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Reachable Set Estimation of Periodic Piecewise Polynomial Systems Using Bernstein Polynomials

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Abstract: In this work, a Bernstein polynomial approach is first applied to the estimation of reachable set for a class of periodic piecewise polynomial systems, whose subsystems are time-varying and can be expanded to Bernstein polynomial forms. A lemma on the negativity/positivity for a class of Bernstein polynomial matrix functions is presented. Based on the integration of the presented lemma and the theory of matrix polynomials, two tractable sufficient conditions are developed. Four sets of constraints with different conservatism are derived and compared. The effectiveness and lower conservatism of the proposed approach in reachable set estimation are illustrated.

Key Words: Bernstein polynomial, Periodic piecewise polynomial systems, Reachable set estimation, Time-varying systems

1 Introduction

Continuous-time periodic systems are common in practice, and usually more difficult to be tackled than discrete-time ones [1]. The periodic piecewise polynomial system (PPPS) is inspired by intuitive characterization of periodic time-varying dynamics in polynomial forms. Dividing the fundamental period into several subintervals, it is favorable to approximate a periodic system by a number of time-varying polynomial subsystems, where each subsystem is described as a matrix polynomial function with a prescribed degree. Driven by the needs in safety monitoring and verification, the reachable set estimation of periodic systems has drawn increasing attention, but mostly in the discrete-time domain [2]. In the authors' previous work [3], a reachable set estimation approach is developed for continuous-time periodic piecewise linear time-varying systems. However, the approach cannot deal with PPPSs involving high-order polynomial subsystems. Aimed at tackling PPPSs, this work develops a Bernstein polynomial approach by proposing a lemma based on the Bernstein polynomial basis [4], enabling lower conservatism than the existing method [5, 6]. Two tractable sufficient conditions are developed, and four sets of constraints are derived. For comparison of conservatism, the reachable set estimation is achieved through optimizing the ellipsoidal bounding region.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{N}$ and $\mathbb{N}^+$ denote the set of natural numbers (including zero) and the set of positive integers, respectively. For $n \in \mathbb{N}$, $n!$ denotes the factorial of n. I and 0 denote an $n \times n$ identity matrix and a zero matrix, respectively (if the subscript is omitted, the dimension is consistent with the context). P^T and P^{-1} are the transpose and inverse of matrix P, respectively, and $\mathbf{sym}(P) = P^T + P$. For real symmetric matrices P and Q, the notation $P \geq Q$ (resp., $P > Q$) means that the matrix $P - Q$ is positive semi-definite (resp., positive definite). $\mathscr{P}_N^{m\times n}[\tau_1, \tau_2]$ denotes the set of $m \times n$ matrix polynomials of degree no greater than N over the interval $[\tau_1, \tau_2]$.

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2 Problem Formulation and Preliminaries

Consider a T_p-periodic time-varying system:

$$\dot{x}(t) = \mathcal{A}(t)x(t) + \Omega(t)\omega(t), \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $\omega(t) \in \mathbb{R}^{n_\omega}$ are the state vector and the disturbance vector, respectively; $\mathcal{A}(t) = \mathcal{A}(t + T_p)$, $\Omega(t) = \Omega(t + T_p)$ are continuously periodic time-varying matrix functions for $t \geq 0$. With a known scalar bound $\bar{\omega} > 0$, peak-bounded disturbance $\omega(t)$ satisfies

$$\omega^T(t)\omega(t) \leq \bar{\omega}, \quad \forall t \geq 0. \tag{2}$$

Partitioning each time interval of fundamental period T_p into S subintervals denoted as $[lT_p + t_{i-1}, lT_p + t_i)$, $l = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, S$, where $t_0 = 0$, $t_S = T_p$. The dwell time of the i-th subinterval is defined as $T_i \triangleq t_i - t_{i-1}$, $i \in \mathcal{S} \triangleq \{1, 2, \ldots, S\}$, and $\sum_{i=1}^{S} T_i = T_p$. Periodic system (1) is approximated by the following PPPS:

$$\dot{x}(t) = \mathcal{A}_i(t)x(t) + \Omega_i(t)\omega(t), t \in [lT_p + t_{i-1}, lT_p + t_i), \tag{3}$$

where the matrix functions over the i-th subinterval satisfy:

$$\begin{aligned}\mathcal{A}_i(t) &= A_{i,0} + \sigma_i(t)A_{i,1} + \cdots + \sigma_i^{N_i}(t)A_{i,N_i} \\ &= \sum_{j=0}^{N_i} \sigma_i^j(t)A_{i,j} \in \mathscr{P}_{N_i}^{n_x\times n_x}[lT_p + t_{i-1}, lT_p + t_i),\end{aligned} \tag{4}$$

$$\Omega_i(t) = \sum_{j=0}^{N_i} \sigma_i^j(t)\Omega_{i,j} \in \mathscr{P}_{N_i}^{n_x\times n_\omega}[lT_p + t_{i-1}, lT_p + t_i), \tag{5}$$

with $\sigma_i(t) \triangleq (t - lT_p - t_{i-1})/T_i$ and constant matrices $A_{i,j} \in \mathbb{R}^{n_x\times n_x}$, $\Omega_{i,j} \in \mathbb{R}^{n_x\times n_\omega}$, $i \in \mathcal{S}$, $j \in \mathcal{N}_i \triangleq \{0, 1, \ldots, N_i\}$, $N_i \in \mathbb{N}$. Over the i-th subinterval of the period from lT_p to $(l + 1)T_p$, $\mathcal{A}(t) = \mathcal{A}_i(t)$, $\Omega(t) = \Omega_i(t)$, meanwhile $\mathcal{A}_i(t)$ and $\Omega_i(t)$ respectively described by (4) and (5) are right continuous. Moreover, take $\mathcal{A}(t)$ for example, if $A_{1,0} = \sum_{j=0}^{N_S} A_{S,j}$, $A_{i,0} = \sum_{j=0}^{N_{i-1}} A_{i-1,j}$, $i =$

$2,3,\ldots,S$, then $\mathcal{A}(t)$ will be continuous at all switching instants for $t\in[0,\infty)$ with $\lim_{t\to lT_p+t_i^-}\mathcal{A}_i(t)=\mathcal{A}_{i+1}(lT_p+t_i)$, $i\in\mathcal{S}$. One may choose the continuity of $\mathcal{A}(t)$ and/or $\Omega(t)$ at switching instants based on the requirements of analysis and synthesis in practice. The state of PPPS (3) is continuous for $t\geq 0$, with reachable set

$$\mathcal{R}_x \triangleq \{x\in\mathbb{R}^{n_x} \mid x(0)=0, x(t) \text{ and } \omega(t) \text{ satisfy (3) and (2)}, \forall t\geq 0\}. \quad (6)$$

To estimate the reachable set of PPPS (3), an intuitive method is sought for a bounding region as small as possible for $\mathcal{R}_x$, which can be described by $\mathscr{R}\triangleq\bigcup_{0\leq t\leq T_p}\mathcal{E}(\mathcal{P}(t))$, where

$$\mathcal{E}(\mathcal{P}(t))\triangleq\{x\in\mathbb{R}^{n_x}\mid x^T\mathcal{P}(t)x\leq 1, \mathcal{P}(t)>0\}, \quad (7)$$

with a continuous time-varying matrix function $\mathcal{P}(t)$.

Lemma 1 *(Negativity/positivity property for a class of matrix polynomials [5, 6]) Consider a bounded n-th degree symmetric matrix polynomial function $f:[0,1]\to\mathbb{R}^{d\times d}$ as*

$$f(\beta)=\Xi_0+\beta\Xi_1+\beta^2\Xi_2+\cdots+\beta^n\Xi_n, \quad (8)$$

where $\beta\in[0,1]$ is a scalar, and $\Xi_k\in\mathbb{R}^{d\times d}$ are real symmetric matrices, $k=0,1,\ldots,n$, $n\in\mathbb{N}$, $d\in\mathbb{N}^+$. Symmetric matrix polynomial function $f(\beta)<0$ (resp., >0) if

$$\sum_{q=0}^{k}\Xi_q<0 \; (\textit{resp.}, >0), \;\; k=0,1,\ldots,n. \quad (9)$$

3 Main Results

Consider a scalar $\beta\in[0,1]$ and constant matrices $\Xi_k\in\mathbb{R}^{d\times d}$, $k=0,1,\ldots,n$, $n\in\mathbb{N}$, $d\in\mathbb{N}^+$. Any n-th degree matrix polynomial in the following form can be expanded to a Bernstein polynomial:

$$f(\beta)=\sum_{k=0}^{n}\beta^k\Xi_k=\sum_{k=0}^{n}B_k(\beta)\Lambda_k, \quad (10)$$

where the Bernstein polynomial basis can be characterized by

$$B_k(\beta)=\binom{n}{k}\beta^k(1-\beta)^{n-k} \quad (11)$$

with binomial coefficient $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, and Bernstein coefficient matrices Λ_k, $k=0,1,\ldots,n$, can be obtained by

$$\Lambda_k=\sum_{q=0}^{k}\frac{\binom{k}{q}}{\binom{n}{q}}\Xi_q,\; k=0,1,\ldots,n. \quad (12)$$

Based on the previous studies on Bernstein polynomials [7, 8], one can derive the following lemma.

Lemma 2 *(Negativity/positivity property on the Bernstein polynomial basis) Consider an n-th degree symmetric matrix polynomial function $f:[0,1]\to\mathbb{R}^{d\times d}$ defined in (8) with scalar $\beta\in[0,1]$ and real symmetric matrices $\Xi_k\in\mathbb{R}^{d\times d}$, $k=0,1,\ldots,n$, $n\in\mathbb{N}$, $d\in\mathbb{N}^+$. Symmetric matrix polynomial function $f(\beta)<0$ (resp., >0) if*

$$\sum_{q=0}^{k}\frac{\binom{k}{q}}{\binom{n}{q}}\Xi_q<0 \; (\textit{resp.}, >0),\; k=0,1,\ldots,n. \quad (13)$$

Construct a Lyapunov function $V(t)=x^T(t)\mathcal{P}(t)x(t)$, where $\mathcal{P}(t)=\mathcal{P}(t+T_p)>0$. For $t\in[lT_p+t_{i-1},lT_p+t_i)$, suppose $V(t)=V_i(t)=x^T(t)\mathcal{P}_i(t)x(t)$, $\mathcal{P}(t)=\mathcal{P}_i(t)=\sum_{m=0}^{M_i}\sigma_i^m(t)P_{i,m}\in\mathscr{P}_{M_i}^{n_x\times n_x}[lT_p+t_{i-1},lT_p+t_i)$ with real symmetric matrices $P_{i,m}\in\mathbb{R}^{n_x\times n_x}$, $i\in\mathcal{S}$, $m\in\mathcal{M}_i\triangleq\{0,1,\ldots,M_i\}$, $M_i\in\mathbb{N}$, and

$$P_{1,0}=\sum_{m=0}^{M_i}P_{S,m}, \quad (14)$$

$$P_{i,0}=\sum_{m=0}^{M_{i-1}}P_{i-1,m},\; i=2,3,\ldots,S, \quad (15)$$

$$\sum_{i=1}^{S}\sum_{m=1}^{M_i}P_{i,m}=0, \quad (16)$$

which guarantee the continuity of $\mathcal{P}(t)$, $\forall t\geq 0$. Hence, $\mathcal{P}(t)$ is a continuous and Dini-differentiable T_p-periodic piecewise matrix polynomial function for $t\geq 0$. To guarantee $\mathcal{P}(t)>0$ while minimizing the bounding region of $\mathcal{R}_x$, one may use either a scalar upper bound $\varepsilon>0$ or a real symmetric matrix $\hat{P}$ to characterize the region. Consider

$$\begin{bmatrix}\hat{P} & I\\ I & \epsilon I\end{bmatrix}\geq 0, \quad (17)$$

where $\epsilon=\varepsilon^{-1}>0$, and $\hat{P}>0$. Based on Lemma 2, one has

$$\sum_{m=0}^{r}\frac{\binom{r}{m}}{\binom{M_i}{m}}P_{i,m}-\hat{P}\geq 0,\; r=0,1,\ldots,M_i,\; i\in\mathcal{S}. \quad (18)$$

Alternatively, based on the existing Lemma 1 one has

$$\sum_{m=0}^{r}P_{i,m}-\hat{P}\geq 0,\; r=0,1,\ldots,M_i,\; i\in\mathcal{S}. \quad (19)$$

Apply Schur complement equivalence to (17) and combine the resulting inequality with (18) or (19). Either set of constraints (17) and (18), or (17) and (19), enables that for all $i\in\mathcal{S}$, $\mathcal{P}(t)=\mathcal{P}_i(t)\geq\hat{P}\geq\varepsilon I>0$, which ensures $\varepsilon x^T(t)x(t)\leq x^T(t)\hat{P}x(t)\leq x^T(t)\mathcal{P}(t)x(t)\leq 1$ to satisfy (7).

Theorem 1 *Consider PPPS (3) with $N_i\geq 1$, $M_i\geq 1$, $i\in\mathcal{S}$, and peak-bounded disturbance $\omega(t)$ satisfying (2). Given scalars $\alpha_i>0$, $i\in\mathcal{S}$, the system is asymptotically stable with bounded reachable set satisfying (7), if there exist scalar $\epsilon>0$, matrix $\hat{P}>0$, and symmetric matrices $P_{i,m}$, $i\in\mathcal{S}$, $m\in\mathcal{M}_i$, such that conditions (14)–(17), condition (18) or (19), and the following inequalities hold:*

$$\sum_{q=0}^{k}\frac{\binom{k}{q}}{\binom{M_i+N_i}{q}}\Theta_{i,q}<0,\; k=0,1,\ldots,M_i+N_i, \quad (20)$$

where

$$\Theta_{i,k}=\begin{bmatrix}\Delta_{i,k} & \Upsilon_{i,k}\\ \Upsilon_{i,k}^T & \Phi_{i,k}\end{bmatrix}, \quad (21)$$

and

$$\Delta_{i,0}=\alpha_iP_{i,0}+\frac{1}{T_i}P_{i,1}+\textbf{sym}(P_{i,0}A_{i,0}), \quad (22)$$

$$\Delta_{i,k} = \alpha_i P_{i,k} + \frac{k+1}{T_i} P_{i,k+1} + \sum_{\substack{j+m=k \\ j\in\mathcal{N}_i, m\in\mathcal{M}_i}} \mathbf{sym}\left(P_{i,m}A_{i,j}\right), \quad k = 1, 2, \ldots, M_i - 1, \tag{23}$$

$$\Delta_{i,M_i} = \alpha_i P_{i,M_i} + \sum_{\substack{j+m=M_i \\ j\in\mathcal{N}_i, m\in\mathcal{M}_i}} \mathbf{sym}\left(P_{i,m}A_{i,j}\right), \tag{24}$$

$$\Delta_{i,k} = \sum_{\substack{j+m=k \\ j\in\mathcal{N}_i, m\in\mathcal{M}_i}} \mathbf{sym}\left(P_{i,m}A_{i,j}\right), \ k = M_i + 1, \ldots, M_i + N_i; \tag{25}$$

$$\Upsilon_{i,0} = P_{i,0}\Omega_{i,0}, \tag{26}$$

$$\Upsilon_{i,k} = \sum_{\substack{j+m=k \\ j\in\mathcal{N}_i, m\in\mathcal{M}_i}} P_{i,m}\Omega_{i,j}, \ k = 1, 2, \ldots, M_i + N_i; \tag{27}$$

$$\Phi_{i,0} = -\frac{\alpha_i}{\bar{\omega}} I, \tag{28}$$

$$\Phi_{i,k} = 0, \ k = 1, 2, \ldots, M_i + N_i. \tag{29}$$

Theorem 2 *Consider PPPS (3) with peak-bounded disturbance $\omega(t)$ satisfying (2). Given scalars $\alpha_i > 0$, $i \in \mathcal{S}$, the system is asymptotically stable with bounded reachable set satisfying (7), if there exist exist scalar $\epsilon > 0$, matrix $\hat{P} > 0$, and matrices $P_{i,m}$, $i \in \mathcal{S}$, $m \in \mathcal{M}_i$, such that conditions (14)–(16), condition (18) or (19), and the following inequalities hold:*

$$\sum_{q=0}^{k} \Theta_{i,q} < 0, \ k = 0, 1, \ldots, M_i + N_i, \tag{30}$$

where $\Theta_{i,k} < 0$, $k = 0, 1, \ldots, M_i + N_i$, are described by (21)–(29).

Combine Theorem 1 and Theorem 2 with $\varepsilon = \epsilon^{-1}$. Given a scalar $\mu \in \{0, 1\}$, the optimization problem of reachable set estimation can be solved subjected to four sets of constraints:

$$\text{Minimize} \quad (1-\mu)\epsilon - \mu \ln(\det(\hat{P})) \quad \text{subject to}$$
$$\begin{cases} \text{Case 1: (14)–(16), (18), (20)} \\ \text{Case 2: (14)–(16), (19), (20)} \\ \text{Case 3: (14)–(16), (18), (30)} \\ \text{Case 4: (14)–(16), (19), (30)} \end{cases}$$

The comparative results are illustrated in Figs. 1–2. It can be observed that the bounding ellipsoids for Case 1 (resp. Case 2) are smaller than those for Case 3 (resp. Case 4), showing the lower conservatism in reachable set estimation achieved by condition (20) based on Lemma 2, compared to condition (30) based on Lemma 1.

4 Conclusion

This work first uses a Bernstein polynomial approach to tackle the reachable set estimation problem for PPPSs. A useful lemma is proposed, and two tractable sufficient conditions are given in terms of linear matrix inequalities. The optimization of bounding region for reachable set can be solved subject to four sets of constraints. The proposed approach not only enables lower conservatism in PPPS reachable set estimation, but also provides an intuitive route to tackle time-varying parameter products with high powers. A complete version of this work was presented in [9].

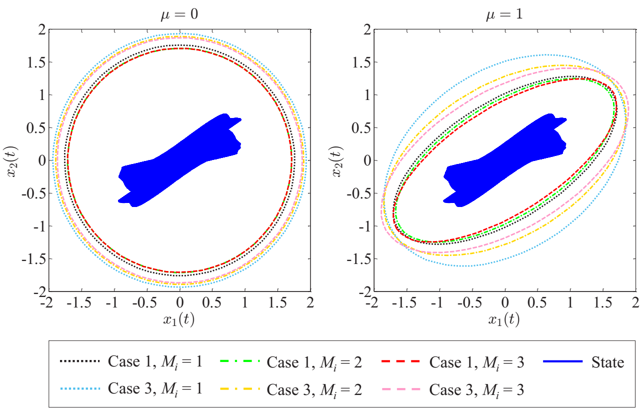

Fig. 1: Bounding ellipsoids of reachable sets for Cases 1, 3

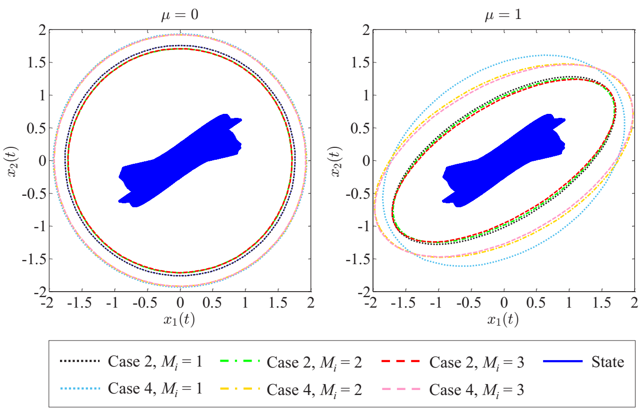

Fig. 2: Bounding ellipsoids of reachable sets for Cases 2, 4

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