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PD control of positive interval continuous-time systems with time-varying delay

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ABSTRACT

This article aims to design proportional-derivative (PD) controllers for interval positive linear systems in the continuous-time domain, which still remains a widely-discussed open problem in positive systems theory. The specific objective is to design a PD controller for the system with interval uncertain parameters and time-varying delay, which simultaneously ensures closed-loop system stability and preserves positivity. The work proposes a systematic framework, with the aim of finding PD controller gains for positive robust stabilization. The methodology and algorithm are presented first in the study, and the performance of such methods is instantiated by numerical examples.

1. Introduction

In general, a positive system is a special type of dynamic systems whose state and output variables have to be positive, or at least non-negative, throughout its entire evolutionary horizon. The research on positive linear systems traces back to David G. Luenberger, who systematically introduced the concept of such class of systems in a fundamental book [23]. Since then, positive systems theory has seen broad applications in many industrial problems, such as biochemical engineering and traffic control [7,36]. For many real-world physical systems, the descriptor variables usually have intrinsically positive or non-negative features [11]. For instance, the amount of electric charges stored in a capacitor must always remain non-negative. Meanwhile, positive systems theory has also been broadly applied in stochastic processes since probabilities also have non-negative features, more specifically, Markov chains [29], Poisson processes [15] and other probabilistic models can be regarded as special types of positive systems. Due to the recent progress in non-negative matrices [6,12] and co-positive programming [9,14], an increasing number of mathematical tools have been utilized in the research of positive systems theory, which signifies its particularity and importance compared with other dynamic systems. Current research on positive systems, especially positive linear systems, could be roughly divided into three areas, namely, positive controllability and controller design [10,38], positive observability and observer design [8,20], and positive realization [5]. In recent years, positive systems theory has also been combined with other branches of control theory, such as multi-agent systems [40] and switched systems [43].

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Much attention on positive systems has been devoted to fairly different analysis and synthesis issues, in particular, positive stabilization and system performance [32,33], the Bounded Real Lemma [34], the Kalman-Yakubovic-Popov Lemma [26], and decentralized and distributed control [17,16,18]. Admittedly, a lot of results have been obtained.However, the PD controller design, which is a fundamental methodology in output-feedback control systems [4], still has not been developed for such kind of systems (even without delay). This motivates us to investigate the PD controller design problem for systems with positivity constraints. It is known that stability of time-delay systems is a fundamental issue from both theoretical and practical points of view [35,41,42]. In recent years, there are research activities on the analysis and synthesis problems of positive systems with time-varying delay. For instance, Liu *et al.* have established necessary and sufficient stability criteria for positive systems with continuous and bounded time-varying delay [21,22]. Shen *et al.* have shown that the L_∞-gain of positive fractional-order systems depends on neither the magnitude of time-varying delay nor the fractional order of the system [31]. Ait Rami *et al.* have addressed the interval estimation issue for linear positive systems with time-varying unknown delay in [3]. The state-bounding problem for positive singular discrete-time systems with time-varying delay and bounded disturbances was studied by Sau *et al.* in [28]. *Contrasted to the existing works, for the first time, this paper investigates the PD controller design problem for interval continuous-time positive linear systems with time-varying delay.*In addition, the interval uncertainties in system matrices are also considered in the design of robust controller gains for positive systems. *The major challenge of this problem stems from the difficulty in guaranteeing the positivity of the differentiator dynamics under the positive system framework.* More specifically, the input signal of the differentiator is not guaranteed to be monotonic, resulting in a sign-indefinite output signal. Therefore, the design of an appropriate derivative gain preserving the overall positivity has become the key issue. This issue is further complicated due to the significant coupling between the centralized multivariable proportional and derivative gains in the synthesis process, and the interval uncertainties of system matrices as well as the time-varying delay.

This paper studies the PD control of positive systems with time-varying delay, which is still an open problem. The motivations for this research are two-fold. First, the PD controller design has been an important topic in communication systems, robot manipulation and marine engineering, to name just a few, for its simple implementation and reliable performance, such as shorter settling time. Most of these industrial processes involve nonnegative physical quantities, and unexpected issues may happen if the positivity is not preserved. Second, the PD control of positive systems was one of the open problems in the field of positive systems theory. The research in this paper will work as an integral part of the complete answer to this open problem and propose useful results that could be extended to study relevant issues. The main results and contributions of this work are summarized as follows: A systematic framework is proposed for designing the robust PD controller of positive interval continuous-time systems with time-varying delay, and a tractable semi-definite programming algorithm is developed for solution.Since the proposed design framework is systematic and tractable, and the analysis and synthesis conditions can be represented in the form of convex programming for designing output-feedback PD controllers, it is believed that, our approaches could be easily extended with different control strategies, such as hybrid control [27,37], fuzzy and cascade control [13,24,25], and integral control [19] for practical applications. Furthermore, it can be also straightforwardly extended to the case in which different types of disturbances, such as L_1 and L_∞ [30], are considered.

2. Preliminaries

2.1. Notations

The notations employed in this paper are standard. For matrix $A \in \mathbb{R}^{n\times n}$, we use $[A]_{ij}$ to denote the entry located at the i-th row and the j-th column. Matrix A is called Metzler if all of its off-diagonal entries are non-negative. Matrix A is called Hurwitz if all of its eigenvalues have strictly negative real parts. $\mathrm{diag}(v_1, v_2, \ldots, v_n)$ is a diagonal matrix with diagonal entries being the entries of vector $v := [v_1, v_2, \ldots, v_n]^{\mathrm{T}}$. $\mathscr{C}(D, R)$ denotes the continuous functions whose domain is D and range is R. The other notations employed in this paper are summarized in the following table.

Notation	Type	Description
$\mathbb{R}$	set	real numbers
$\mathbb{R}^n$	set	n-dimension Euclidean space
$\mathbb{R}_+$	set	non-negative real numbers
$\mathbb{M}$	set	Metzler matrices, whose off-diagonal entries are all non-negative
$\mathbb{H}$	set	Hurwitz matrices, whose eigenvalues all have strictly negative real parts
$\alpha(X)$	scalar	spectral abscissa of X
X^{T}	matrix	matrix transpose of X
$\mathrm{sym}(X)$	matrix	symmetric matrix $X^{\mathrm{T}} + X$
I (or I_n)	matrix	$(n \times n)$ identity matrix
K_{P}	matrix	proportional controller gain
K_{D}	matrix	derivative controller gain

$G(s)$	function	transfer function
$\vert\cdot\vert$	operator	Euclidean norm for vectors
$X \succ$ (or $\succeq$) 0	operator	X is positive (semi-) definite
$X \prec$ (or $\preceq$) 0	operator	X is negative (semi-) definite
$X >$(or $\geqslant$) 0	operator	$\forall i,j,[X]_{ij} > 0$ (or $\geqslant 0$)
$X \succ$(or $\succeq$) Y	operator	$X - Y \succ 0$(or $\succeq 0$)
$X >$(or $\geqslant$) Y	operator	$X - Y > 0$(or $\geqslant 0$)

2.2. Positive systems theory

Consider the following linear continuous-time system:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_\tau x(t-\tau(t)) + Bu(t), \\ y(t) = Cx(t) + C_\tau x(t-\tau(t)), \\ x(t) = \varphi(t),\ t \in [-\overline{\tau}, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the input, and $y(t) \in \mathbb{R}^q$ is the output. Furthermore, matrices A, A_τ, B, C and C_τ are real constant matrices with appropriate dimensions. The time-varying delay is assumed to be bounded and continuous with respect to t with

$$0 \leqslant \tau(t) \leqslant \overline{\tau},\ t \geqslant 0. \tag{2}$$

The function $\varphi(t) \in \mathscr{C}([-\overline{\tau}, 0], \mathbb{R}^n)$ is a continuous vector-valued function specifying the initial system state. To pave the way for further analysis, some useful results [1,2,11,22] are provided as follows.

Definition 1. System (1) is said to be positive if, for any $\varphi(t) \in \mathscr{C}\left([-\overline{\tau}, 0], \mathbb{R}^n_+\right)$ and every $u(t) \geq 0$, we have $x(t) \geqslant 0$ and $y(t) \geqslant 0$ for all $t \geqslant 0$.

Lemma 1. *For any Metzler matrices $M_1 \in \mathbb{R}^{n\times n}$ and $M_2 \in \mathbb{R}^{n\times n}$, if $M_1 \leqslant M_2$, then it holds that $\alpha(M_1) \leqslant \alpha(M_2)$.*

Lemma 2. *System (1) is positive if and only if A is Metzler, $A_\tau \geqslant 0, B \geqslant 0, C \geqslant 0$ and $C_\tau \geqslant 0$.*

The asymptotic stability condition for the time-varying delay positive system in (1) is given in Lemma 3 [2,39].

Lemma 3. *Assuming that system (1) with $u(t) = 0$ is positive, then it is asymptotically stable for all $\varphi(t) \in \mathscr{C}\left([-\overline{\tau}, 0], \mathbb{R}^n_+\right)$ and any time-varying delay $\overline{\tau} \geqslant \tau(t) \geqslant 0$, if and only if one of the following conditions hold:*

1) *There exists a diagonal matrix $P \succ 0$ such that*

$$P(A + A_\tau) + (A + A_\tau)^{\mathrm{T}} P \prec 0;$$

2) *$(A + A_\tau)$ is a Hurwitz matrix: the real parts of its eigenvalues are strictly negative.*

Through using the above fundamental results on matrix theory and positive systems theory, the PD controller design of the interval continuous-time positive linear system in (1) will be investigated in the following sections.

3. Main results

In this section, we first propose a systematic formulation for PD controller design of interval positive linear systems with unbounded delay, and then provide several positivity and stability analysis results. Based on positive systems theory and Lyapunov theory, the positivity and stability design of PD controllers is derived, and the corresponding semi-definite programming algorithm is developed.

3.1. Formulation of Robust PD Controller

Let us consider the following positive system with interval uncertainties and time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_\tau x(t-\tau(t)) + Bu(t), \\ y(t) = Cx(t) + C_\tau x(t-\tau(t)), \\ x(t) = \varphi(t) \geqslant 0, t \in [-\overline{\tau}, 0], \end{cases} \tag{3}$$

where matrices A, A_τ, B, C and C_τ are unknown and belong to the uncertainty set $\Xi = \{(A, A_\tau, B, C, C_\tau) : A \in [\underline{A}, \overline{A}], A_\tau \in [\underline{A}_\tau, \overline{A}_\tau], B \in [\underline{B}, \overline{B}], C \in [\underline{C}, \overline{C}], C_\tau \in [\underline{C}_\tau, \overline{C}_\tau]\}$, where $\underline{A} \in \mathbb{R}^{n\times n}$ is Metzler, and $\underline{A}_\tau \in \mathbb{R}_+^{n\times n}, \underline{B} \in \mathbb{R}_+^{n\times p}, \underline{C} \in \mathbb{R}_+^{q\times n}, \underline{C}_\tau \in \mathbb{R}_+^{q\times n}, 0 \leqslant \tau(t) \leqslant \overline{\tau}, t \geqslant 0$, and $\varphi(t) \in \mathscr{C}([-\overline{\tau}, 0], \mathbb{R}_+^n)$.

The objective of this subsection is to provide a systematic framework for synthesizing the gains of the following multivariable PD controller:

$$u(t) = K_{\mathrm{P}} y(t) + K_{\mathrm{D}} \hat{y}(t) \tag{4}$$

where $\hat{y}(t) = \dot{\check{y}}(t)$ is the derivative of the output signal of the low past filter: $\Phi\dot{\check{y}}(t) = -\check{y}(t) + y(t)$ and $\Phi := \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_m)$. The positive time constant matrix Φ, in the filtered derivative term, is fixed according to the required controller bandwidth of each channel [4]. The control input in (4) can be expressed as

$$u(s) = K_{\mathrm{P}} y(s) + K_{\mathrm{D}} G_{\mathrm{D}}(s) y(s)$$

where $G_{\mathrm{D}}(s) := \mathrm{diag}((s/(1+\tau_1 s)), (s/(1+\tau_2 s)), \ldots, (s/(1+\tau_m s)))$ denotes the transfer function matrix of the differentiator [4]. To transform the computation of the controller parameters into an SOF control problem, the transfer function matrix $G_{\mathrm{D}}(s)$ admits the following state-space form:

$$\begin{cases} \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}y(t) \\ \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}y(t) \end{cases}$$

with $\widehat{A} = -\Phi^{-1}, \widehat{B} = \Phi^{-1}, \widehat{C} = -\Phi^{-1}$ and $\widehat{D} = \Phi^{-1}$, which is a state-space realization given in an explicit form. Therefore, the closed-loop system in (3) with the PD controller in (4) can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A + BK_{\mathrm{P}}C + BK_{\mathrm{D}}\widehat{D}C & BK_{\mathrm{D}}\widehat{C} \\ \widehat{B}C & \widehat{A} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} A_\tau + BK_{\mathrm{P}}C_\tau + BK_{\mathrm{D}}\widehat{D}C_\tau & 0 \\ \widehat{B}C_\tau & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau(t)) \\ \hat{x}(t-\tau(t)) \end{bmatrix}. \tag{5}$$

Hence, the computation of the PD controller parameters for the positive linear system in (3) is reduced to finding an SOF controller gain matrix $K = [K_{\mathrm{P}} \quad K_{\mathrm{D}}]$ for the overall closed-loop system in (5). In order to formulate a systematic procedure to determining the SOF controller gain K, we further define that

$$\widetilde{A} = \begin{bmatrix} A & 0 \\ \widehat{B}C & \widehat{A} \end{bmatrix}, \quad \widetilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} C & 0 \\ \widehat{D}C & \widehat{C} \end{bmatrix},$$

$$\widetilde{A}_\tau = \begin{bmatrix} A_\tau & 0 \\ \widehat{B}C_\tau & 0 \end{bmatrix}, \quad \widetilde{C}_\tau = \begin{bmatrix} C_\tau & 0 \\ \widehat{D}C_\tau & 0 \end{bmatrix}, \quad \tilde{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}, \quad \tilde{x}(t-\tau(t)) = \begin{bmatrix} x(t-\tau(t)) \\ \hat{x}(t-\tau(t)) \end{bmatrix},$$

then the closed-loop system in (3) can be represented in the following compact form:

$$\dot{\tilde{x}}(t) = \left(\widetilde{A} + \widetilde{B}K\widetilde{C}\right)\tilde{x}(t) + \left(\widetilde{A}_\tau + \widetilde{B}K\widetilde{C}_\tau\right)\tilde{x}(t-\tau(t)). \tag{6}$$

Based on the above discussions, the problem to be solved in this paper is presented as follows.

Problem PDIPSD (PD Controller Design of Interval Positive Systems with Delay): Design a robust PD controller gain in (4), that is, K_{P} and K_{D}, for the positive linear system in (3) such that the closed-loop system in (6) is asymptotically stable, and the system state $\tilde{x}(t)$ always stays in the non-negative orthant, that is, $\tilde{x}(t) \geqslant 0$ for $t \geqslant 0$.

The key point is how to choose the appropriate derivative gains for the differentiator. One well-known fact of the traditional PD controller design is that it cannot guarantee the positivity during the dynamic process while only preserving the stability. The uncertainty stems from the input signal of the differentiator, which might not be monotonically increasing or decreasing, attributed to the indefiniteness of the sign of the output signal. The situation becomes much more complicated when significant coupling between the proportional and derivative gains in the centralized multivariable PD controllers. The main obstacle of the design is to preserve stability and positivity of the positive linear system in (3) simultaneously. In the following subsections, the **Problem PDIPSD** is analyzed and solved by utilizing positive systems theory and Lyapunov theory.

3.2. Positivity and stability analysis

Theorem 1. *The system in (6) is positive and asymptotically stable if the following conditions hold:*

1) $\underline{A} + \underline{B}K_{\mathrm{P}}\underline{C} + \overline{B}K_{\mathrm{D}}\widehat{D}\overline{C}$ *is Metzler;*
2) $\underline{A}_\tau + \underline{B}K_{\mathrm{P}}\underline{C}_\tau + \overline{B}K_{\mathrm{D}}\widehat{D}\overline{C}_\tau$ *is non-negative;*
3) K_{P} *is non-negative;*
4) K_{D} *is non-positive;*
5) *The following matrix is Hurwitz:*

$$\begin{bmatrix} \overline{A}+\overline{A}_{\tau}+\overline{B}K_{\mathrm{P}}\left(\overline{C}+\overline{C}_{\tau}\right)+\underline{B}K_{\mathrm{D}}\widehat{D}(\underline{C}+\underline{C}_{\tau}) & \overline{B}K_{\mathrm{D}}\widehat{C} \\ \widehat{B}\left(\overline{C}+\overline{C}_{\tau}\right) & \widehat{A} \end{bmatrix}. \tag{7}$$

Proof. We first prove the positivity of the system in (6). Notice that $\widehat{A}=-\Phi^{-1}$ is Metzler and $\widehat{B}C=\Phi^{-1}C$ is a non-negative matrix. By condition 3), $B\in[\underline{B},\overline{B}], \underline{B}>0$, and $C\in\left[\underline{C},\overline{C}\right], \underline{C}>0$, we have $BK_{\mathrm{P}}C\geqslant\underline{B}K_{\mathrm{P}}\underline{C}\geqslant 0$. By conditions 4), we further have that $BK_{\mathrm{D}}\widehat{C}=-BK_{\mathrm{D}}\Phi^{-1}$ is non-negative. By condition 4) and $\widehat{D}=\Phi^{-1}, BK_{\mathrm{D}}\widehat{D}C>\overline{B}K_{\mathrm{D}}\widehat{D}\overline{C}$. We further derive that $A+BK_{\mathrm{P}}C+BK_{\mathrm{D}}\widehat{D}C\geqslant\underline{A}+\underline{B}K_{\mathrm{P}}\underline{C}+\overline{B}K_{\mathrm{D}}\widehat{D}\overline{C}$. Along with condition 1), $A+BK_{\mathrm{P}}C+BK_{\mathrm{D}}\widehat{D}C$ is Metzler in the given interval. Then we conclude that

$$\begin{bmatrix} A+BK_{\mathrm{P}}C+BK_{\mathrm{D}}\widehat{D}C & BK_{\mathrm{D}}\widehat{C} \\ \widehat{B}C & \widehat{A} \end{bmatrix}\in\mathbb{M}.$$

Analogously, by condition 2), $K_{\mathrm{D}}\leqslant 0$and $K_{\mathrm{P}}\geqslant 0, A_{\tau}+BK_{\mathrm{P}}C_{\tau}+BK_{\mathrm{D}}\widehat{D}C_{\tau}\geqslant A_{\tau}+\underline{B}K_{\mathrm{P}}\underline{C}_{\tau}+\overline{B}K_{\mathrm{D}}\widehat{D}\overline{C}_{\tau}\geqslant 0$, and $\widehat{B}C_{\tau}=\Phi^{-1}C_{\tau}\geqslant 0$, we conclude that

$$\begin{bmatrix} A_{\tau}+BK_{\mathrm{P}}C_{\tau}+BK_{\mathrm{D}}\widehat{D}C_{\tau} & 0 \\ \widehat{B}C_{\tau} & 0 \end{bmatrix}\geqslant 0.$$

Then by Lemma 2, the positivity of the system is derived.

Then it suffices to show the asymptotic stability of the system in (5). Denote $\widetilde{A}+\widetilde{B}K\widetilde{C}+\widetilde{A}_{\tau}+\widetilde{B}K\widetilde{C}_{\tau}$ by M, and denote matrix (7) by $\overline{M}$. Note that $A+A_{\tau}+BK_{P}(C+C_{\tau})+BK_{\mathrm{D}}\widehat{D}(C+C_{\tau})\leqslant\overline{A}+\overline{A}_{\tau}+\overline{B}K_{\mathrm{P}}\left(\overline{C}+\overline{C}_{\tau}\right)+\underline{B}K_{\mathrm{D}}\widehat{D}(\underline{C}+\underline{C}_{\tau}), B(C+C_{\tau})\leqslant\widehat{B}\left(\overline{C}+\overline{C}_{\tau}\right)$, and $BK_{\mathrm{D}}\widehat{C}\leqslant\overline{B}K_{\mathrm{D}}\widehat{C}$, hence $M\leqslant\overline{M}$. It's trivial that M and $\overline{M}$ are Metzler. By Lemma 1 and condition 5),

$$\begin{bmatrix} A+A_{\tau}+BK_{\mathrm{P}}(C+C_{\tau})+BK_{\mathrm{D}}\widehat{D}(C+C_{\tau}) & BK_{\mathrm{D}}\widehat{C} \\ \widehat{B}(C+C_{\tau}) & \widehat{A} \end{bmatrix} \tag{8}$$

is Hurwitz.

Then by Lemma 3 3), the asymptotic stability of system (5) is proved. □

3.3. Positivity and Stability Design

In the following theorem, we will derive a useful lemma as an extension of Theorem 1 for solution.

Lemma 4. *The condition 5) in Theorem 1 is equivalent to the following condition: There exist diagonal matrices $P_1\succ 0, P_2\succ 0$, a scalar $\gamma>0$ such that*

$$\Gamma(\gamma,P_1,P_2,K_{\mathrm{P}},K_{\mathrm{D}}):=\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ * & \mathrm{sym}\left(\widehat{A}P_2\right) & 0 & 0 & P_2\widehat{C}^{\mathrm{T}} \\ * & * & -\gamma I & 0 & 0 \\ * & * & * & -\gamma I & 0 \\ * & * & * & * & -\gamma I \end{bmatrix}\prec 0 \tag{9}$$

$$\begin{aligned} \text{where}\quad \Gamma_{11}= &\ \mathrm{sym}\left(\overline{A}P_1+\overline{A}_{\tau}P_1\right)-\gamma\overline{B}K_{\mathrm{P}}K_P^{\mathrm{T}}\overline{B}^{\mathrm{T}}-\gamma\underline{B}K_{\mathrm{D}}K_{\mathrm{D}}^{\mathrm{T}}\underline{B}^{\mathrm{T}}-\gamma\overline{B}K_{\mathrm{D}}K_{\mathrm{D}}^{\mathrm{T}}\overline{B}^{\mathrm{T}}, \\ \Gamma_{12}= &\ P_1\left(\overline{C}+\overline{C}_{\tau}\right)^{\mathrm{T}}\widehat{B}^{\mathrm{T}}, \\ \Gamma_{13}= &\ \gamma\overline{B}K_{\mathrm{P}}+P_1\left(\overline{C}+\overline{C}_{\tau}\right)^{\mathrm{T}}, \\ \Gamma_{14}= &\ \gamma\underline{B}K_{\mathrm{D}}+P_1(\underline{C}+\underline{C}_{\tau})^{\mathrm{T}}\widehat{D}^{\mathrm{T}}, \\ \Gamma_{15}= &\ \gamma\overline{B}K_{\mathrm{D}}. \end{aligned}$$

Proof. Define the non-singular matrix as

$$T := \begin{bmatrix} I & 0 & \overline{B}K_{\mathrm{P}} & \underline{B}K_{\mathrm{D}} & \overline{B}K_{\mathrm{D}} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Pre- and post-multiplying Γ by T and T^{T}, respectively, leads to

$$T\Gamma T^{\mathrm{T}} = \begin{bmatrix} \Gamma_{11'} & \Gamma_{12'} & \Gamma_{13'} & \Gamma_{14'} & 0 \\ * & \mathrm{sym}\left(\widehat{A}P_2\right) & 0 & 0 & P_2\widehat{C}^{\mathrm{T}} \\ * & * & -\gamma I & 0 & 0 \\ * & * & 0 & -\gamma I & 0 \\ * & * & 0 & 0 & -\gamma I \end{bmatrix} \prec 0$$

$$\begin{aligned} \text{where}\quad \Gamma_{11'} =\ & \mathrm{sym}\left(\left(\overline{A} + \overline{A}_{\tau} + \overline{B}K_{\mathrm{P}}\left(\overline{C} + \overline{C}_{\tau}\right) + \underline{B}K_{D}\widehat{D}(\underline{C} + \underline{C}_{\tau})\right)P_1\right), \\ \Gamma_{12'} =\ & P_1\left(\overline{C} + \overline{C}_{\tau}\right)^{\mathrm{T}}\widehat{B}^{\mathrm{T}} + \overline{B}K_{\mathrm{D}}\widehat{C}P_2, \\ \Gamma_{13'} =\ & P_1\left(\overline{C} + \overline{C}_{\tau}\right)^{\mathrm{T}}\widehat{D}^{\mathrm{T}}, \\ \Gamma_{14'} =\ & P_1(\underline{C} + \underline{C}_{\tau})^{\mathrm{T}}, \end{aligned}$$

from which we can see that

$$\Omega := \begin{bmatrix} \Gamma_{11'} & \Gamma_{12'} \\ * & \mathrm{sym}\left(\widehat{A}P_2\right) \end{bmatrix} \prec 0. \tag{10}$$

This inequality guarantees that the matrix in (7) is Hurwitz.
Let

$$\Sigma := \begin{bmatrix} \Gamma_{13'} & \Gamma_{14'} & 0 \\ 0 & 0 & P_2\widehat{C}^{\mathrm{T}} \end{bmatrix}.$$

By the above definition,

$$T\Gamma T^{\mathrm{T}} = \begin{bmatrix} \Omega & \Sigma \\ \Sigma^{\mathrm{T}} & -\gamma I \end{bmatrix}.$$

If the matrix in (7) is Hurwitz, there always exist matrices $P_1 \succ 0$ and $P_2 \succ 0$ such that (10) holds. Therefore, one can always find a sufficiently large scalar $\gamma > 0$ such that

$$-\gamma I - \Sigma\Omega^{-1}\Sigma^{\mathrm{T}} \prec 0. \tag{11}$$

Through Schur complement equivalence, (11) is equivalent to $T\Gamma T^{\mathrm{T}} \prec 0$, which further indicates that $\Gamma \prec 0$.
Therefore, the condition in Lemma 4 is equivalent to condition 5) in Theorem 1. The proof is completed. □

Theorem 2. Problem PDIPSD *is solvable if there exist diagonal matrices $P_1 \succ 0, P_2 \succ 0$, scalar $\gamma > 0$, and matrices $L_{\mathrm{P}}, L_{\mathrm{D}}, M_{\mathrm{P}}, M_{\mathrm{D}_1}$ and M_{D_2} such that the following conditions hold:*

1) $\gamma\underline{A} + \underline{B}L_{\mathrm{P}}\underline{C} + \overline{B}L_{\mathrm{D}}\widehat{D}\overline{C}$ *is Metzler;*
2) $\gamma\underline{A}_{\tau} + \underline{B}L_{\mathrm{P}}\underline{C}_{\tau} + \overline{B}L_{\mathrm{D}}\widehat{D}\overline{C}_{\tau}$ *is non-negative;*
3) $L_{\mathrm{D}} \leqslant 0$;
4) $L_{\mathrm{P}} \geqslant 0$;
5)

$$\Lambda(\gamma, P_1, P_2, L_{\mathrm{P}}, L_{\mathrm{D}}, M_{\mathrm{P}}, M_{\mathrm{D}_1}, M_{\mathrm{D}_2}) := \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ * & \mathrm{sym}\left(\widehat{A}P_2\right) & 0 & 0 & P_2\widehat{C}^{\mathrm{T}} \\ * & * & -\gamma I & 0 & 0 \\ * & * & 0 & -\gamma I & 0 \\ * & * & 0 & 0 & -\gamma I \end{bmatrix} \prec 0 \tag{12}$$

$$
\begin{aligned}
\text{where}\quad \Lambda_{11} = {} & \mathrm{sym}\left(\overline{A}P_1 + \overline{A}_\tau P_1\right) - \overline{B}L_\mathrm{P}M_\mathrm{P}^\mathrm{T} - M_P L_P^\mathrm{T}\overline{B}^\mathrm{T} + \gamma M_\mathrm{P}M_\mathrm{P}^\mathrm{T} - \underline{B}L_D M_{\mathrm{D}_1}^\mathrm{T} \\
& - M_{\mathrm{D}_1}L_\mathrm{D}^\mathrm{T}\underline{B}^\mathrm{T} + \gamma M_{D_1}M_{\mathrm{D}_1}^\mathrm{T} - \overline{B}L_\mathrm{D}M_{\mathrm{D}_2}^\mathrm{T} - M_{\mathrm{D}_2}L_\mathrm{D}^\mathrm{T}\overline{B}^\mathrm{T} + \gamma M_{\mathrm{D}_2}M_{\mathrm{D}_2}^\mathrm{T}, \\
\Lambda_{12} = {} & P_1\left(\overline{C} + \overline{C}_\tau\right)^\mathrm{T}\widehat{B}^\mathrm{T}, \\
\Lambda_{13} = {} & \overline{B}L_\mathrm{P} + P_1\left(\overline{C} + \overline{C}_\tau\right)^\mathrm{T}, \\
\Lambda_{14} = {} & \underline{B}L_\mathrm{D} + P_1(\underline{C} + \underline{C}_\tau)^\mathrm{T}\widehat{D}^\mathrm{T}, \\
\Lambda_{15} = {} & \overline{B}L_\mathrm{D}.
\end{aligned}
$$

When the above conditions hold, the PD controller gains can be obtained by $K_\mathrm{P} = (1/\gamma)L_\mathrm{P}$ *and* $K_\mathrm{D} = (1/\gamma)L_\mathrm{D}$.

Proof. In the following, we will give a proof to show that Theorem 2 is equivalent to Theorem 1. Substituting $L_\mathrm{P} = \gamma K_\mathrm{P}$ and $L_\mathrm{D} = \gamma K_\mathrm{D}$ into Theorem 2 1)–4), since $\gamma > 0$, it follows that Theorem 1 1)–4) are equivalent to Theorem 2 1)–4). Then it suffices to show $\Lambda \prec 0$ is equivalent to condition 5) in Theorem 1.

Assume $\Lambda \prec 0$. Substituting $L_\mathrm{P} = \gamma K_\mathrm{P}$ and $L_\mathrm{D} = \gamma K_\mathrm{D}$ into Eqn. (12), we have $\Lambda_{13} = \overline{B}L_\mathrm{P} + P_1\left(\overline{C} + \overline{C}_\tau\right)^\mathrm{T} = \gamma\overline{B}K_\mathrm{P} + P_1\left(\overline{C} + \overline{C}_\tau\right)^\mathrm{T} = \Gamma_{13}, \Lambda_{14} = \underline{B}L_\mathrm{P} + P_1(\underline{C} + \underline{C}_\tau)^\mathrm{T} = \gamma\underline{B}K_\mathrm{P} + P_1(\underline{C} + \underline{C}_\tau)^\mathrm{T} = \Gamma_{14}$, and $\Lambda_{15} = \gamma\overline{B}K_\mathrm{P} = \Gamma_{15}$. We also notice that:

$$
\Gamma_{11} - \Lambda_{11} = -\gamma\left(\overline{B}K_\mathrm{P} - M_\mathrm{P}\right)\left(\overline{B}K_\mathrm{P} - M_\mathrm{P}\right)^\mathrm{T} - \gamma\left(\underline{B}K_\mathrm{D} - M_{\mathrm{D}_1}\right)\left(\underline{B}K_\mathrm{D} - M_{\mathrm{D}_1}\right)^\mathrm{T} - \gamma\left(\overline{B}K_\mathrm{D} - M_{\mathrm{D}_2}\right)\left(\overline{B}K_\mathrm{D} - M_{\mathrm{D}_2}\right)^\mathrm{T} \preceq 0
$$

which implies that $\Gamma_{11} \prec 0$ hence $\Gamma \prec 0$.

On the other hand, we assume $\Gamma \prec 0$. If $\Gamma \prec 0$, there exist $M_\mathrm{P} = \overline{B}K_\mathrm{P}, M_{\mathrm{D}_1} = \underline{B}K_\mathrm{D}$, and $M_{\mathrm{D}_2} = \overline{B}K_\mathrm{D}$, such that $\Lambda = \Gamma \prec 0$. This proves $\Gamma \prec 0$ implies $\Lambda \prec 0$.

Hence, we prove that $\Lambda \prec 0$ is equivalent to $\Gamma \prec 0$. By Lemma 4, we further have $\Lambda \prec 0$ is equivalent to condition 5) in Theorem 1. The proof is completed. □

3.4. Algorithmic solution

Based on the discussions and derivations in the previous subsections, in particular, Theorem 2, an iterative algorithm is constructed to design the robust PD controller gains for the positive linear system in (3).

Algorithm: Robust PD Controller Design (RPDCD)

Step 1: Set $k = 1$ and $\epsilon^{(0)} = 0$. Select initial matrices $M_\mathrm{P}^{(1)}, M_{\mathrm{D}_1}^{(1)}$, and $M_{\mathrm{D}_2}^{(1)}$ such that matrix (7) is Hurwitz and the conditions 1)–4) hold .

Step 2: For fixed $M_\mathrm{P} = M_\mathrm{P}^{(k)}, M_{\mathrm{D}_1} = M_{\mathrm{D}_1}^{(k)}, M_{\mathrm{D}_2} = M_{\mathrm{D}_2}^{(k)}$, solve the following convex optimization problem with respective to $\gamma > 0, P_1 \succ 0, P_2 \succ 0$, L_P and L_D: minimize $\epsilon^{(k)}$ subject to

$$
\begin{cases}
\gamma\underline{A} + \underline{B}L_\mathrm{P}\underline{C} + \overline{B}L_\mathrm{D}\widehat{D}\overline{C}\text{isMetzler}, \\
\gamma\underline{A}_\tau + \underline{B}L_\mathrm{P}\underline{C}_\tau + \overline{B}L_\mathrm{D}\widehat{D}\overline{C}_\tau \leqslant 0, \\
L_\mathrm{D} \leqslant 0, \\
L_\mathrm{P} \geqslant 0, \\
\Lambda\left(\gamma, P_1, P_2, L_\mathrm{P}, L_\mathrm{D}, M_\mathrm{P}, M_{\mathrm{D}_1}, M_{\mathrm{D}_2}\right) \prec \epsilon^{(k)}I.
\end{cases}
$$

Step 3: If $\epsilon^{(k)} \leqslant 0$, STOP, a solution is obtained as $K_\mathrm{P}^* = (1/\gamma)L_\mathrm{P}$ and $K_\mathrm{D}^* = (1/\gamma)L_\mathrm{D}$. Otherwise, go to next step.

Step 4: If $|\epsilon^{(k)} - \epsilon^{(k-1)}|/\epsilon^{(k)} < \theta$which is a prescribed tolerance, STOP. Otherwise, update $k = k+1, M_\mathrm{P}^{(k)} = (1/\gamma)\overline{B}L_\mathrm{P}, M_{\mathrm{D}_1}^{(k)} = (1/\gamma)\underline{B}L_\mathrm{D}$ and $M_{\mathrm{D}_2}^{(k)} = (1/\gamma)\overline{B}L_\mathrm{D}$, then go to Step 2

Remark 1. In Step 1, the initial matrices $M_\mathrm{P}^{(1)}, M_{\mathrm{D}_1}^{(1)}$, and $M_{\mathrm{D}_2}^{(1)}$ can be obtained by solving a structured observer gain matrix:

$$
M := \begin{bmatrix} M_\mathrm{P}^{(1)} & M_{\mathrm{D}_1}^{(1)} & M_{\mathrm{D}_2}^{(1)} \\ 0 & 0 & 0 \end{bmatrix} \tag{13}
$$

for the closed-loop system $\check{A}+M\check{C}$ to be positive and Hurwitz where

$$\check{A}=\begin{bmatrix}\overline{A}+\overline{A}_\tau & 0\\ \widehat{B}\left(\overline{C}+\overline{C}_\tau\right) & \widehat{A}\end{bmatrix},\quad \check{C}=\begin{bmatrix}\overline{C}+\overline{C}_\tau & 0\\ \underline{C}+\underline{C}_\tau & 0\\ 0 & \widehat{C}\end{bmatrix}. \tag{14}$$

This will lead to the solvability of a positive observer design problem [20]. Solving the following LMIs with respect to two diagonal matrices $Q_1>0$ and $Q_2>0$, and three matrix variables $S_{\mathrm{P}}, S_{\mathrm{D}_1}$, and S_{D_2}:

- $S_{D_2}\geqslant 0$,
- $Q_1\left(\overline{A}+\overline{A}_\tau\right)+S_P\left(\overline{C}+\overline{C}_\tau\right)+S_{D_1}(\underline{C}+\underline{C}_\tau)$ is Metzler,
-

$$\begin{bmatrix}\mathrm{sym}\left(Q_1\left(\overline{A}+\overline{A}_\tau\right)+S_{\mathrm{P}}\left(\overline{C}+\overline{C}_\tau\right)+S_{D_1}(\underline{C}+\underline{C}_\tau)\right) & \left(\overline{C}+\overline{C}_\tau\right)^{\mathrm{T}}\widehat{B}^{\mathrm{T}}Q_2+S_{\mathrm{D}_2}\widehat{C}\\ Q_2\widehat{B}\left(\overline{C}+\overline{C}_\tau\right)+\widehat{C}^{\mathrm{T}}S_{\mathrm{D}_2}^{\mathrm{T}} & \mathrm{sym}\left(Q_2\widehat{A}\right)\end{bmatrix}\prec 0. \tag{15}$$

Then $M_{\mathrm{P}}^{(1)}=Q_1^{-1}S_{\mathrm{P}}, M_{\mathrm{D}_1}^{(1)}=Q_1^{-1}S_{D_1}$, and $M_{\mathrm{D}_2}^{(1)}=Q_1^{-1}S_{D_2}$.

4. Illustrative examples

In this section, we will verify the effectiveness of our theoretical results and algorithms through using two illustrative examples. The filtered derivative term is set to be $\tau_i=0.1$ $(i=1,2,\ldots,m)$. In the simulation, two types of controllers, that is, the centralized and the decentralized, will be designed for verification. Consider the positive interval continuous-time system in (3) with the following system matrices:

$$\underline{A}=\begin{bmatrix}-4.2200 & 0.9400 & 2\\ 1.1800 & -4.2000 & 2\\ 0.8000 & 1 & -6\end{bmatrix},\quad \overline{A}=\begin{bmatrix}-4.2064 & 0.9600 & 2\\ 1.2030 & -4.0370 & 2\\ 1 & 2 & -5\end{bmatrix},$$

$$\underline{A}_\tau=\begin{bmatrix}0.3100 & 0.1000 & 0.5250\\ 0.2400 & 0.1500 & 0.5000\\ 0.2450 & 0.2000 & 0.5000\end{bmatrix},\quad \overline{A}_\tau=\begin{bmatrix}0.3152 & 0.1000 & 0.5500\\ 0.2488 & 0.2500 & 0.5000\\ 0.2550 & 0.3000 & 0.5000\end{bmatrix},$$

$$\underline{B}=\begin{bmatrix}1.500 & 0.2\\ 1 & 0.1600\\ 0.2 & 0.2\end{bmatrix},\quad \overline{B}=\begin{bmatrix}1.400 & 0.1\\ 1 & 0.1400\\ 0.2 & 0.1\end{bmatrix},$$

$$\underline{C}=\begin{bmatrix}0.4800 & 0 & 0\\ 0.08 & 0.49 & 1\end{bmatrix},\quad \overline{C}=\begin{bmatrix}0.4900 & 0 & 0\\ 0.09 & 0.5 & 1\end{bmatrix},$$

$$\underline{C}_\tau=\begin{bmatrix}0.4800 & 0 & 0\\ 0.08 & 0.49 & 1\end{bmatrix},\quad \overline{C}_\tau=\begin{bmatrix}0.4900 & 0 & 0\\ 0.09 & 0.5 & 1\end{bmatrix}.$$

4.1. Centralized PD Controller

By solvig the constraints in Remark 1, the initial matrices can be calculated that

$$M_{\mathrm{P}}^{(1)}=\begin{bmatrix}0.1329 & 0.0377\\ 0.0918 & 0.0295\\ 0.0805 & 0.0318\end{bmatrix},\quad M_{\mathrm{D}_1}^{(1)}=\begin{bmatrix}-0.0290 & -0.0125\\ -0.0284 & -0.0154\\ -0.0237 & -0.0224\end{bmatrix},\quad M_{\mathrm{D}_2}^{(1)}=\begin{bmatrix}-0.0290 & -0.0125\\ -0.0284 & -0.0154\\ -0.0237 & -0.0224\end{bmatrix}.$$

By implementing Algorithm RPDCD using the YALMIP and Sedumi with MATLAB R2020b, a feasible solution is obtained as

$$K_{\mathrm{P}}=\begin{bmatrix}0.1106 & 0.0287\\ 0.0287 & 0.0426\end{bmatrix},\quad K_{\mathrm{D}}=\begin{bmatrix}-0.0293 & -0.0139\\ -0.0139 & -0.0106\end{bmatrix}. \tag{16}$$

Substituting (16) into the conditions 1) and 2) of Theorem 1, we can verify that

$$\underline{A} + \underline{B}K_{\mathrm{P}}\underline{C} + \overline{B}K_{\mathrm{P}}\widehat{D}\overline{C} = \begin{bmatrix} -4.3907 & 0.8469 & 1.8147 \\ 1.0691 & -4.2610 & 1.8787 \\ 0.7660 & 1.9804 & -6.0390 \end{bmatrix}$$

is a Metzler matrix and

$$\underline{A}_{\tau} + \underline{B}K_{\mathrm{P}}\underline{C}_{\tau} + \quad \overline{B}K_{\mathrm{P}}\widehat{D}\overline{C}_{\tau} = \begin{bmatrix} 0.1393 & 0.0069 & 0.3397 \\ 0.1291 & 0.0890 & 0.3787 \\ 0.2110 & 0.1804 & 0.4610 \end{bmatrix} \geqslant 0.$$

The matrix in (7) is

$$\begin{bmatrix} -4.1542 & 0.9104 & 2.2427 & 0.4678 & 0.2297 \\ 1.2462 & -3.9023 & 2.2633 & 0.3156 & 0.1560 \\ 1.2091 & 2.2766 & -4.5483 & 0.0865 & 0.0490 \\ 9.8000 & 0 & 0 & -10.0000 & 0 \\ 1.8000 & 10.0000 & 20.0000 & 0 & -10.0000 \end{bmatrix} \tag{17}$$

which have preserved the positivity of system (3). Furthermore, the eigenvalues of the corresponding matrix (17) are $\{-0.1114, -10.8005, -10.0231, -5.1187, -6.5509\}$, which have guaranteed the stability according to Theorem 1.

To show the robustness of the solution (16) in the system (6) with system matrices mentioned above

$$\dot{\tilde{x}}(t) = \left(\tilde{A}^{(n)} + \tilde{B}^{(n)}K\tilde{C}^{(n)}\right)\tilde{x}(t) + \left(\tilde{A}_{\tau}^{(n)} + \tilde{B}^{(n)}K\tilde{C}_{\tau}^{(n)}\right)\tilde{x}(t - \tau(t)) \tag{18}$$

where

$$\tilde{A}^{(n)} = \begin{bmatrix} A^{(n)} & 0 \\ \widehat{B}C^{(n)} & \widehat{A} \end{bmatrix}, \quad \tilde{B}^{(n)} = \begin{bmatrix} B^{(n)} \\ 0 \end{bmatrix}, \quad \tilde{C}^{(n)} = \begin{bmatrix} C^{(n)} & 0 \\ \widehat{D}C^{(n)} & \widehat{C} \end{bmatrix},$$

$$\tilde{A}_{\tau}^{(n)} = \begin{bmatrix} A_{\tau}^{(n)} & 0 \\ \widehat{B}C_{\tau}^{(n)} & 0 \end{bmatrix}, \quad \tilde{C}_{\tau}^{(n)} = \begin{bmatrix} C_{\tau}^{(n)} & 0 \\ \widehat{D}C_{\tau}^{(n)} & 0 \end{bmatrix}.$$

We select 100 systems with every system matrix value equally distributed in the given interval. For example, $A^{(n)} = \underline{A} + (n/100)\left(\overline{A} - \underline{A}\right)$. We generate corresponding system matrices, and hence we have 100 different systems for (18).

To simulate, the initial condition is chosen to be

$$\tilde{x}(t) = \begin{bmatrix} 0.1\sin(2t) + 1 \\ 0.2\cos(0.5t) + 1 \\ 0.3\sin(t) + 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in [-\overline{\tau}, 0].$$

We take $\tau_1(t) = 2 + \sin(t)$ and $\tau_2(t) = 10 + \sin(t)$ (see Fig. 1), respectively, and the state responses of 100 systems (6) with controller (16) are shown in Fig. 2 and Fig. 3. We can see that system states converge to zero for the 100 systems, hence the gain matrix in (16) guarantees the positivity and asymptotic stability of the system.

4.2. Decentralized PD Controller

The same system also has a diagonal feasible solution that

$$K_{\mathrm{P}} = \begin{bmatrix} 0.1186 & 0 \\ 0 & 0.1191 \end{bmatrix}, \quad K_{\mathrm{D}} = \begin{bmatrix} -0.0302 & 0 \\ 0 & -0.0679 \end{bmatrix}. \tag{19}$$

Substituting (19) into the conditions 1) and 2) of Theorem 1, we can verify that

$$\underline{A} + \underline{B}K_{\mathrm{P}}\underline{C} + \overline{B}K_{\mathrm{P}}\widehat{D}\overline{C} = \begin{bmatrix} -3.2461 & 1.0649 & 2.2501 \\ 1.8366 & -4.0966 & 2.2072 \\ 0.9500 & 2.1249 & -5.7499 \end{bmatrix}$$

is a Metzler matrix and

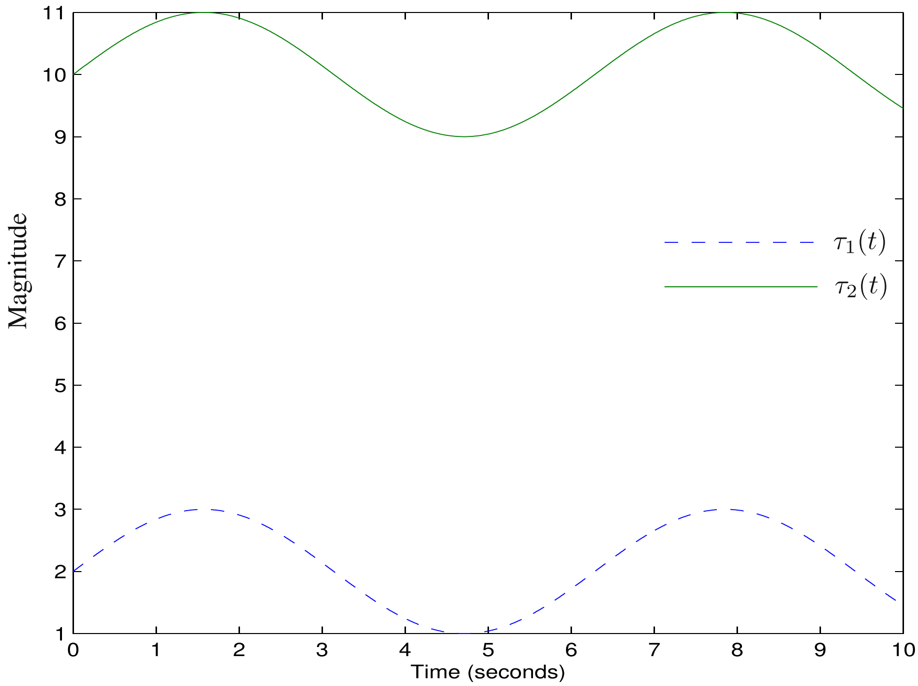

Fig. 1. Time-varying delays $\tau_1(t)$ and $\tau_2(t)$.

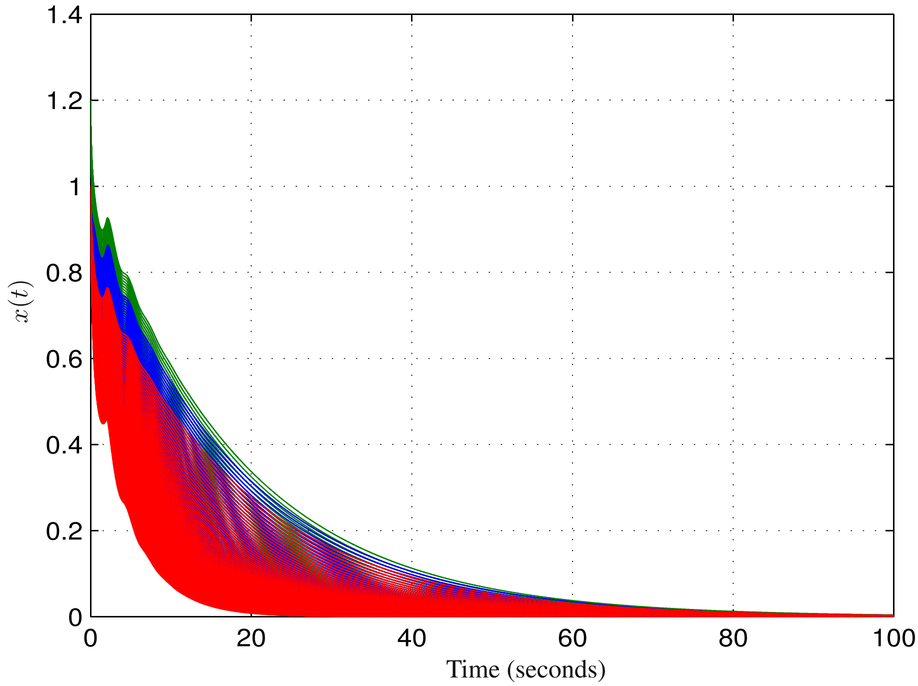

Fig. 2. State responses of system (1) with controller (16) and $\tau_1(t) = 2 + \sin(t)$.

$$\underline{A}_\tau + \underline{B}K_P\underline{C}_\tau + \quad \overline{B}K_P\widehat{D}\overline{C}_\tau = \begin{bmatrix} 1.2839 & 0.2249 & 0.7751 \\ 0.8966 & 0.2534 & 0.7072 \\ 0.3950 & 0.3249 & 0.7501 \end{bmatrix} \geqslant 0.$$

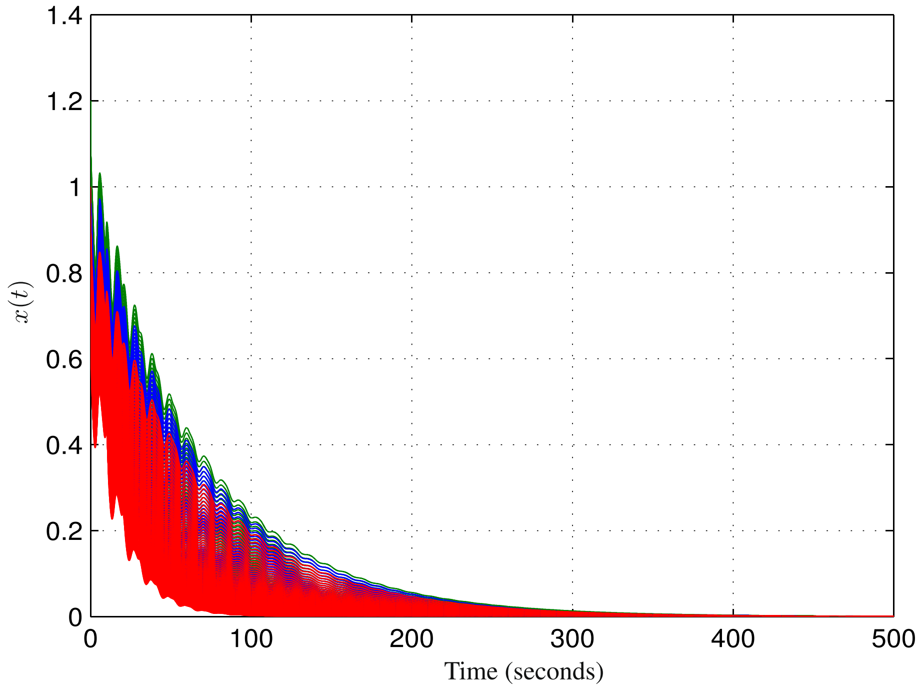

Fig. 3. State responses of system (1) with controller (16) and $\tau_2(t) = 10 + \sin(t)$.

The matrix in (7) is

$$\begin{bmatrix} -4.1296 & 1.0173 & 2.4618 & 0.4534 & 0.1358 \\ 1.2661 & -3.8611 & 2.3480 & 0.3022 & 0.1086 \\ 1.2136 & 2.2573 & -4.5882 & 0.0604 & 0.1358 \\ 9.8000 & 0 & 0 & -10.0000 & 0 \\ 1.8000 & 10.0000 & 20.0000 & 0 & -10.0000 \end{bmatrix} \tag{20}$$

which has preserved the positivity of system (3). Furthermore, the eigenvalues of the corresponding matrix in (20) are $\{-0.0406, -5.1326, -6.2617, -10.4027, -10.7412\}$, which have guaranteed the stability according to Theorem 1.

To show the robustness of the solution (16) in the system (6) with system matrices mentioned above

$$\dot{\tilde{x}}(t) = \left(\tilde{A}^{(n)} + \tilde{B}^{(n)} K \tilde{C}^{(n)}\right)\tilde{x}(t) + \left(\tilde{A}_\tau^{(n)} + \tilde{B}^{(n)} K \tilde{C}_\tau^{(n)}\right)\tilde{x}(t - \tau(t)). \tag{21}$$

where

$$\tilde{A}^{(n)} = \begin{bmatrix} A^{(n)} & 0 \\ \hat{B}C^{(n)} & \hat{A} \end{bmatrix}, \quad \tilde{B}^{(n)} = \begin{bmatrix} B^{(n)} \\ 0 \end{bmatrix}, \quad \tilde{C}^{(n)} = \begin{bmatrix} C^{(n)} & 0 \\ \hat{D}C^{(n)} & \hat{C} \end{bmatrix},$$

$$\tilde{A}_\tau^{(n)} = \begin{bmatrix} A_\tau^{(n)} & 0 \\ \hat{B}C_\tau^{(n)} & 0 \end{bmatrix}, \quad \tilde{C}_\tau^{(n)} = \begin{bmatrix} C_\tau^{(n)} & 0 \\ \hat{D}C_\tau^{(n)} & 0 \end{bmatrix}.$$

We select 100 systems with every system matrix value equally distributed in the given interval. For example, $A^{(n)} = \underline{A} + (n/100)\left(\overline{A} - \underline{A}\right)$. We generate corresponding system matrices, and hence we have 100 different systems for (18).

To simulate, we choose the same initial condition $\varphi(t)$ and time-varying delay $\tau(t)$ as those in the centralized case. Correspondingly, the state response of system (3) with controller (19) is shown in Figs. 3 and 4 and Figs. 4 and 5. We can see that system states converge to zero for the 100 systems, hence the gain matrix in (19) guarantees the positivity and asymptotic stability of the system.

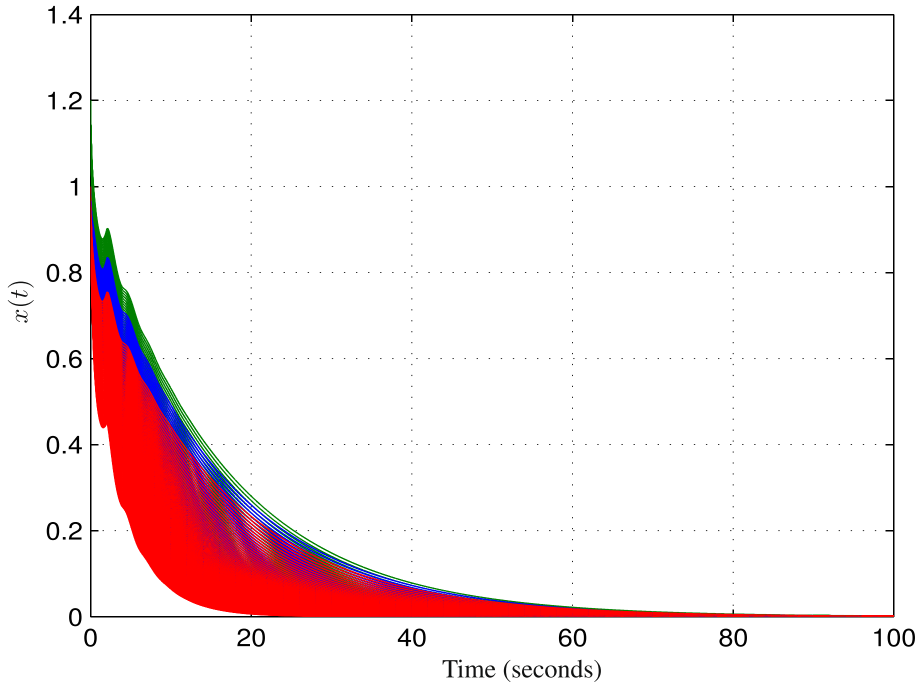

Fig. 4. State responses of system (1) with controller (19) and $\tau_1(t) = 2 + \sin(t)$.

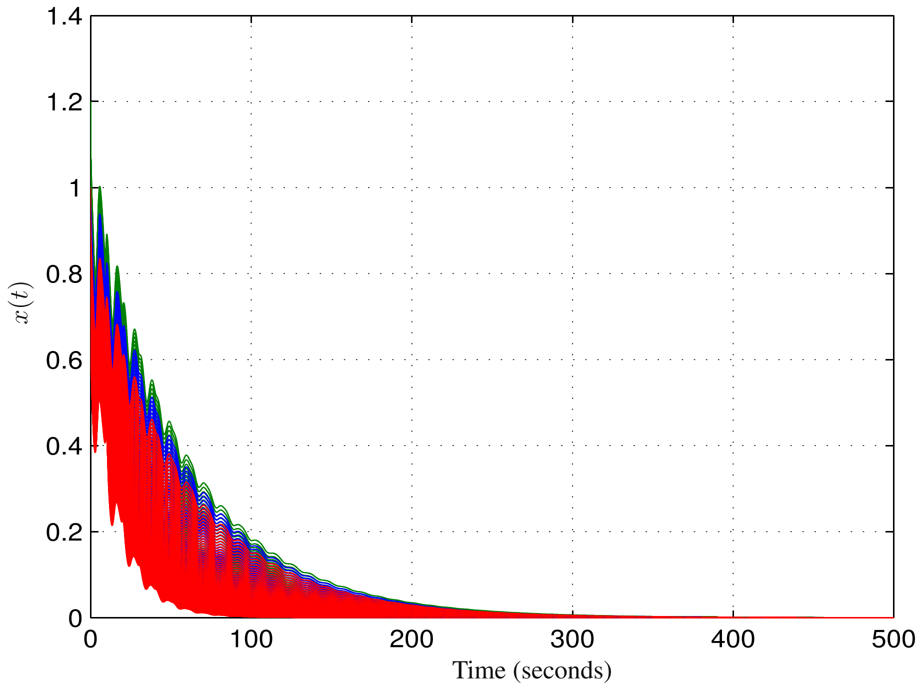

Fig. 5. State responses of system (1) with controller (19) and $\tau_2(t) = 10 + \sin(t)$.

5. Conclusion

In this paper, we have addressed and solved the robust PD control design problem for interval positive systems with time-varying delay. A systematic framework has been proposed for designing the multivariable PD controller that can preserve the stability and positivity. A semi-definite programming algorithm has been developed for multi-input positive systems. Two illustrative examples attested the numerical efficiency of the proposed PD control design methodologies. In the future, we will extend our approaches to address the synthesis issues of time-delay systems without positivity.

CRediT authorship contribution statement

Jason Jinrong Liu: Methodology, Conceptualization, Formal analysis, Writing - original draft, Software. **Maoqi Zhang:** Writing - original draft, Software, Methodology, Conceptualization, Formal analysis, Supervision, Funding acquisition, Writing - review & editing. **James Lam:** Conceptualization, Formal analysis, Funding acquisition. **Baozhu Du:** Conceptualization, Formal analysis, Funding acquisition. **Ka-Wai Kwok:** Conceptualization, Formal analysis, Supervision, Funding acquisition, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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