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Proportional-derivative controller design of continuous-time positive linear systems

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Abstract

This article focuses on the design of proportional-derivative (PD) controllers for positive linear systems in the continuous-time domain, which is a well-known open problem in positive systems theory. The main objective is to design a PD controller subject to interval gain variations, which simultaneously preserves the closed-loop system stability and positivity. The provided results fill the literature gap by considering both the PD controller structure and positivity. Although various feedback control strategies of positive systems have been developed recently, the work provides, for the first time, a systematic and tractable framework for finding nonfragile PD controller gains for positive stabilization. Finally, the theory and algorithm performance are evaluated and illustrated by numerical examples.

KEYWORDS

nonfragile control, PD controller, positive linear systems, static output-feedback control

1 | INTRODUCTION

1.1 | Background

Generally speaking, a positive system is a special dynamic system whose state and output variables have to be positive, or at least non-negative, throughout its entire evolutionary horizon. The research on positive systems can be traced back to Luenberger who, for the first time, introduced the concept of such class of systems in a fundamental book.[1] Since then, positive systems theory has seen a lot of applications in many industrial problems, such as biochemical engineering and traffic control, to name just a few.[2,3] The reason is that, for many real-world physical systems, the descriptor variables are usually intrinsically positive or non-negative, otherwise the system state will lose its physical attributes.[4] For example, the amount of electric charges (electrons) stored in a capacitor must always remain non-negative. Meanwhile, positive systems theory has also seen significant applications in stochastic processes since probabilities should be non-negative, more specifically, Markov chains, Poisson processes and other probabilistic models can be regarded as special types of positive systems.[5] Due to the recent progresses in non-negative matrices[6] and co-positive programming,[7] more and more mathematical tools are utilized in the research of positive systems theory, which identifies its particularity and significance compared with other dynamic systems. Current research on positive systems, especially positive linear systems, could be roughly divided into three types, that is, positive controllability and controller design,[8,9] positive observability and observer design,[10] and positive realization.[11] In recent years, positive systems theory has also been combined with other branches of control theory, such as cooperative control[12,13] and time-delay systems.[14]

More recently, the research interest of positive systems has focused on fairly different control issues, in particular, robust positive stabilization and system performance[15,16], the bounded real lemma[17] and the Kalman–Yakubovic–Popov lemma[18] for positive systems, decentralized and distributed control.[19] In addition to the aforementioned control problems of positive systems, the PD controller design, which is a fundamental methodology in feedback control systems, is a new problem in the field of positive systems theory.[20-22] The major challenge of this problem stems from the difficulty in guaranteeing the positivity of the differentiator dynamics under the positive system framework. More specifically, since the input signal of the differentiator may not be monotonically increasing or decreasing, the output signal of it will be sign-indefinite. Therefore, how to choose an appropriate derivative gain for preserving the overall positivity has become the key issue. This issue is further complicated by the significant coupling between the centralized multivariable proportional and derivative gains in the synthesis process.

1.2 | Related works

In the literature, a lot of feedback control strategies have been applied to controlling positive systems in order to fulfill various constraints and performance indices. For instance, an asynchronous state-feedback controller was designed for positive Markovian jump systems by applying Lyapunov–Krasovskii functional approach and recursive matrix inequality methods.[23] Necessary and sufficient conditions of state-feedback controller design were established in linear programming for positive delay systems with semi-Markov process.[24] Static output-feedback (SOF) controllers were designed for continuous-time and discrete-time positive linear systems.[25,26] Distributed observer-based output-feedback controllers were employed for consensus of networked control systems with positivity constraint.[27-29] Positive stabilization of linear singular systems through PD control was discussed recently.[20] The PI controller design for positive systems was also discussed.[30] In a recent work,[22] PID control for SISO positive systems was investigated under a strictly Metzlerian assumption. In contrast to the existing works, the PD controller design problem is investigated for continuous-time positive LTI systems in this article, and new results are obtained. In addition, the interval gain variations are also considered in the design of nonfragile controllers for positive systems. It is worth mentioning that there have been very few results dedicated to the design of nonfragile controllers for positive systems to date.

1.3 | Contributions

The main results and contributions of this work are summarized as follows:

I. The PD controller design problem is investigated for continuous-time positive linear systems;
II. A systematic framework is proposed for designing the nonfragile PD controller of positive linear systems;
III. Tractable linear programming and semidefinite programming algorithms are developed for solution.

Since the proposed design framework is systematic and tractable, and the analysis and synthesis conditions can be represented in the form of convex programming for designing output-feedback or state-feedback PD controllers, it is believed that our approaches can be readily extended to solve the control problems for various kinds of positive systems with different performance indices, such as positive switched systems and delay systems. Regarding the broad and successful applications of PID control in practical implementations, the authors also believe that, the positivity-preserving PD controller design framework as well as the analysis and approaches proposed in this article will have significant influence in both theoretical research and real-world industries in the near future.

1.4 | Paper outline

The remainder of this article is organized in the following order. In Section 2, the notations used in this article are provided and some preliminary knowledge on positive systems theory is introduced. In Section 3, a systematic formulation for positivity-preserving PD controller design is presented, and several conditions on the positivity and stability are derived using positive systems theory and Lyapunov theory. Then the positivity and stability design of PD controllers is provided and the corresponding linear programming and semidefinite programming algorithms are developed. In Section 4, numerical examples are included to illustrate the effectiveness of our proposed results on single-input positive systems and multi-input positive systems, respectively. The nonfragility of the PD controller design is also verified through using randomly generated gain variations. In Section 5, the whole article is summarized and concluded.

2 | PRELIMINARIES

2.1 | Notations

The notations used in this article are included in the following table.

Notation	Type	Description
$\mathbb{R}$	Set	Set of real numbers
$\mathbb{C}$	Set	Set of complex numbers
$\alpha(X)$	Scalar	Spectral abscissa
$\beta(X)$	Scalar	Maximal diagonal entry
$[A]_{ij}$	Scalar	Entry of matrix A
$x(t)$	Vector	System state
$u(t)$	Vector	System input
$y(t)$	Vector	System output
X^{T}	Matrix	Matrix transpose
$\mathrm{sym}(X)$	Matrix	Symmetric matrix $X^{\mathrm{T}} + X$
$\mathrm{diag}(a_1, a_2, \dots, a_n)$	Matrix	Diagonal matrix
I (or I_n)	Matrix	$(n \times n)$ identity matrix
K_{P}	Matrix	Proportional controller gain
K_{D}	Matrix	Derivative controller gain
A, B, C	Matrix	State, input, output matrices
$G(s)$	Function	Transfer function
$X \succ$ (or $\succcurlyeq$) 0	Operator	Positive (semi) definite
$X >$ (or $\geq$) 0	Operator	$\forall i, j,\ [X]_{ij} > 0$ (or ≥ 0)
$X \succ$ (or $\succcurlyeq$) Y	Operator	$X - Y \succ 0$ (or $\succcurlyeq 0$)
$X >$ (or $\geq$) Y	Operator	$X - Y > 0$ (or ≥ 0)

2.2 | Positive systems theory

Consider the following continuous-time positive linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, and $y(t) \in \mathbb{R}^m$ denote the system state, input, and output, respectively. The state, input, and output matrices are denoted by A, B, and C. Moreover, it is assumed that (A, B) is stabilizable and (C, A) is detectable in this article. To pave the way for further analysis, some useful results[4] are provided as follows.

Definition 1. The system in (1) is called a positive linear system if for any non-negative initial state $x(0) \geq 0$ and inputs $u(t) \geq 0,\ \forall t \geq 0$, the state $x(t)$ and output $y(t)$ are always non-negative, that is, $x(t) \geq 0$ and $y(t) \geq 0,\ \forall t \geq 0$.

Lemma 1. *The linear dynamic system in (1) is positive if and only if matrix $A \in \mathbb{R}^{n\times n}$ is Metzler, meanwhile matrices $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{m\times n}$ are non-negative.*

Lemma 2. *For any Metzler matrices $M_1 \in \mathbb{R}^{n\times n}$ and $M_2 \in \mathbb{R}^{n\times n}$, if $M_1 \leq M_2$, then it holds that $\alpha(M_1) \leq \alpha(M_2)$.*

Lemma 3. *For any Metzler matrix $M \in \mathbb{R}^{n\times n}$, it is Hurwitz if and only if there exists a diagonal matrix $D \succ 0$ such that*

$$DM + M^{\mathrm{T}}D \prec 0 \quad \text{or} \quad MD + DM^{\mathrm{T}} \prec 0.$$

Lemma 4. *For any Metzler matrix $M \in \mathbb{R}^{n\times n}$, it is Hurwitz if and only if there exists a vector $v > 0$ such that*

$$v^{\mathrm{T}}M < 0 \quad \text{or} \quad Mv < 0.$$

Through using the above fundamental results on matrix theory and positive systems theory, the PD controller design of the continuous-time positive linear system in (1) will be investigated in the following sections.

3 | MAIN RESULTS

In this section, we first propose a systematic formulation for PD controller design of the positive linear system in (1), and then provide several positivity and stability analysis results. Based on positive systems theory and Lyapunov theory, the positivity and stability design of PD controllers is derived and the corresponding linear programming and semidefinite programming algorithms are developed.

3.1 | Formulation of PD controller

The objective of this subsection is to provide a systematic framework for tuning the gains of the following multivariable PD controller:

$$u(t) = K_{\mathrm{P}}y(t) + K_{\mathrm{D}}\hat{y}(t), \tag{2}$$

where $\hat{y}(t) = \dot{\check{y}}(t)$ is the derivative of the output signal of the low past filter: $\Phi\dot{\check{y}}(t) = -\check{y}(t) + y(t)$ and $\Phi := \mathrm{diag}(\tau_1, \tau_2, \dots, \tau_m)$. The positive time constant matrix Φ, in the filtered derivative term, is fixed according to the required controller bandwidth of each channel.[31] The control input in (2) can be expressed as

$$u(s) = K_{\mathrm{P}}y(s) + K_{\mathrm{D}}G_{\mathrm{D}}(s)y(s), \tag{3}$$

where $G_{\mathrm{D}}(s) := \mathrm{diag}((s/(1+\tau_1 s)), (s/(1+\tau_2 s)), \dots, (s/(1+\tau_m s)))$ denotes the transfer function matrix of the differentiator.[31] To transform the tuning of controller parameters into an SOF control problem, we reformulate the transfer function matrix $G_{\mathrm{D}}(s)$ into the following state-space form:

$$\begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}y(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}y(t) \end{cases}, \tag{4}$$

with $\hat{A} = -\Phi^{-1}$, $\hat{B} = \Phi^{-1}$, $\hat{C} = -\Phi^{-1}$, and $\hat{D} = \Phi^{-1}$, which is a state-space realization given in an explicit form. Therefore, the closed-loop system in (1) with the PD controller in (2) can be reformulated as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A + BK_{\mathrm{P}}C + BK_{\mathrm{D}}\hat{D}C & BK_{\mathrm{D}}\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}. \tag{5}$$

Hence, the tuning of the PD controller parameters for the positive linear system in (1) is reduced to finding an SOF controller gain matrix $K = \begin{bmatrix} K_{\mathrm{P}} & K_{\mathrm{D}} \end{bmatrix}$ for the overall closed-loop system in (5). In order to formulate a systematic procedure to determining the SOF controller gain K, we further define that

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \hat{B}C & \hat{A} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C & 0 \\ \hat{D}C & \hat{C} \end{bmatrix},$$

and $\tilde{x}(t) = \begin{bmatrix} x(t) & \hat{x}(t) \end{bmatrix}^{\mathrm{T}}$, then the closed-loop system in (1) can be represented in the following compact form:

$$\dot{\tilde{x}}(t) = (\tilde{A} + \tilde{B}K\tilde{C})\tilde{x}(t).$$

In the above derivations, the gain variations of the PD controller are not considered, while in real situations, gain variations always exist. With the gain variations, the control input in (2) becomes

$$u(t) = (K_{\mathrm{P}} + \Delta_{\mathrm{P}})y(t) + (K_{\mathrm{D}} + \Delta_{\mathrm{D}})\hat{y}(t), \tag{6}$$

where Δ_{P} and Δ_{D} denote the gain variations of K_{P} and K_{D}, respectively. The nonfragile PD controller in (6) has interval gain variations described as

$$-\underline{\Delta}_{\mathrm{P}} \leq \Delta_{\mathrm{P}} \leq \overline{\Delta}_{\mathrm{P}} \quad \text{and} \quad -\underline{\Delta}_{\mathrm{D}} \leq \Delta_{\mathrm{D}} \leq \overline{\Delta}_{\mathrm{D}}, \tag{7}$$

where $\underline{\Delta}_{\mathrm{P}}$, $\overline{\Delta}_{\mathrm{P}}$, $\underline{\Delta}_{\mathrm{D}}$, and $\overline{\Delta}_{\mathrm{D}}$ are non-negative matrices with compatible dimensions. Define $\Delta := \begin{bmatrix} \Delta_{\mathrm{P}} & \Delta_{\mathrm{D}} \end{bmatrix}$. Using the nonfragile PD controller in (6) to the system in (1) leads to the following closed-loop system:

$$\dot{\tilde{x}}(t) = (\tilde{A} + \tilde{B}(K + \Delta)\tilde{C})\tilde{x}(t). \tag{8}$$

Based on the above discussions, the problem to be solved in this article is presented as follows.

Problem PDPLS (PD controller design of positive linear system) Subject to the gain variations in (7), design the nonfragile PD controller gain in (6), that is, K_{P} and K_{D}, for the positive linear system in (1) such that the closed-loop system in (8) is asymptotically stable and the system state $\tilde{x}(t)$ always stays in the non-negative orthant, that is, $\tilde{x}(t) \geq 0$ for $t \geq 0$.

It is well known that the traditional PD controller design can only guarantee the stability of the system, and the positivity is not preserved during the dynamic process. More specifically, since the input signal of the differentiator may not be monotonically increasing or decreasing, the output signal of it will be sign-indefinite. Therefore, the key point is how to choose the appropriate derivative gain for the differentiator. The situation becomes much more complicated when significant coupling between the proportional and derivative gains in the centralized multivariable PD controllers. The main difficulty of the above problem is to simultaneously achieve the stability and positivity of the positive linear system in (1). Through utilizing positive systems theory and Lyapunov theory, the **Problem PDPLS** is analyzed and solved in the following subsections.

Remark 1. The PD control strategy, as shown in (2), (3), and (4), provides more degrees of freedom for parameter tuning to stabilize or optimize certain performances such as disturbance rejection[32] and transient deviation.[33] This also exhibits the advantages of PD controller in practical implementations. For example, it is well known that a suitable PD controller usually has a shorter settling time than a P controller.[31] Moreover, since industrial systems are much more complicated, a simple P controller is usually not a good practical choice, such examples can be found in wireless communications[34] and robotic manipulators.[35] Although one can obtain a similar state-space representation corresponding to the PID type controller, the challenge of this problem is that the positivity and stability of systems may not be preserved at the same time. To address this problem, one alternative choice may be the generalized PID controller, which is an approximate version of the traditional PID controller. This problem will be left to our future work

3.2 | Positivity and stability analysis

Proposition 1. *The system in (5) is positive and asymptotically stable if and only if*

1) *$A + BK_{\mathrm{P}}C + BK_{\mathrm{D}}\hat{D}C$ is Metzler;*
2) *BK_{D} is nonpositive;*
3) *The following matrix is Hurwitz:*

$$\begin{bmatrix} A + BK_{\mathrm{P}}C + BK_{\mathrm{D}}\hat{D}C & BK_{\mathrm{D}}\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix}. \tag{9}$$

Proof. The asymptotic stability of the system in (5) follows from condition 3). Thus it suffices to show that, the system in (5) is a positive linear system. Notice that $\hat{A} = -\Phi^{-1}$ is Metzler and $\hat{B}C = \Phi^{-1}C$ is a non-negative matrix. By conditions 1) and 2), we further have that $BK_D\hat{C} = -BK_D\Phi^{-1}$ is non-negative and $A + BK_PC + BK_D\hat{D}C$ is Metzler.

Based on the above results, we can conclude that the matrix in (9) is Metzler. Hence the dynamic system in (5) is a positive linear system. ■

With interval gain variations considered in the design, we will derive the conditions for positivity and stability analysis in the following theorem.

Theorem 1. Problem PDPLS *is solvable if and only if all of the following conditions hold,*

1) $A + B(K_P - \underline{\Delta}_P)C + B(K_D - \underline{\Delta}_D)\hat{D}C$ *is Metzler;*
2) $B(K_D + \overline{\Delta}_D)$ *is non-positive;*
3) *The following matrix is Hurwitz:*

$$\begin{bmatrix} A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C & B(K_D - \underline{\Delta}_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix}. \tag{10}$$

Proof. Regarding the proof, the necessity part is obvious since the three conditions always hold if **Problem PDPLS** is solvable. We will give the sufficiency part in the following.

Notice that, (1) is a positive linear system, and thus matrix A is Metzler, and matrices B and C are non-negative. The system matrix in (8) can be expressed as

$$\tilde{A} + \tilde{B}(K + \Delta)\tilde{C} = \begin{bmatrix} A + B(K_P + \Delta_P)C + B(K_D + \Delta_D)\hat{D}C & B(K_D + \Delta_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix}.$$

For any $-\underline{\Delta}_P \leq \Delta_P \leq \overline{\Delta}_P$ and $-\underline{\Delta}_D \leq \Delta_D \leq \overline{\Delta}_D$, it is easy to see that

$$A + B(K_P + \Delta_P)C + B(K_D + \Delta_D)\hat{D}C \geq A + B(K_P - \underline{\Delta}_P)C + B(K_D - \underline{\Delta}_D)\hat{D}C.$$

By condition 1), we further have that $A + B(K_P + \Delta_P)C + B(K_D + \Delta_D)\hat{D}C$ is Metzler. For condition 2), since $B(K_D + \overline{\Delta}_D)$ is a non-positive matrix, then $B(K_D + \Delta_D)\hat{C} \geq B(K_D + \overline{\Delta}_D)\hat{C} \geq 0$ which implies that $B(K_D + \Delta_D)\hat{C}$ is a non-negative matrix. Hence we have that $\tilde{A} + \tilde{B}(K + \Delta)\tilde{C}$ is a Metzler matrix and thus the system in (8) is a positive linear system. Then it remains to prove the stability of the system in (8). Notice that, from condition 3),

$$\tilde{A} + \tilde{B}(K + \Delta)\tilde{C} \leq \begin{bmatrix} A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C & B(K_D - \underline{\Delta}_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix},$$

which follows from $B(\overline{\Delta}_P - \Delta_P)C + B(\overline{\Delta}_D - \Delta_D)\hat{D}C \geq 0$ and $B(\underline{\Delta}_D + \Delta_D) \geq 0$. By Lemma 2, we further have that

$$\alpha(\tilde{A} + \tilde{B}(K + \Delta)\tilde{C}) \leq \alpha\left(\begin{bmatrix} A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C & B(K_D - \underline{\Delta}_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix}\right) < 0,$$

which implies that $\tilde{A} + \tilde{B}(K + \Delta)\tilde{C}$ is Hurwitz. Hence the system in (8) is asymptotically stable. ■

Remark 2. The above analysis provides several conditions for the solvability of **Problem PDPLS**. Based on the positivity and stability analyses in Proposition 1 and Theorem 1, the positivity and stability design on gain matrices K_P and K_P of the PD controller in (2) are derived in Section 3.3. ■

3.3 | Positivity and stability design

Before we solve the **Problem PDPLS** in the multi-input case, we first provide the following theorem for the single-input case, which is able to lead to convex linear-programming-based conditions.

Theorem 2. *Assume that (1) is a single-input system.* **Problem PDPLS** *is solvable if and only if there exist vectors* $p_1 \succ 0$ *and* $p_2 \succ 0$*, and matrices* K_P *and* K_D *such that the following linear program holds:*

1) $A + B(K_P - \underline{\Delta}_P)C + B(K_D - \underline{\Delta}_D)\hat{D}C$ *is Metzler;*
2) $B(K_D + \overline{\Delta}_D) \le 0$*;*
3) $p_1^T A + K_P C + \overline{\Delta}_P C + K_D \hat{D}C + \overline{\Delta}_D \hat{D}C + p_2^T \hat{B}C < 0$*;*
4) $K_D \hat{C} - \underline{\Delta}_D \hat{C} + p_2^T \hat{A} < 0$*;*
5) $p_1^T B = 1$.

When the linear program holds, the PD controller gains K_P *and* K_D *can be obtained.*

Proof. In the following, we will give a proof to show that the conditions 3) to 5) in Theorem 2 are equivalent to the condition 3) in Theorem 1, since the conditions 1) and 2) in Theorems 2 and 1 are identical.

In Theorem 2, since 5) $p_1^T B = 1$ is a scalar, we substitute it into conditions 3) and 4), and they, respectively, become

$$p_1^T A + p_1^T B K_P C + p_1^T B \overline{\Delta}_P C + p_1^T B K_D \hat{D}C + p_1^T B \overline{\Delta}_D \hat{D}C + p_2^T \hat{B}C < 0, \tag{11}$$

and

$$p_1^T B K_D \hat{C} - p_1^T B \underline{\Delta}_D \hat{C} + p_2^T \hat{A} < 0. \tag{12}$$

We rewrite (11) and (12) into a compact form:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C & B(K_D - \underline{\Delta}_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix} < 0. \tag{13}$$

By Lemma 4, we can conclude from (13) that the matrix in (10) is Hurwitz.

It follows from the condition 3) of Theorem 1 that, if the matrix in (10) is Hurwitz, according to Lemma 4, there must exist vectors $\tilde{p}_1 > 0$ and $\tilde{p}_2 > 0$ such that

$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}^T \begin{bmatrix} A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C & B(K_D - \underline{\Delta}_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix} < 0.$$

Noticing that $\tilde{p}_1^T B$ is a positive scalar, the following inequality also holds:

$$\frac{1}{\tilde{p}_1^T B} \times \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{bmatrix}^T \begin{bmatrix} A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C & B(K_D - \underline{\Delta}_D)\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix} < 0,$$

which means that there exist vectors $p_1 = \tilde{p}_1/(\tilde{p}_1^T B) > 0$ (or equivalently, $p_1^T B = (\tilde{p}_1^T B)/(\tilde{p}_1^T B) = 1$), and $p_2 = \tilde{p}_2^T/(\tilde{p}_1^T B) > 0$ such that conditions 3) to 5) hold. The whole proof is completed. ∎

In Theorem 2, the design conditions are represented in the form of linear programming for the single-input case. In the following theorem, we will consider the problem in the multi-input case, and derive a necessary and sufficient condition of Theorem 1 for solution.

Theorem 3. **Problem PDPLS** *is solvable if and only if there exist diagonal matrices* $P_1 \succ 0$, $P_2 \succ 0$*, a scalar* $\gamma > 0$*, and matrices* K_P *and* K_D *such that the following conditions hold:*

1) $A + B(K_P - \underline{\Delta}_P)C + B(K_D - \underline{\Delta}_D)\hat{D}C$ *is Metzler;*
2) $B(K_D + \overline{\Delta}_D) \le 0$*;*

3) $\Gamma(\gamma, P_1, P_2, K_P, K_D) :=$

$$\begin{bmatrix} \left\{ \begin{matrix} \mathrm{sym}(AP_1 + B\overline{\Delta}_P CP_1 + B\overline{\Delta}_D \hat{D}CP_1) \\ -\gamma BK_P K_P^T B^T - \gamma BK_D K_D^T B^T \end{matrix} \right\} & P_1 C^T \hat{B}^T - B\underline{\Delta}_D \hat{C}P_2 & \gamma BK_P + P_1 C^T & \gamma BK_D + P_1 C^T \hat{D}^T \\ \# & \mathrm{sym}(\hat{A}P_2) & 0 & P_2 \hat{C}^T \\ \# & \# & -\gamma I & 0 \\ \# & \# & 0 & -\gamma I \end{bmatrix} \prec 0.$$

Proof. Define the nonsingular matrix as

$$T = \begin{bmatrix} I & 0 & BK_P & BK_D \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Pre and postmultiplying Γ by T and T^T, respectively, leads to

$$T\Gamma T^T = \begin{bmatrix} \left\{ \begin{matrix} \mathrm{sym}((A + B(K_P + \overline{\Delta}_P)C \\ + B(K_D + \overline{\Delta}_D)\hat{D}C)P_1) \end{matrix} \right\} & P_1 C^T \hat{B}^T + B(K_D - \underline{\Delta}_D)\hat{C}P_2 & P_1 C^T & P_1 C^T \hat{D}^T \\ \# & \mathrm{sym}(\hat{A}P_2) & 0 & P_2 \hat{C}^T \\ \# & \# & -\gamma I & 0 \\ \# & \# & 0 & -\gamma I \end{bmatrix} \prec 0,$$

from which we can see that

$$\begin{bmatrix} \mathrm{sym}((A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C)P_1) & P_1 C^T \hat{B}^T + B(K_D - \underline{\Delta}_D)\hat{C}P_2 \\ \# & \mathrm{sym}(\hat{A}P_2) \end{bmatrix} \prec 0. \tag{14}$$

This inequality guarantees that the matrix in (10) is Hurwitz.

If the matrix in (10) is Hurwitz, there are matrices $P_1 \succ 0$ and $P_2 \succ 0$ such that (14) holds. Therefore, one can always find a sufficiently large scalar $\gamma > 0$ such that

$$-\begin{bmatrix} CP_1 & 0 \\ \hat{D}CP_1 & \hat{C}P_2 \end{bmatrix} \begin{bmatrix} \mathrm{sym}((A + B(K_P + \overline{\Delta}_P)C + B(K_D + \overline{\Delta}_D)\hat{D}C)P_1) & P_1 C^T \hat{B}^T + B(K_D - \underline{\Delta}_D)\hat{C}P_2 \\ \# & \mathrm{sym}(\hat{A}P_2) \end{bmatrix}^{-1} \begin{bmatrix} P_1 C^T & P_1 C^T \hat{D}^T \\ 0 & P_2 \hat{C}^T \end{bmatrix} + \begin{bmatrix} -\gamma I & 0 \\ 0 & -\gamma I \end{bmatrix} \prec 0. \tag{15}$$

Through Schur complement equivalence, (15) is equivalent to $T\Gamma T^T \prec 0$ which further indicates that $\Gamma \prec 0$. Therefore, the conditions in Theorem 2 are equivalent to the conditions in Theorem 3. The whole proof is completed. ■

Some equivalent conditions of Theorem 3 will be derived in the following theorem.

Theorem 4. Problem PDPLS *is solvable if and only if there exist diagonal matrices* $P_1 \succ 0$, $P_2 \succ 0$, *a scalar* $\gamma > 0$, *and matrices* L_P, L_D, M_P, *and* M_D *such that the following conditions hold:*

1) $\gamma A + B(L_P - \gamma\underline{\Delta}_P)C + B(L_D - \gamma\underline{\Delta}_D)\hat{D}C$ *is Metzler;*
2) $B(L_D + \gamma\overline{\Delta}_D) \leq 0$;

3) $\Lambda(\gamma, P_1, P_2, L_P, L_D, M_P, M_D) :=$

$$\begin{bmatrix} \left\{\begin{matrix} \mathrm{sym}(AP_1 + B\overline{\Delta}_P CP_1 + B\overline{\Delta}_D \hat{D}CP_1) - BL_P M_P^T B^T \\ -BM_P L_P^T B^T + \gamma BM_P M_P^T B^T - BL_D M_D^T B^T \\ -BM_D L_D^T B^T + \gamma BM_D M_D^T B^T \end{matrix}\right\} & P_1 C^T \hat{B}^T - B\underline{\Delta}_D \hat{C} P_2 & BL_P + P_1 C^T & BL_D + P_1 C^T \hat{D}^T \\ \# & \mathrm{sym}(\hat{A}P_2) & 0 & P_2 \hat{C}^T \\ \# & \# & -\gamma I & 0 \\ \# & \# & 0 & -\gamma I \end{bmatrix} \prec 0. \quad (16)$$

When the above conditions hold, the PD controller gains can be obtained by $K_P = (1/\gamma)L_P$ *and* $K_D = (1/\gamma)L_D$.

Proof. In the following, we will give a proof to show that the conditions in Theorem 4 are equivalent to the conditions in Theorem 3. Substituting $L_P = \gamma K_P$ and $L_D = \gamma K_D$ into (16), we have $BL_P - P_1C^T = \gamma BK_P - P_1C^T$ and $BL_D - P_1C^T\hat{D}^T = \gamma BK_D - P_1C^T\hat{D}^T$. Then it suffices to show that $\Lambda \prec 0$ if and only if $\Gamma \prec 0$. If $\Gamma \prec 0$, there exist $M_P = K_P$ and $M_D = K_D$ such that $\Lambda = \Gamma \prec 0$. On the other hand, if $\Lambda \prec 0$, notice that

$$\left\{\begin{matrix} \mathrm{sym}(AP_1 + B\overline{\Delta}_P CP_1 + B\overline{\Delta}_D \hat{D}CP_1) \\ -\gamma BK_P K_P^T B^T - \gamma BK_D K_D^T B^T \end{matrix}\right\} - \left\{\begin{matrix} \mathrm{sym}(AP_1 + B\overline{\Delta}_P CP_1 + B\overline{\Delta}_D \hat{D}CP_1) - BL_P M_P^T B^T \\ -BM_P L_P^T B^T + \gamma BM_P M_P^T B^T - BL_D M_D^T B^T \\ -BM_D L_D^T B^T + \gamma BM_D M_D^T B^T \end{matrix}\right\}$$
$$= -\gamma B(K_P - M_P)(K_P - M_P)^T B^T - \gamma B(K_D - M_D)(K_D - M_D)^T B^T \preccurlyeq 0.$$

which implies that $\Gamma \prec 0$. The whole proof is completed. ■

Remark 3. For the system in (1), if $p = m$, one can also design a decentralized PD controller of which the gains have the following diagonal structure:

$$K_P = \begin{bmatrix} [K_P]_{11} & 0 & \dots & 0 \\ 0 & [K_P]_{22} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & [K_P]_{pp} \end{bmatrix}, \quad K_D = \begin{bmatrix} [K_D]_{11} & 0 & \dots & 0 \\ 0 & [K_D]_{22} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & [K_D]_{pp} \end{bmatrix},$$

using Theorem 4. In this case, the corresponding design variables in the conditions of Theorem 4 have the following diagonal structure:

$$L_P = \begin{bmatrix} [L_P]_{11} & 0 & \dots & 0 \\ 0 & [L_P]_{22} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & [L_P]_{pp} \end{bmatrix}, \quad L_D = \begin{bmatrix} [L_D]_{11} & 0 & \dots & 0 \\ 0 & [L_D]_{22} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & [L_D]_{pp} \end{bmatrix}. \quad (17)$$

■

In this subsection, through using the Lyapunov theory of positive systems, Theorem 2 provides a linear-programming-based method to solve **Problem PDPLS** in the single-input case. Then an LMI-based PD controller design is given in Theorems 3 and 4 which can be utilized to tackle the problem in the multi-input case. From the computational point of view, the linear-programming-based approach is more efficient than the LMI-based approach. Based on the above result in Theorem 4, a tractable algorithmic solution to **Problem PDPLS** will be developed in the following subsection.

3.4 Algorithmic solution

Based on the discussions and derivations in the previous subsections, in particular, Theorems 3 and 4, an iterative algorithm is constructed to design the nonfragile PD controller gains for the positive linear system in (1).

Algorithm: Nonfragile PD Controller Design (NPDCD)

Setp 1: Set $k = 1$ *and* $\epsilon^{(0)} = 0$. *Select initial values* $K_{\mathrm{P}}^{(1)}$ *and* $K_{\mathrm{D}}^{(1)}$ *such that matrix (10) is Hurwitz.*

Step 2: For fixed $M_{\mathrm{P}} = K_{\mathrm{P}}^{(k)}$ *and* $M_{\mathrm{D}} = K_{\mathrm{D}}^{(k)}$, *solve the following convex optimization problem with respective to* $\gamma > 0$, $P_1 \succ 0$, $P_2 \succ 0$, L_{P} *and* L_{D}:

minimize $\epsilon^{(k)}$ *subject to*

$$\begin{cases} \gamma A + B(L_{\mathrm{P}} - \gamma \underline{\Delta}_{\mathrm{P}})C + B(L_{\mathrm{D}} - \gamma \underline{\Delta}_{\mathrm{D}})\hat{D}C \text{ is Metzler}, \\ B(L_{\mathrm{D}} + \gamma \overline{\Delta}_{\mathrm{D}}) \leq 0, \\ \Lambda(\gamma, P_1, P_2, L_{\mathrm{P}}, L_{\mathrm{D}}, M_{\mathrm{P}}, M_{\mathrm{D}}) \prec \epsilon^{(k)} I. \end{cases}$$

Step 3: If $\epsilon^{(k)} \leq 0$, *STOP, a solution is obtained as* $K_{\mathrm{P}}^* = (1/\gamma)L_{\mathrm{P}}$ *and* $K_{\mathrm{D}}^* = (1/\gamma)L_{\mathrm{D}}$. *Otherwise, go to next step.*

Step 4: If $|\epsilon^{(k)} - \epsilon^{(k-1)}|/\epsilon^{(k)} < \theta$ *which is a prescribed tolerance, STOP. Otherwise, update* $k = k + 1$, $K_{\mathrm{P}}^{(k)} = (1/\gamma)L_{\mathrm{P}}$, *and* $K_{\mathrm{D}}^{(k)} = (1/\gamma)L_{\mathrm{D}}$, *then go to Step 2.*

Remark 4. In Step 1, the initial values of K_{P} and K_{D} can be obtained by solving an SOF controller $K = \begin{bmatrix} K_{\mathrm{P}} & K_{\mathrm{D}} \end{bmatrix}$ for the closed-loop system $\bar{A} + \overline{B}K\overline{C}$ where

$$\bar{A} = \begin{bmatrix} A + B\overline{\Delta}_{\mathrm{P}}C + B\overline{\Delta}_{\mathrm{D}}\hat{D}C & -B\underline{\Delta}_{\mathrm{D}}\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad \text{and} \quad \overline{C} = \begin{bmatrix} C & 0 \\ \hat{D}C & \hat{C} \end{bmatrix}.$$

This is a very standard SOF control problem that can be solved by various design approaches in the literature.[36] ■

4 SIMULATIONS

In this section, we will verify the effectiveness of our theoretical results and algorithms through using illustrative examples. The filtered derivative term is set to be $\tau_i = 0.1$, $i = 1, 2, \ldots, m$, in the numerical simulations.

4.1 Single-input positive system

Consider the positive linear system in (1) with the following system matrices:

$$A = \begin{bmatrix} -0.15 & 1.90 & 1.55 \\ 0.50 & -0.3 & 0.10 \\ 0.20 & 0.50 & -2.55 \end{bmatrix}, \quad B = \begin{bmatrix} 0.055 \\ 0.169 \\ 0.059 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

and the state-space realization corresponding to differentiator G_{D}:

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} -10 & 0 & 10 & 0 \\ 0 & -10 & 0 & 10 \\ -10 & 0 & 10 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix}. \tag{18}$$

The controller gain variations have the values as follows:

$$\underline{\Delta}_{\mathrm{P}} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad \overline{\Delta}_{\mathrm{P}} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad \underline{\Delta}_{\mathrm{D}} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad \text{and} \quad \overline{\Delta}_{\mathrm{D}} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}.$$

This is a single-input positive linear system. By implementing the program in Theorem 2 using YALMIP with MATLAB R2020b, a feasible solution was obtained as

$$K_{\mathrm{P}}^{*} = \begin{bmatrix} -26.2373 & -0.8230 \end{bmatrix} \quad \text{and} \quad K_{\mathrm{D}}^{*} = \begin{bmatrix} -0.2282 & -0.2223 \end{bmatrix}. \tag{19}$$

Substituting (19) into the conditions 1) and 2) of Proposition 1, we can verify that

$$A + BK_{\mathrm{P}}^{*}C + BK_{\mathrm{D}}^{*}\hat{D}C = \begin{bmatrix} -0.3069 & 1.7431 & 1.5332 \\ 0.0180 & -0.7820 & 0.0485 \\ 0.0317 & 0.3317 & -2.5680 \end{bmatrix},$$

is a Metzler matrix and

$$BK_{\mathrm{D}}^{*} = \begin{bmatrix} -0.1255 & -0.1222 \\ -0.3856 & -0.3756 \\ -0.1346 & -0.1311 \end{bmatrix} < 0,$$

which have preserved the positivity of system (1). Furthermore, the eigenvalues of the corresponding matrix (9) are $\{-0.0867, -0.9172, -2.5928, -10.0601, -10.0000\}$ which have guaranteed the stability according to Proposition 1.

To show the controller nonfragility, we let the gain variations be

$$\Delta_{\mathrm{P}} = \begin{bmatrix} -0.25 & 0.5 \end{bmatrix} \quad \text{and} \quad \Delta_{\mathrm{D}} = \begin{bmatrix} 0.075 & -0.025 \end{bmatrix}. \tag{20}$$

Under the gain variations (20), we have

$$A + B(K_{\mathrm{P}}^{*} + \Delta_{\mathrm{P}})C + B(K_{\mathrm{D}}^{*} + \Delta_{\mathrm{D}})\hat{D}C = \begin{bmatrix} -0.3041 & 1.7459 & 1.5346 \\ 0.0265 & -0.7735 & 0.0528 \\ 0.0347 & 0.3347 & -2.5665 \end{bmatrix},$$

is a Metzler matrix and

$$B(K_{\mathrm{D}}^{*} + \Delta_{\mathrm{D}}) = \begin{bmatrix} -0.0842 & -0.1360 \\ -0.2589 & -0.4179 \\ -0.0904 & -0.1459 \end{bmatrix} < 0,$$

which have preserved the positivity of system (1). Meanwhile, the eigenvalues of the corresponding matrix in (9) are $\{-0.0992, -0.9075, -2.5902, -10.0473, -10.0000\}$ which have guaranteed the stability according to Proposition 1.

4.2 | Multi-input positive system

Consider the positive linear system in (1) with the following system matrices:

$$A = \begin{bmatrix} -1 & 1 & 3 \\ 1 & -1 & 3 \\ 0.5 & 2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.1 & 0.5 & 0 \end{bmatrix},$$

and the state-space realization corresponding to the differentiator $G_{\rm D}$:

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} -10 & 10 \\ -10 & 10 \end{bmatrix}. \tag{21}$$

The controller gain variations have the values as follows:

$$\underline{\Delta}_{\rm P} = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}, \quad \overline{\Delta}_{\rm P} = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}, \quad \underline{\Delta}_{\rm D} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad \text{and} \quad \overline{\Delta}_{\rm D} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

By implementing **Algorithm NPDCD**, the PD controller gains were initialized as

$$K_{\rm P}^{(1)} = \begin{bmatrix} -7.4994 \\ -4.0750 \end{bmatrix} \quad \text{and} \quad K_{\rm D}^{(1)} = \begin{bmatrix} -4.6186 \\ -2.8065 \end{bmatrix}.$$

After several iterations, we obtained $\epsilon^{(3)} = -1.00 < 0$ and **Algorithm NPDCD** returned the PD controller gains as

$$K_{\rm P}^{*} = \begin{bmatrix} -9.0185 \\ 0.2990 \end{bmatrix} \quad \text{and} \quad K_{\rm D}^{*} = \begin{bmatrix} -2.4113 \\ -0.2988 \end{bmatrix}. \tag{22}$$

Substituting (22) into the conditions 1) and 2) of Proposition 1, we can verify that

$$A + BK_{\rm P}^{*}C + BK_{\rm D}^{*}\hat{D}C = \begin{bmatrix} -1.1143 & 0.4286 & 3.0000 \\ 0.4285 & -3.8574 & 3.0000 \\ 0.1571 & 0.2855 & -5.0000 \end{bmatrix},$$

is a Metzler matrix and $BK_{\rm D}^{*} = \begin{bmatrix} -0.3905 & -1.2355 & -0.7234 \end{bmatrix}^{\rm T}$ is nonpositive, which have preserved the positivity of system (1). Meanwhile, we can verify that

$$\begin{bmatrix} A + BK_{\rm P}^{*}C + BK_{\rm D}^{*}\hat{D}C & BK_{\rm D}^{*}\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix} = \begin{bmatrix} -1.1143 & 0.4286 & 3.0000 & 0.3905 \\ 0.4285 & -3.8574 & 3.0000 & 1.2355 \\ 0.1571 & 0.2855 & -5.0000 & 0.7234 \\ 0.1000 & 0.5000 & 0.0000 & -1.0000 \end{bmatrix},$$

which has eigenvalues $\{-0.3610, -1.3020, -3.8461, -5.4627\}$ and thus is a Hurwitz matrix.

To show the controller nonfragility, we randomly generated 100 pairs $(\Delta_{\rm P}, \Delta_{\rm D})$ satisfying $\underline{\Delta}_{\rm P} \le \Delta_{\rm P} \le \overline{\Delta}_{\rm P}$ and $\underline{\Delta}_{\rm D} \le \Delta_{\rm D} \le \overline{\Delta}_{\rm D}$. The state responses of system (1) with controller (6) are shown in Figure 1.

4.3 | Decentralized PD controller

Consider the positive linear system in (1) with the following system matrices:

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & -4 & 3 \\ 0.5 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix},$$

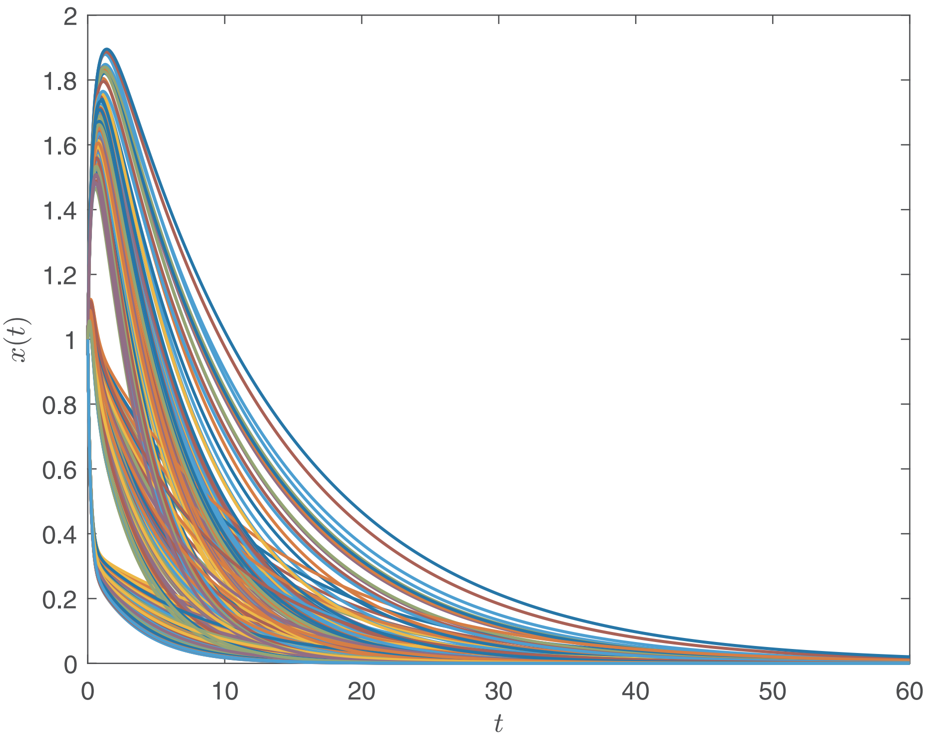

FIGURE 1 State responses of system (1) subject to gain variations ($x(0) = [1.0\ 1.0\ 1.0]^{\mathrm{T}}$)

and the state-space realization corresponding to the differentiator G_{D}:

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} -10 & 0 & 10 & 0 \\ 0 & -10 & 0 & 10 \\ -10 & 0 & 10 & 0 \\ 0 & -10 & 0 & 10 \end{bmatrix}. \tag{23}$$

The controller gain variations have the values as follows:

$$\underline{\Delta}_{\mathrm{P}} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad \overline{\Delta}_{\mathrm{P}} = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \underline{\Delta}_{\mathrm{D}} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \text{and} \quad \overline{\Delta}_{\mathrm{D}} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

By implementing **Algorithm NPDCD** with L_{P} and L_{D} having the structure in (17), the PD controller gains were initialized as

$$K_{\mathrm{P}}^{(1)} = \begin{bmatrix} -2.8469 & -3.1249 \\ -0.4411 & -3.1574 \end{bmatrix} \quad \text{and} \quad K_{\mathrm{D}}^{(1)} = \begin{bmatrix} -2.9027 & -0.5846 \\ -0.3374 & 0.0012 \end{bmatrix}.$$

After several iterations, we obtained $\epsilon^{(3)} = -1.00 < 0$ and **Algorithm NPDCD** returned the PD controller gains as

$$K_{\mathrm{P}}^{*} = \begin{bmatrix} -4.4586 & 0 \\ 0 & -3.2725 \end{bmatrix} \quad \text{and} \quad K_{\mathrm{D}}^{*} = \begin{bmatrix} -0.1235 & 0 \\ 0 & -0.1091 \end{bmatrix}. \tag{24}$$

Substituting (24) into the conditions 1) and 2) of Proposition 1, we can verify that

$$A + BK_{\mathrm{P}}^{*}C + BK_{\mathrm{D}}^{*}\hat{D}C = \begin{bmatrix} -2.2751 & 0.2789 & 2.7818 \\ 0.6717 & -5.5107 & 2.9564 \\ 0.3292 & 1.1459 & -3.0000 \end{bmatrix},$$

is a Metzler matrix and

$$BK_\mathrm{D}^* = \begin{bmatrix} -0.0124 & -0.0546 \\ -0.0618 & -0.0109 \\ -0.0371 & 0 \end{bmatrix},$$

is nonpositive, which have preserved the positivity of system (1). Meanwhile, we can verify that

$$\begin{bmatrix} A + BK_\mathrm{P}^*C + BK_\mathrm{D}^*\hat{D}C & BK_\mathrm{D}^*\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix} = \begin{bmatrix} -2.2751 & 0.2789 & 2.7818 & 0.1235 & 0.5456 \\ 0.6717 & -5.5107 & 2.9564 & 0.6176 & 0.1091 \\ 0.3292 & 1.1459 & -3.0000 & 0.3706 & 0 \\ 1.0000 & 5.0000 & 0 & -10.0000 & 0 \\ 1.0000 & 2.0000 & 1.0000 & 0 & -10.0000 \end{bmatrix},$$

which has eigenvalues $\{-0.7371, -3.3277, -6.1144, -10.0243, -10.5825\}$ and thus is a Hurwitz matrix.

5 | CONCLUSION

In this article, we have addressed and solved the PD control design problem for positive systems. A systematic framework has been proposed for designing the multivariable PD controller that can preserve the stability and positivity. Controller gain variations have also been considered in the design. Linear programming and semidefinite programming algorithms have been developed for single-input and multi-input positive systems, respectively. Illustrative examples attested the numerical efficiency of the proposed PD control design methodologies.

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CONFLICT OF INTEREST

The authors declared that they have no conflict of interest to this work.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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