.}, >0). \tag{30}$$

Using the property of Bernstein polynomial (Rokne, 1979) and Lemma 3 in Xie et al. (2021), it is immediate to obtain $p(\epsilon) < 0$ (resp., >0), which completes the proof. □

For comparison, we also display the inequality conditions based on Lemma 6 to guarantee $p(\epsilon) < 0$, $n = 2,3$, in the column of **Bernstein Polynomial Case,** Table 1.

Comparing Lemmas 5 and 3, it is clear that symmetric matrix polynomial (12) in Lemma 3 is another special case of matrix polynomial (21) in Lemma 5, with $\Theta_{k,q} = 0$, $k \in \overline{1,n}$, $q = 2,3,\dots,\binom{n}{k}$. Take $n = 3$ for example, to guarantee $p(\varepsilon_1, \varepsilon_2, \varepsilon_3) < 0$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in [0,1]$, the special case means that the symmetric matrices in Lemma 3 satisfy $\Lambda_0 = \Theta_0$, $\Lambda_1 = \Theta_{1,1}$, $\Lambda_2 = \Theta_{2,1}$, $\Theta_{1,2} = \Theta_{1,3} = \Theta_{2,2} = \Theta_{2,3} = 0$, and thus the inequality constraints will reduce to those listed in the column of **Special Case,** Table 1 ($n = 3$), which are corresponding to (13). Since both Lemmas 5 and 3 are necessary and sufficient conditions, it can be concluded that *Lemma* 5 *covers the result in Lemma* 3.

Remark 4. Note that the orders of ε_k, $k \in \overline{1,N}$, in symmetric matrix polynomial $p(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ do not exceed one, which easily preserves the multi-convexity of $p(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ (Shen et al., 2017). This property is different from that for another type of symmetric matrix polynomials given as $\rho : [0,1]^2 \to \mathbb{R}^{d\times d}$,

$$\rho(\varepsilon_1, \varepsilon_2) = \Delta_0 + \varepsilon_1\Delta_1 + \varepsilon_2\Delta_2 + \varepsilon_1^2\Delta_{11} + \varepsilon_2^2\Delta_{22} + \varepsilon_1\varepsilon_2\Delta_{12},$$

where $\varepsilon_1 \in [0,1]$, $\varepsilon_2 \in [0,1]$, and symmetric matrices Δ_0, Δ_1, Δ_2, Δ_{11}, Δ_{22}, $\Delta_{12} \in \mathbb{R}^{d\times d}$. The existence of $\varepsilon_1^2\Delta_{11}$ and $\varepsilon_2^2\Delta_{22}$ results in the following Hessian matrix:

$$\mathbf{H}_{\rho(\varepsilon_1,\varepsilon_2)} = \begin{bmatrix} \frac{\partial^2\rho}{\partial\varepsilon_1^2} & \frac{\partial^2\rho}{\partial\varepsilon_1\partial\varepsilon_2} \\ \frac{\partial^2\rho}{\partial\varepsilon_2\partial\varepsilon_1} & \frac{\partial^2\rho}{\partial\varepsilon_2^2} \end{bmatrix} = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12} & \Delta_{22} \end{bmatrix}, \tag{31}$$

which can preserve the multi-convexity of $\rho(\varepsilon_1, \varepsilon_2)$ if $\Delta_{11} \geq 0$, $\Delta_{22} \geq 0$ and $\Delta_{11}\Delta_{22} - \Delta_{12}^2 \geq 0$ to guarantee $\mathbf{H}_{\rho(\varepsilon_1,\varepsilon_2)} \geq 0$. Conditions on the negativity of $\rho(\varepsilon_1, \varepsilon_2)$ can be given by the parameterized LMI technique in Proposition 4.1 of Apkarian and Tuan (2000).

3.3. Tractable conditions for controller design

Regarding time-varying controller design, we revisit the segmentation approach that has been widely applied in studies on periodic piecewise time-varying systems (Xie et al., 2020) and switched systems (Allerhand & Shaked, 2011; Xiang, 2015). The core idea is partitioning a known time interval into M segments, using LTV matrix functions to establish piecewise Lyapunov function candidates. This approach has been found effective and easy-to-use in time-varying controller synthesis, but may be limited when there exist more than two time-varying coefficients over non-identical time intervals (Xie et al., 2020). To solve this problem, we employ the proposed Lemma 5 to tackle controller design and enable more flexibility.

Before the controller design, we give a sufficient condition for the stability of system (1) with formulation (3), using a periodic USF $\mu(t)$ based on Lemmas 1–2.

Theorem 1 (*GUES Criterion*)**.** *Consider periodic time-varying system (1) with formulation (3) and $u(t) = 0$. The system is GUES if there exist a scalar $c_1 > 0$, a periodic USF $\mu(t) = \mu(t + T_p)$ and a Dini-differentiable continuous periodic symmetric matrix function $\mathcal{P}(t) = \mathcal{P}(t + T_p) > 0$, such that the following conditions hold:*

$$\textbf{\textit{sym}}(\mathcal{P}(t)\mathcal{A}(t)) + \mathcal{D}^+\mathcal{P}(t) - \mu(t)\mathcal{P}(t) \leq 0, \tag{32}$$

$$\int_t^{t+T_p} \mu(s)ds \leq -c_1. \tag{33}$$

Proof. Consider Lyapunov function (5) with a Dini-differentiable continuous periodic symmetric matrix function $\mathcal{P}(t) = \mathcal{P}(t + T_p) > 0$. Since $\mathcal{P}(t) > 0$ and USF $\mu(t)$ are periodic and naturally bounded for $t \geq 0$, (11) can be satisfied. If (33) holds, there always exists a scalar $c_2 > 0$ such that $\int_t^{t+\theta} \mu(s)ds \leq c_2, \forall\theta \in [0, T_p]$. For $t \in [lT_p, (l+1)T_p)$, $l = 0, 1, \ldots$, we can always find a scalar $\varrho \leq T_p$ satisfying $t = lT_p + \varrho$, such that $-l \leq -\left(\frac{t}{T_p} - 1\right) = 1 - \frac{t}{T_p}$. When (32) holds, we have $\mathcal{D}^+V(t) \leq \mu(t)V(t)$, which indicates that

$$V(t) \leq e^{\int_{lT_p}^{t} \mu(s)ds + \int_{(l-1)T_p}^{lT_p} \mu(s)ds + \cdots + \int_0^{T_p} \mu(s)ds} V(0)$$
$$\leq e^{c_1 + c_2 - \frac{c_1}{T_p}t} V(0), \tag{34}$$

and thus we have $\|x(t)\| \leq \varphi e^{-\chi t}\|x(0)\|$, $\forall t \geq 0$, where $\varphi = e^{\frac{1}{2}(c_1+c_2)}\sqrt{\frac{\overline{\lambda}(\mathcal{P}(0))}{\underline{\lambda}(\mathcal{P}(0))}} > 0$, $\chi = \frac{c_1}{2T_p} > 0$. By Definition 1, the system is GUES. The proof is complete. □

Compared with Lemma 2, Theorem 1 specializes $\mu(t)$ as a periodic USF. Such an assumption not only reduces the need of conditions (9) and (11), but also facilitates a tractable stabilizing controller design that follows. To simplify the condition related to $\mu(t)$ in practice, we can directly use a periodic piecewise constant USF $\mu(t) = \mu(t + T_p) = \mu_m$, $t \in \mathcal{T}_m \triangleq [lT_p + (m-1)\delta, lT_p + m\delta)$, to improve the tractability in controller design.

Theorem 2 (*Tractable Criterion for Periodic Stabilizing Controller Design*)**.** *Consider periodic time-varying system (1) with formulation (3) and periodic controller (6). Given a periodic piecewise USF* $\mu(t) = \mu_m$, $t \in \mathcal{T}_m$, $m \in \overline{1, M}$, *the system is GUES if there exist a scalar* $c_1 > 0$, *symmetric matrices* $Q_m > 0$ *and* U_m, $m \in \overline{1, M}$, *such that the following conditions hold:*

$$\Omega_{0,m} < 0, \tag{35}$$

$$\Omega_{0,m} + \sum_{\nu=1}^{k} \sum_{\substack{q\in\mathcal{S}_\nu \\ \mathcal{S}_\nu\subseteq\overline{1,n} \\ |\mathcal{S}_\nu|=\binom{k}{\nu}}} \Omega_{\nu,q,m} < 0,$$
$$k = 1, 2, \ldots, N + S + 1, \tag{36}$$

$$\frac{T_p}{M}\sum_{m=1}^{M} \mu_m \leq -c_1, \tag{37}$$

where

$$\Omega_{0,m} = \textbf{sym}(A_0Q_m + B_0U_m) - \delta^{-1}\tilde{Q}_m - \mu_m Q_m,$$
$$\Omega_{1,i,m} = \textbf{sym}(A_iQ_m),\ i = 1, 2, \ldots, N,$$
$$\Omega_{1,N+j,m} = \textbf{sym}(B_jU_m),\ j = 1, 2, \ldots, S,$$
$$\Omega_{1,N+S+1,m} = \textbf{sym}(A_0\tilde{Q}_m + B_0\tilde{U}_m) - \mu_m\tilde{Q}_m,$$
$$\Omega_{2,i,m} = \textbf{sym}(A_i\tilde{Q}_m),\ i = 1, 2, \ldots, N,$$
$$\Omega_{2,N+j,m} = \textbf{sym}(B_i\tilde{U}_m),\ j = 1, 2, \ldots, S,$$
$$\Omega_{2,N+S+\vartheta,m} = 0_{n_x},\ \vartheta = 1, 2, \ldots, \binom{N+S}{2},$$
$$\Omega_{k,q,m} = 0_{n_x},\ k = 3, 4, \ldots, N + S + 1,$$
$$q = 1, 2, \ldots, \binom{N+S+1}{k},$$

and $\delta \triangleq \frac{T_p}{M}$. *Over each period, the periodic time-varying controller gains are calculated by*

$$\mathcal{K}(t) = \mathcal{U}(t)\mathcal{Q}^{-1}(t),\ t \in [lT_p, (l+1)T_p), \tag{38}$$

where for $t \in \mathcal{T}_m$, $m \in \overline{1, M}$, *time-varying matrix functions* $\mathcal{Q}(t)$ *and* $\mathcal{U}(t)$ *are obtained by*

$$\mathcal{Q}(t) = Q_m + \sigma(t)\tilde{Q}_m, \tag{39}$$

$$\mathcal{U}(t) = U_m + \sigma(t)\tilde{U}_m, \tag{40}$$

and $\tilde{Q}_m \triangleq Q_{m+1} - Q_m$, $Q_{M+1} = Q_1$, $\tilde{U}_m \triangleq U_{m+1} - U_m$, $\sigma(t) = \frac{(t - lT_p - (m-1)\delta)}{\delta} \in [0, 1)$, $m \in \overline{1, M}$.

Proof. Combining the system formulation in Section 2 with (39), (40) and $\delta = T_p/M$, it follows that $\alpha_i(t), \beta_j(t) \in [0, 1]$, $i \in \overline{1, N}$, $j \in \overline{1, S}$, and $\sigma(t) \in [0, 1) \subset [0, 1]$. For $t \in \mathcal{T}_m$, $m \in \overline{1, M}$, the upper right Dini derivative of $\mathcal{Q}(t)$ is given by

$$\mathcal{D}^+\mathcal{Q}(t) = \frac{Q_{m+1} - Q_m}{\delta} = \delta^{-1}\tilde{Q}_m. \tag{41}$$

where $Q_{M+1} = Q_1$. When conditions (35) and (36) hold, by the polynomial blossoming-inspired Lemma 5, for $t \in \mathcal{T}_m$, we can guarantee the negativity of the following symmetric matrix polynomial:

$$\begin{aligned}
&\textbf{sym}(\mathcal{A}(t)\mathcal{Q}(t) + \mathcal{B}(t)\mathcal{U}(t)) - \mathcal{D}^+\mathcal{Q}(t) - \mu(t)\mathcal{Q}(t) \\
&= \Omega_{0,m} + \sum_{i=1}^{N}\alpha_i(t)\Omega_{1,i,m} + \sum_{j=1}^{S}\beta_j(t)\Omega_{1,N+j,m} \\
&\quad + \sigma(t)\Omega_{1,N+S+1,m} + \sigma(t)\sum_{i=1}^{N}\alpha_i(t)\Omega_{2,i,m} \\
&\quad + \sigma(t)\sum_{j=1}^{S}\beta_j(t)\Omega_{2,N+j,m} + \sum_{\vartheta=1}^{\binom{N+S}{2}}\Upsilon(\Phi, 2, \vartheta)\Omega_{2,N+S+\vartheta,m} \\
&\quad + \sum_{k=3}^{N+S+1}\sum_{q=1}^{\binom{N+S+1}{k}}\Upsilon(\Psi, k, q)\Omega_{k,q,m} \\
&< 0,
\end{aligned} \tag{42}$$

with two sets consisting of the [0, 1]-bounded time-varying coefficients given as

$$\Phi \triangleq \left\{\alpha_i(t), \beta_j(t), \mid i \in \overline{1, N}, j \in \overline{1, S}\right\}, \tag{43}$$
$$\Psi \triangleq \left\{\alpha_i(t), \beta_j(t), \sigma(t) \mid i \in \overline{1, N}, j \in \overline{1, S}\right\}. \tag{44}$$

Since continuous matrix function $\mathcal{Q}(t) = \mathcal{Q}(t + T_p) > 0$, for any $t \geq 0$, we have $\mathcal{Q}^{-1}(t) = \mathcal{Q}^{-1}(t + T_p) > 0$, and

$$\min_t \overline{\lambda}(\mathcal{Q}(t))I_{n_x} \leq \mathcal{Q}^{-1}(t) \leq \max_t \underline{\lambda}(\mathcal{Q}(t))I_{n_x}. \tag{45}$$

Next, multiplying both sides of (42) by $\mathcal{Q}^{-1}(t)$, based on (38) and the fact $\mathcal{D}^+\mathcal{Q}^{-1}(t) = -\mathcal{Q}^{-1}(t)\mathcal{D}^+\mathcal{Q}(t)\mathcal{Q}^{-1}(t)$ it follows that for $t \in \mathcal{T}_m$,

$$\textbf{sym}(\mathcal{Q}^{-1}(t)\mathcal{A}_c(t)) + \mathcal{D}^+\mathcal{Q}^{-1}(t) - \mu(t)\mathcal{Q}^{-1}(t) < 0, \tag{46}$$

where $\mathcal{A}_c(t) \triangleq \mathcal{A}(t) + \mathcal{B}(t)\mathcal{K}(t)$. Similar to (5), we construct a Lyapunov function:

$$V(t) = x^T(t)\mathcal{Q}^{-1}(t)x(t),\ t \geq 0. \tag{47}$$

It follows that $\mathcal{D}^+V(t) < \mu(t)V(t) = \mu_mV(t)$, $t \in \mathcal{T}_m$.

Based on the proof of Theorem 1, inequality (46) corresponds to condition (32). Moreover, when (37) holds, the condition of $\mu(t)$ in (33) is satisfied. Thus, periodic time-varying system (1) with formulation (3) and periodic controller (6) is GUES. The time-varying controller gains can be calculated by (38)–(40). □

Remark 5. The periodic time-varying controller gains $\mathcal{K}(t)$ in Theorem 2 may be discontinuous at the switching instants upon the end of each period, that is, $t = lT_p$, $l = 1, 2, \ldots$. According to previous studies (Li et al., 2019; Xie et al., 2022), $\mathcal{K}(t)$ can be imposed to be continuous at all the switching instants if we suppose $\mathcal{U}(t)$ as a continuous function by letting $U_{M+1} = U_1$. The

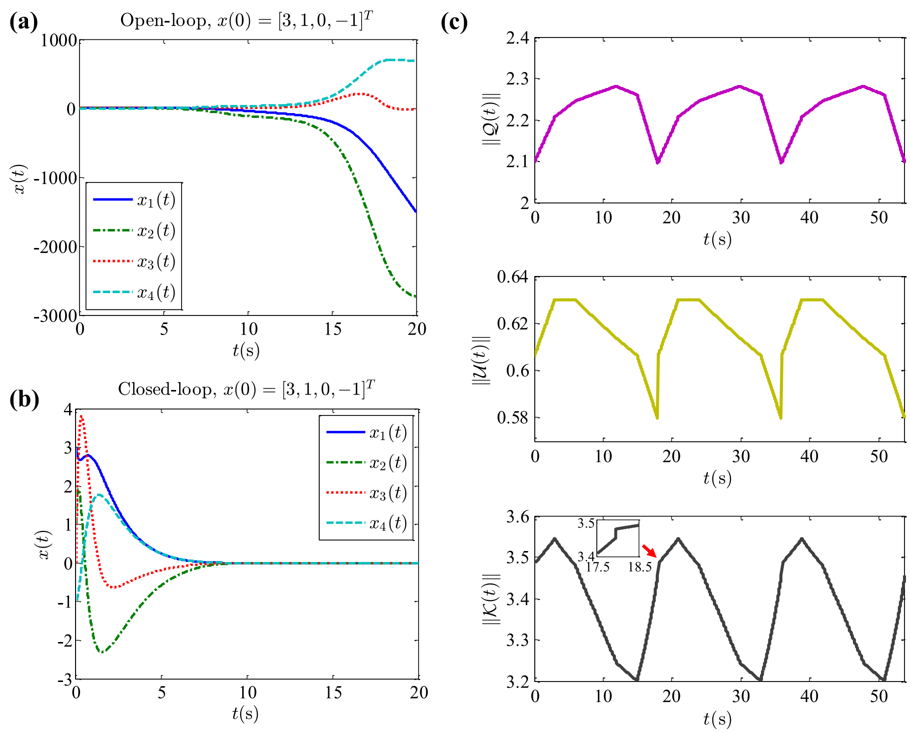

Fig. 1. Illustrative results of *Example 1*: (a) Open-loop system state trajectory; (b) Closed-loop system state trajectory; (c) Variations of $\|\mathcal{Q}(t)\|$, $\|\mathcal{U}(t)\|$ and $\|\mathcal{K}(t)\|$ demonstrated over 3 periods (discontinuous controller gains).

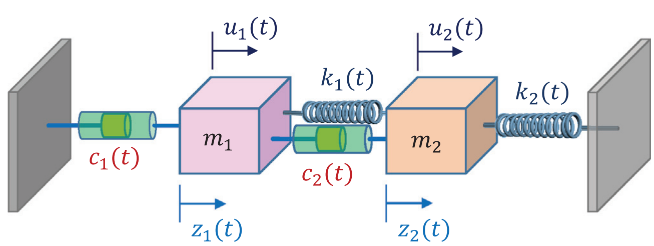

Fig. 2. Equivalent mass–spring–damper system with time-periodic coefficients.

total number of decision variables in Theorem 2 is $Mn_x(\frac{n_x+1}{2} + n_u) + n_x n_u$, which will become $Mn_x(\frac{n_x+1}{2} + n_u)$ when $\mathcal{U}(t)$ is continuous at the end of each period. When M increases, more matrix variables will be introduced to improve the feasibility. Note that in this study, we do not need a very large M. We just need to choose an appropriate M to ensure the feasibility, in order to keep the computational complexity from getting too high.

4. Simulation verification

4.1. Example 1

Consider a helicopter model adapted from Hooshmandi, Bayat, Jahed-Motlagh, and Jalali (2020) and Narendra and Tripathi (1973), with time-varying parameters due to air speed variation:

$$\dot{x}(t) = \begin{bmatrix} -0.036 & 0.0271 & 0.018 & -0.455 \\ 0.048 & -1.01 & 0.002 & -4.02 \\ 0.1 & a_{32}(t) & -0.707 & a_{34}(t) \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.4422 & b_{21}(t) & -5.52 & 0 \\ 0.1761 & -7.59 & 4.99 & 0 \end{bmatrix}^T u(t), \tag{48}$$

where state variable $x(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]^T$ is composed of horizontal velocity, vertical velocity, pitch rate and pitch angle; control input $u(t) = [u_1(t), u_2(t)]^T$ consists of collective pitch control and longitudinal cyclic, respectively; parameters $a_{32}(t)$, $a_{34}(t)$ and $b_{21}(t)$ are periodic time-varying as follows:

$$\begin{aligned} a_{32}(t) &= 0.066 + |0.42\sin(0.35t)|, \\ a_{34}(t) &= 0.11 + |2.25\sin(0.35t)|, \\ b_{21}(t) &= 0.97 + |4.1\sin(0.35t)|. \end{aligned} \tag{49}$$

Note that (49) is just used for simulation, and we do not need the exact periodic time-varying coefficient $|\sin(0.35t)|$ in controller design or assume the same coefficients in the controller gains. The system and control input matrices in (48) are formulated as $\mathcal{A}(t) = A_0 + \alpha_1(t)A_1$, $\mathcal{B}(t) = B_0 + \beta_1(t)B_1$, where $\alpha_1(t) = \beta_1(t) = |\sin(0.35t)| \in [0, 1]$.

We use Theorem 2 to solve the stabilizing controller gains with $\sigma(t)$, $M = 6$, and μ_m sampled from a USF $-0.5\cos(0.35t)$, $m \in \overline{1, M}$. The results of open-loop/closed-loop system state trajectories, and variations of $\|\mathcal{Q}(t)\|$, $\|\mathcal{U}(t)\|$ and $\|\mathcal{K}(t)\|$ over 3 periods are shown in Fig. 1(a)–(c), which demonstrate the stabilizing control effects under $\mathcal{K}(t)$ with controller gains discontinuous at the end of each fundamental period.

4.2. Example 2

Moreover, we consider an equivalent mass–spring–damper system model that involves time-periodic spring stiffness and damping coefficients, as shown in Fig. 2. The system comprises of two masses m_1, m_2, two spring elements with stiffness coefficients $k_1(t), k_2(t)$, two damping elements with coefficients $c_1(t), c_2(t)$, and two external force inputs $u_1(t), u_2(t)$; $z_1(t)$ and $z_2(t)$ are the displacements of m_1 and m_2, respectively. Let $z(t) = [z_1(t), z_2(t)]^T$, $u(t) = [u_1(t), u_2(t)]^T$, for $t \geq 0$ we have the

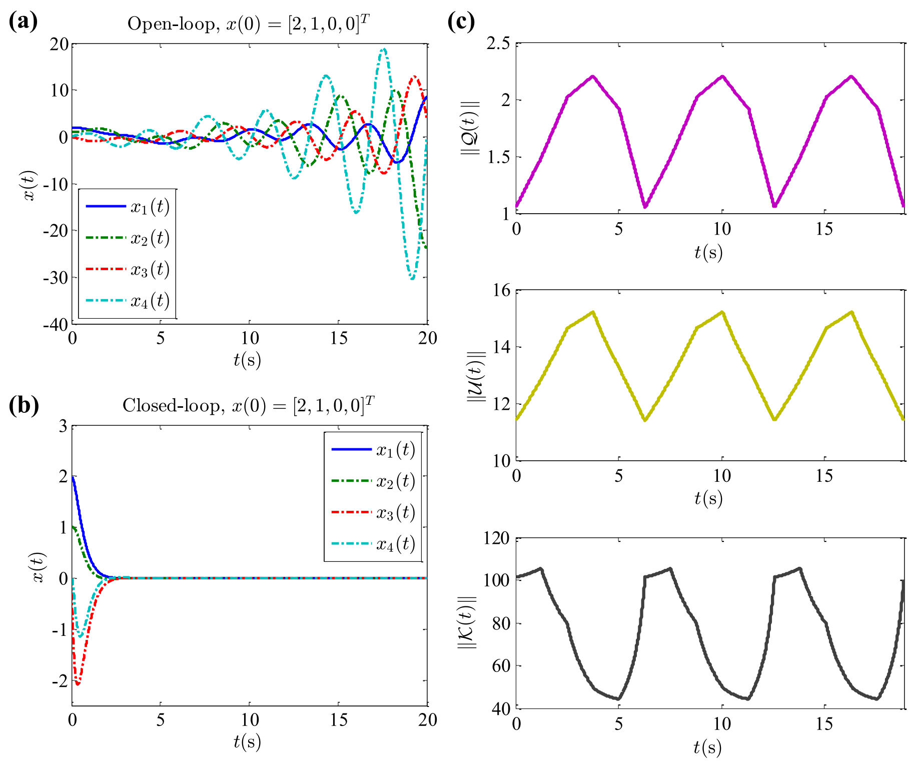

Fig. 3. Illustrative results of Example 2: (a) Open-loop system state trajectory; (b) Closed-loop system state trajectory; (c) Variations of $\|\mathcal{Q}(t)\|$, $\|\mathcal{U}(t)\|$ and $\|\mathcal{K}(t)\|$ demonstrated over 3 periods (continuous controller gains).

following dynamic system:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{z}(t) + \begin{bmatrix} c_1(t)+c_2(t) & -c_2(t) \\ -c_2(t) & c_2(t) \end{bmatrix} \dot{z}(t) + \begin{bmatrix} k_1(t) & -k_1(t) \\ -k_1(t) & k_1(t)+k_2(t) \end{bmatrix} z(t) = u(t), \tag{50}$$

Let $x(t) = [z_1(t), z_2(t), \dot{z}_1(t), \dot{z}_2(t)]^T$, (50) is rewritten as $\dot{x}(t) = \mathcal{A}(t)x(t) + B_0 u(t)$, where

$$\mathcal{A}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1(t)}{m_1} & \frac{k_1(t)}{m_1} & -\frac{c_1(t)+c_2(t)}{m_1} & \frac{c_2(t)}{m_1} \\ \frac{k_1(t)}{m_2} & -\frac{k_1(t)+k_2(t)}{m_2} & \frac{c_2(t)}{m_2} & -\frac{c_2(t)}{m_2} \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \end{bmatrix}^T.$$

For this system, we assume that the approximation in form of (3) is available, and set the parameter values as listed in Table 2 for simulation purpose. Thus, $\mathcal{A}(t)$ can be formulated by $\mathcal{A}(t) = A_0 + \alpha_1(t)A_1 + \alpha_2(t)A_2$, where

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1.25 & 1.25 & -0.1875 & 0 \\ 2.5 & -3.875 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \\ -1 & 1.75 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.625 & -0.25 \\ 0 & 0 & -0.5 & 0.5 \end{bmatrix},$$

Table 2
Parameter values.

Parameter	Value	Unit
m_1	8	kg
m_2	4	kg
$k_1(t)$	$8 - 2\sin(t)$	N/m
$k_2(t)$	$4 - 1.5\sin(t)$	N/m
$c_1(t)$	$-1.5\cos(t)$	N s/m
$c_2(t)$	$-1 - \cos(t)$	N s/m

and $\alpha_1(t) = 0.5(\sin(t) + 1) \in [0, 1]$, $\alpha_2(t) = 0.5(\cos(t) + 1) \in [0, 1]$, $T_p = 2\pi$ s. Let $M = 5$ and μ_m sampled from a USF $-0.5 - \sin(t)$, we can stabilize the open-loop unstable system by Theorem 2. Using $\sigma(t)$ for controller design, there are three [0, 1]-bounded time-varying coefficients in total. The results of open-loop/closed-loop system state trajectories, and variations of $\|\mathcal{Q}(t)\|$, $\|\mathcal{U}(t)\|$ and $\|\mathcal{K}(t)\|$ demonstrated over 3 periods are shown in Fig. 3(a)–(c). In this example, we use continuous controller gains to satisfy the practical requirement in smoothness of state variations.

5. Conclusions

A polynomial blossoming approach to stabilizing a class of periodic time-varying systems is developed in this paper. Based on the multi-convexity of symmetric matrix polynomial blossoming (or polar) form, the proposed approach can transfer optimization constraints with [0, 1]-bounded time-varying coefficients over non-identical time intervals into LMI conditions. It also generalizes the existing matrix polynomial approaches in Li et al. (2019) and Xie et al. (2021), and therefore provides an alternative approach capable of tackling systems with complex time-varying structures. We have applied the proposed approach to the stabilizing controller design for a class of periodic time-varying systems. The obtained controller involves periodic time-varying gains either continuous or discontinuous at switching instants (ends of period intervals), and can guarantee the globally uniformly exponentially stability of the closed-loop system. Benefiting from the polynomial blossoming approach, we do not need to use the same time-varying coefficients in the controller as in the system model, enabling less requirement in modelling accuracy. The proposed approach is validated via two simulation examples. Our future work will be devoted to improving the performance of polynomial-based controller synthesis while reducing the computation complexity.

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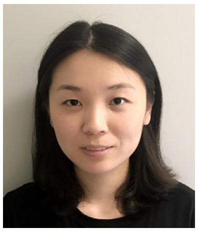

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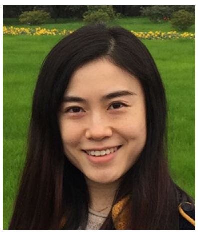

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