Brief paper

On positive realness, negative imaginariness, and H_∞ control of state-space symmetric systems☆

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ABSTRACT

This paper studies positive realness and negative imaginariness of state-space symmetric systems. First, a necessary and sufficient condition is provided for a state-space symmetric system to be, respectively, positive real and negative imaginary. Then an explicit formulation for the infimum of H_∞ norm under a mixed H_∞ and negative imaginary performance and an explicit solution for the associated optimal static output feedback control gain are developed based on symmetric negative imaginary theorem. Finally, three examples are provided to illustrate the main results of the paper.

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1. Introduction

Dissipativity is a notion relating energy storage and energy dissipation in dynamic systems (Brogliato, Lozano, Maschke, & Egeland, 2007; Meisami-Azad, Mohammadpour, & Grigoriadis, 2009). Dissipativity theory can be used in many areas of science and engineering, and it provides a framework to design and analyze control systems by using an input–output energy approach (Brogliato et al., 2007). In the past three decades, linear positive real (or nonlinear passivity-based) control systems, which can be considered as special situations of dissipative systems, have obtained enormous achievements both in theory and in practice (Brogliato et al., 2007). In particular, there are great interests in the problems of mixed H_∞ and output feedback synthesis for positive real systems. For example, the strictly positive real synthesis problems by using constant output feedback has been studied in Barkana (2004) and Huang, Ioannou, Maroulas, and Safonov (1999). The authors of Covacic, Teixeira, Assunção, and Gaino (2012) have proposed a linear matrix inequality based algorithm to find a constant output feedback matrix, such that the closed-loop system is strictly positive real. Research article (Chen & Wen, 1995) has addressed the positive realness preserving model reduction problem with associated H_∞ norm error bounds.

Negative imaginary systems theory, which was first established in Lanzon and Petersen (2008), has emerged as a useful complement to the positive real systems theory (Petersen, 2015, 2017) and attracted much attention among control theorists in recent years; see Bhowmick and Patra (2017), Ferrante and Ntogramatzidis (2013), Liu, Ono, Li, and Wu (2017), Liu and Xiong (2015), Liu and Xiong (2017) and Patra and Lanzon (2011). One of the main differences between positive real and negative imaginary systems is that the Nyquist plot of a Single-Input Single-Output (SISO) positive real transfer function is contained in the right half of the complex plane (Brogliato et al., 2007), while the positive-frequency Nyquist plot of a SISO negative-imaginary transfer function lies below the real axis (Petersen & Lanzon, 2010). Examples of such negative imaginary properties could be found in many practical systems, such as RLC networks (Petersen, 2015), large vehicle platoons (Cai & Hagen, 2010), active vibration systems by appropriately choosing inputs and outputs (Petersen & Lanzon, 2010). In addition, stability results of positive feedback interconnected negative imaginary systems, which are dependent on the system gains at zero and infinite frequency (Lanzon & Chen, 2017; Lanzon & Petersen, 2008), play an important role in robust control problems. Several important applications on negative imaginary stability results could be found in Bhikkaji, Moheimani, and Petersen (2012) and Karvinen and Moheimani (2014). Along this line of research, one of the main issues on negative imaginary systems theory is how to test the negative imaginary property efficiently. Methods based on frequency conditions, minimal state-space realization, spectral conditions and Riccati equations have been studied in Lanzon and Petersen (2008), Liu and Xiong (2015), Mabrok, Kallapur, Petersen, and Lanzon (2014) and Mabrok, Kallapur, Petersen, and Lanzon

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(2015), respectively. Moreover, the negative imaginary synthesis problems based on state feedback controller and output feedback controller have been addressed in Liu and Xiong (2015), Mabrok et al. (2015), Song, Lanzon, Patra, and Petersen (2010) and Xiong, Lam, and Petersen (2016).

State-space symmetric systems, which could lead to, but different from, transfer function matrix symmetry, arise very often in diverse engineering fields (Meisami-Azad et al., 2009; Tan & Grigoriadis, 2001). Physical systems include only one type of energy storage capability, such as structural systems with only elastic potential energy, and circuit networks with only electric energy or magnetic energy (e.g., RL or RC circuits), as well as a special system with zeros interlacing the poles (Liu, Sreeram, & Teo, 1998), can be modeled as such state-space symmetric systems (Meisami-Azad et al., 2009; Tan & Grigoriadis, 2001, 2007). Various problems about state-space symmetric systems have been discussed such as model reduction (Liu et al., 1998), dissipative control (Bara, 2012; Meisami-Azad et al., 2009), decentralized control (Yang, Wang, & Soh, 2001), stabilization and H_∞ control synthesis problems in continuous and discrete time cases (Tan & Grigoriadis, 2001, 2007) and controller failure time analysis problem (Zhai & Lin, 2004). Note that the obtained results show that state-space symmetric systems exhibit many nice properties. For instance, the H_∞ norm of such systems, the optimal static output feedback gain, and the optimal full-order dynamic output controller can be directly computed based on system matrices.

In this paper, we are interested in studying the negative imaginary control for linear time-invariant state-space symmetric systems. The research is motivated by that many negative imaginary systems in practice are symmetric, and partially motivated by the nice properties of state-space symmetric systems. The objective of the paper is to develop a simple and explicit testing method for state-space symmetric negative imaginary and positive real systems, and analytically provide the optimal H_∞ norm and the optimal control gain. The contributions of this paper are twofold: (1) A necessary and sufficient condition based on system matrix properties is provided to test the positive real and negative imaginary properties for state-space symmetric systems, respectively; (2) An explicit expression is obtained to compute the infimum of the closed-loop H_∞ norm under a mixed H_∞ and negative imaginary performance, and an explicit expression for associated optimal control gain is also derived.

The layout of the paper is as follows: In Section 2, some basic concepts and useful properties which will be used in later sections are presented. In Section 3, the main results which contain the positive real and negative imaginary lemmas for state-space symmetric systems are developed. In Section 4, static output feedback control for mixed H_∞ and negative imaginary control synthesis problems are considered. Section 5 presents three examples to illustrate the main results of the paper. Conclusions of the paper are given in Section 6.

Notation: $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrices. $\mathcal{R}^{m \times n}[s]$ denotes the set of $m \times n$ real–rational proper transfer function matrices in s. $\mathcal{R}^n$ denotes n-dimensional Euclidean space. $\text{Re}[\cdot]$ denotes the real part of complex numbers. A^T and A^* denote the transpose and the complex conjugate transpose of a complex matrix A, respectively. $\lambda_{\max}$ denotes the maximum eigenvalue for a square complex matrix with only real eigenvalues. $A > 0$ or $A \geq 0$ denotes a real symmetric positive definite or real symmetric positive semidefinite matrix. $A^\dagger$ denotes the Moore–Penrose generalized inverse of A. If A is full column rank and $A^*A = I$, then $A^\dagger = A^*$. For a matrix $A \in \mathbb{R}^{n \times m}$ with rank r ($r < n$), the orthogonal complement $A^\perp$ is defined as any $(n-r) \times n$ matrix (possibly non-unique) that satisfies $A^\perp A = 0$ and $A^\perp A^{\perp T} > 0$. $G(s) \sim (A, B, C, D)$ denotes that (A, B, C, D) is a state-space realization of $G(s)$. $\|G\|_\infty$ denotes the H_∞ norm of the transfer function matrix $G(s)$. I denotes an identity matrix with compatible dimensions. $\inf \gamma$ denotes the infimum of γ.

2. Preliminaries

In this section, we introduce some basic concepts and useful properties for state-space symmetric systems, positive real and negative imaginary systems.

2.1. State-space symmetric systems

Consider the following class of linear time-invariant systems:
$$\dot{x}(t) = Ax(t) + Bw(t),$$
$$z(t) = Cx(t) + Dw(t), \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, $m \leq n$, $x(t) \in \mathcal{R}^n$ is the state vector, $w(t) \in \mathcal{R}^m$ is the exogenous input vector, $z(t) \in \mathcal{R}^m$ is the system output vector, and (A, B, C, D) are assumed to be a minimal state-space realization of the system.

System (1) is called state-space symmetric (Tan & Grigoriadis, 2001) if the following conditions hold:

$$A = A^T, \quad C = B^T, \quad D = D^T.$$

It is clear that state-space symmetric system (1) implies external symmetry or system symmetry, that is, $G(s) = G^T(s)$, where $G(s) = C(sI - A)^{-1}B + D$ is the transfer function matrix of system (1). However, the converse does not hold, that is, some external symmetric systems do not admit a state-space symmetric realization.

Using a particular solution of the bounded real lemma, the authors of Tan and Grigoriadis (2001) have provided an explicit formula to compute the H_∞ norm of the stable state-space symmetric systems:

$$\|G\|_\infty = \max\{\lambda_{\max}(-D), \lambda_{\max}(G(0))\}. \tag{2}$$

2.2. Positive real and negative imaginary systems

The definitions of positive real and negative imaginary transfer function matrices are introduced in the following.

Definition 1 (*Anderson & Vongpanitlerd, 1973*). A square transfer function matrix $F(s)$ is positive real if

1. all the elements of $F(s)$ are analytic in $\text{Re}[s] > 0$;
2. $F(s)$ is real for real positive s;
3. $F^*(s) + F(s) \geq 0$ for $\text{Re}[s] > 0$.

Definition 2 (*Xiong, Petersen, & Lanzon, 2010*). A square real–rational proper transfer function matrix $G(s)$ is said to be negative imaginary if

1. $G(s)$ has no poles at the origin and in $\text{Re}[s] > 0$;
2. $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω, where $j\omega$ is a pole of $G(s)$;
3. If $s = j\omega_0$, $\omega_0 \in (0, \infty)$, is a pole of $G(s)$, it is at most a simple pole, and the residual matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)jG(s)$ is positive semidefinite Hermitian.

Remark 1. The concept of negative imaginary transfer function matrix has been extended to allow poles at the origin or infinity in Liu and Xiong (2016) and Mabrok et al. (2015). New version of negative imaginary lemma, which allows poles at the origin, has been introduced in Mabrok et al. (2015). In this paper, to better help the controller design and calculate the infimum of H_∞ norm for closed-loop state-space symmetric systems, we only consider real–rational proper negative imaginary systems without poles at the origin.

To determine whether a transfer function matrix is positive real or negative imaginary, the positive real lemma and negative imaginary lemma introduced in the following, provide a necessary and sufficient condition in terms of the minimal state-space realization of the system.

Lemma 1 (*Brogliato et al., 2007; Willems, 1976*). *Let (A, B, C, D) be a minimal state-space realization of a real–rational proper transfer function matrix $F(s) \in \mathcal{R}^{m \times m}[s]$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$. Then $F(s)$ is positive real if and only if there exists matrix $P > 0$, $P \in \mathbb{R}^{n \times n}$, such that*

$$\begin{pmatrix} PA + A^T P & PB - C^T \\ B^T P - C & -D - D^T \end{pmatrix} \leq 0. \tag{3}$$

Lemma 2 (*Lanzon & Petersen, 2008; Xiong et al., 2010*). *Let (A, B, C, D) be a minimal state-space realization of a real–rational proper transfer function matrix $G(s) \in \mathcal{R}^{m \times m}[s]$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$. Then $G(s)$ is negative imaginary if and only if*

1. $\det(A) \neq 0$, $D = D^T$.
2. *There exists a matrix $Y > 0$, $Y \in \mathbb{R}^{n \times n}$ such that*

$$AY + YA^T \leq 0, \quad and \quad B + AYC^T = 0. \tag{4}$$

The following three lemmas are useful to derive the main results of the paper.

Lemma 3 (*Meisami-Azad et al., 2009; Zhou, Doyle, & Glover, 1996*). *Consider real matrices Λ and Γ. Suppose $\Lambda = \Lambda^T > 0$. Then $\Lambda > \Gamma \Gamma^T$ if and only if $\lambda_{\max}(\Gamma^T \Lambda^{-1} \Gamma) < 1$.*

Lemma 4 (*Skelton, Iwasaki, & Grigoriadis, 1998; Tan & Grigoriadis, 2001, Generalized Finsler's Lemma*). *Consider real matrices M and Q such that M has full column rank and $Q = Q^T$. Then the following statements are equivalent:*

(i) *There exists a real matrix $X = X^T$ such that*

$$MXM^T - Q > 0. \tag{5}$$

(ii) *The following condition holds:*

$$M^{\perp} Q M^{\perp T} < 0. \tag{6}$$

If the above statements hold, then all matrices X satisfying (5) have the property

$$X > M^{\dagger}(Q - QM^{\perp T}(M^{\perp}QM^{\perp T})^{-1}M^{\perp}Q)M^{\dagger T}. \tag{7}$$

Lemma 5 (*Skelton et al., 1998, Bounded Real Lemma*). *Consider system (1), and let $\gamma > 0$ be given. Then, system (1) is stable and has H_{∞} norm less than γ if and only if there exists a matrix $P > 0$, $P \in \mathbb{R}^{n \times n}$, such that*

$$\begin{pmatrix} A^T P + PA & PB & C^T \\ B^T P & -\gamma I & D^T \\ C & D & -\gamma I \end{pmatrix} < 0.$$

3. Positive real and negative imaginary lemma for symmetric systems

In this section, we are concerned with the positive real and negative imaginary theorems for state-space symmetric systems. Necessary and sufficient conditions are established to characterize the positive real and negative imaginary properties of state-space symmetric systems.

3.1. State-space symmetric positive real theorem

The state-space symmetric positive real theorem in terms of minimal state-space realization is developed in this subsection.

Theorem 1. *The state-space symmetric system (1) is positive real if and only if*

$$A \leq 0, \quad and \quad D \geq 0.$$

Proof. (Sufficiency) Suppose $A \leq 0$, and $D \geq 0$. Then,

$$\begin{pmatrix} 2A & 0 \\ 0 & -2D \end{pmatrix} = \begin{pmatrix} 2A & B - C^T \\ B^T - C & -2D \end{pmatrix} \leq 0.$$

That is, inequality (3) is satisfied with $P = I$. Hence, according to Lemma 1, system (1) is positive real.

(Necessity) Suppose the state-space symmetric system (1) is positive real. Then, it follows from Lemma 1 that there exists a matrix $P = P^T > 0$ such that

$$\begin{pmatrix} AP + PA & PB - B \\ B^T P - B^T & -2D \end{pmatrix} \leq 0. \tag{8}$$

Next, we will show that $P = I > 0$ is a solution of (8). Since $P > 0$, it has a singular value decomposition as follows:

$$P = U \Sigma_0 U^T, \quad U^T = U^{-1}, \quad \Sigma_0 = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} > 0,$$

where $\sigma_i > 0$, $i = 1, 2, \ldots, n$, are the singular values (or eigenvalues) of P. Let

$$\bar{A} = U^T A U, \quad \bar{B} = U^T B, \quad \bar{C} = CU, \quad \bar{D} = D.$$

It can be found that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is also a minimal realization of system (1) and state-space symmetric. Then, pre-multiplying of (8) by $\begin{pmatrix} U^T & 0 \\ 0 & I \end{pmatrix}$ and post-multiplying of (8) by $\begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$, we obtain the following equivalent condition:

$$\begin{pmatrix} U^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} AP + PA & PB - B \\ B^T P - B^T & -2D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \leq 0,$$

which is equivalent to

$$\begin{pmatrix} \bar{A}\Sigma_0 + \Sigma_0 \bar{A} & \Sigma_0 \bar{B} - \bar{B} \\ \bar{B}^T \Sigma_0 - \bar{B}^T & -2\bar{D} \end{pmatrix} \leq 0. \tag{9}$$

Then, pre- and post-multiplying of (9) by $\begin{pmatrix} \Sigma_0^{-1} & 0 \\ 0 & -I \end{pmatrix}$, where $\Sigma_0^{-1} = \mathrm{diag}\{\sigma_1^{-1}, \sigma_2^{-1}, \ldots, \sigma_n^{-1}\} = (\Sigma_0^{-1})^T > 0$, it follows that

$$\begin{pmatrix} \bar{A}\Sigma_0^{-1} + \Sigma_0^{-1}\bar{A} & \Sigma_0^{-1}\bar{B} - \bar{B} \\ \bar{B}^T \Sigma_0^{-1} - \bar{B}^T & -2\bar{D} \end{pmatrix} \leq 0. \tag{10}$$

Hence, according to inequalities (9) and (10), one has that both Σ_0 and Σ_0^{-1} are solutions of (9). Since $\sigma_1 > 0$, there exists $0 < \lambda_1 < 1$ such that $\lambda_1 \sigma_1 + (1 - \lambda_1)\sigma_1^{-1} = 1$. Then, by computing the linear combination of (9) and (10), that is, $\lambda_1 \times (9) + (1 - \lambda_1) \times (10)$, we obtain

$$\begin{pmatrix} \bar{A}\Sigma_1 + \Sigma_1 \bar{A} & \Sigma_1 \bar{B} - \bar{B} \\ \bar{B}^T \Sigma_1 - \bar{B}^T & -2\bar{D} \end{pmatrix} \leq 0, \tag{11}$$

where

$$\Sigma_1 = \mathrm{diag}\{\lambda_1 \sigma_1 + (1 - \lambda_1)\sigma_1^{-1}, \lambda_1 \sigma_2 + (1 - \lambda_1)\sigma_2^{-1}, \ldots,$$
$$\lambda_1 \sigma_n + (1 - \lambda_1)\sigma_n^{-1}\}$$
$$= \mathrm{diag}\{1, \lambda_1 \sigma_2 + (1 - \lambda_1)\sigma_2^{-1}, \ldots, \lambda_1 \sigma_n + (1 - \lambda_1)\sigma_n^{-1}\}$$
$$\triangleq \mathrm{diag}\{1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n\} > 0.$$

Similar as before, pre- and post-multiplying of (11) by $\begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & -I \end{pmatrix}$, a similar condition for Σ_1^{-1} as in (9) is obtained as follows

$$\begin{pmatrix} \bar{A}\Sigma_1^{-1} + \Sigma_1^{-1}\bar{A} & \Sigma_1^{-1}\bar{B} - \bar{B} \\ \bar{B}^T \Sigma_1^{-1} - \bar{B}^T & -2\bar{D} \end{pmatrix} \leq 0. \tag{12}$$

It follows from (11) and (12) that both Σ_1 and Σ_1^{-1} are solutions of (9). Since $\tilde{\sigma}_2 > 0$, there exists $0 < \lambda_2 < 1$ such that $\lambda_2\tilde{\sigma}_2 + (1 - \lambda_2)\tilde{\sigma}_2^{-1} = 1$. Then, the corresponding linear combination of (11) and (12) results in

$$\begin{pmatrix} \bar{A}\Sigma_2 + \Sigma_2\bar{A} & \Sigma_2\bar{B} - \bar{B} \\ \bar{B}^T\Sigma_2 - \bar{B}^T & -2\bar{D} \end{pmatrix} \leq 0, \tag{13}$$

where

$$\begin{aligned} \Sigma_2 &= \mathrm{diag}\{1, \lambda_2\tilde{\sigma}_2 + (1 - \lambda_2)\tilde{\sigma}_2^{-1}, \lambda_2\tilde{\sigma}_3 + (1 - \lambda_2)\tilde{\sigma}_3^{-1}, \dots, \\ &\qquad \lambda_2\tilde{\sigma}_n + (1 - \lambda_2)\tilde{\sigma}_n^{-1}\} \\ &= \mathrm{diag}\{1, 1, \lambda_2\tilde{\sigma}_3 + (1 - \lambda_2)\tilde{\sigma}_3^{-1}, \dots, \lambda_2\tilde{\sigma}_n + (1 - \lambda_2)\tilde{\sigma}_n^{-1}\} \\ &\triangleq \mathrm{diag}\{1, 1, \hat{\sigma}_3, \dots, \hat{\sigma}_n\} > 0. \end{aligned}$$

By repeating this process, we can construct $\Sigma_n = I$ satisfying (9). That is, $P = U\Sigma_n U^T = I$ is a solution of (8). It follows that

$$\begin{pmatrix} 2A & 0 \\ 0 & -2D \end{pmatrix} \leq 0,$$

and hence $A \leq 0$ and $D \geq 0$. This completes the proof.

Remark 2. Consider the necessity proof of Theorem 1. According to inequality (8), we know that $-2D \leq 0$ and there exists a matrix $P > 0$ such that $AP + PA \leq 0$. Since (A, B, C, D) is a minimal realization of system (1) and $A = A^T$, A has no poles in $\mathrm{Re}[s] > 0$. We can directly obtain that $A \leq 0$ and $D \geq 0$, which does not show that the identity matrix I is a solution of (8). However, the proof in Theorem 1 shows that the identity matrix I is a solution of (8), which is not only a nice property of state-space symmetric positive real theorem, but also will be used in the proof of Theorem 3 later. The idea of Theorem 1 and its proof are motivated by Lemma 2 in Tan and Grigoriadis (2001).

Suppose system (1) has no poles at the origin, we have the following corollary.

Corollary 1. *Suppose the state-space symmetric system* (1) *has no poles at the origin. Then, system* (1) *is positive real if and only if*

$$A < 0, \quad and \quad D \geq 0.$$

Proof. The proof of sufficiency is obvious. For necessity, we know that (A, B, C, D) is a minimal state-space realization of system (1) and system (1) has no poles at the origin, and thus A has no poles at the origin. Moreover, $A = A^T$ implies that all the eigenvalues of matrix A are real numbers. In other words, A has no poles on the imaginary axis. Then, it follows from Theorem 1 that $A < 0$ and $D \geq 0$.

3.2. State-space symmetric negative imaginary theorem

In this subsection, the negative imaginary theorem for state-space symmetric systems is developed in terms of the system state matrix.

Theorem 2. *The state-space symmetric system* (1) *is negative imaginary if and only if*

$$A < 0.$$

Proof. (Sufficiency) Suppose $A < 0$. Then, we have

$$-AA^{-1} - A^{-1}A = -2I < 0,$$
$$B - AA^{-1}B = 0.$$

That is, Condition 2 of Lemma 2 holds with $Y = -A^{-1} > 0$. So, it follows from Lemma 2 that system (1) is negative imaginary.

(Necessity) Suppose the state-space symmetric system (1) is negative imaginary. We know that $A = A^T$, and (A, B, C, D) is a minimal state-space realization. Meanwhile, Condition 1 of 3. implies that system (1) has no poles at the origin and in $\mathrm{Re}[s] > 0$. Hence, $A < 0$.

Remark 3. Theorem 2 cannot be directly proved by using the relationship between positive real and negative imaginary systems, and the state-space symmetric positive real theorem. The reason is that $F(s) = s[G(s) - G(\infty)] \sim (A, B, CA, CB)$ is not state-space symmetric. An equivalent system representation of $F(s) \sim (A, B, CA, CB)$ is given by $F(s) \sim (A^{\frac{1}{2}}AA^{-\frac{1}{2}}, A^{\frac{1}{2}}B, CAA^{-\frac{1}{2}}, CB) \sim (A, A^{\frac{1}{2}}B, B^TA^{\frac{1}{2}}, B^TB)$, which is state-space symmetric, but $A^{\frac{1}{2}}$ could be a complex matrix when $A < 0$.

Remark 4. Based on Theorem 2, it can be found that a stable state-space symmetric system is always negative imaginary, and state-space symmetric positive real system without poles at the origin is also negative imaginary. The H_∞ norm of state-space symmetric negative imaginary systems can be directly calculated by using (2). When removing the minimal realization assumption, the condition in Theorem 2 is sufficient to test the negative imaginary properties of state-space symmetric systems, which can be proved by using Corollary 1 in Song, Lanzon, Patra, and Petersen (2012) and the necessity proof of Theorem 2. When system (1) has poles at the origin, we can use Lemma 2 in Mabrok et al. (2015) to derive the generalized state-space negative imaginary lemma by allowing poles at the origin. Also, note that the state-space symmetric negative imaginary system in this paper is impossible to be lossless negative imaginary, because the system is stable, while all the poles of lossless negative imaginary systems lie on the imaginary axis.

4. Mixed H_∞ and negative imaginary control synthesis problem

Consider a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t), \\ y(t) &= C_2 x(t), \end{aligned} \tag{14}$$

where $x(t) \in \mathcal{R}^n$ is the system state, $w(t) \in \mathcal{R}^m$ is the system input, $u(t) \in \mathcal{R}^p$ is the control input, $p \leq n$, $m \leq n$, $z(t) \in \mathcal{R}^m$ is the system output, $y(t) \in \mathcal{R}^p$ is the measured output. The matrices $A \in \mathbb{R}^{n \times n}$, $B_1 \in \mathbb{R}^{n \times m}$, $B_2 \in \mathbb{R}^{n \times p}$, $C_1 \times \mathbb{R}^{m \times n}$ and $C_2 \in \mathbb{R}^{p \times n}$ are known constant matrices. B_2 is assumed to be of full column rank. The system is chosen to be strictly proper to keep the results tractable and simple. System (14) is called to be state-space symmetric if the following conditions hold:

$$A = A^T, \quad B_1 = C_1^T, \quad B_2 = C_2^T. \tag{15}$$

The aim of this section is to design a symmetric static output feedback control law

$$u(t) = -Fy(t), \tag{16}$$

where $F = F^T$, such that the resulting closed-loop system is negative imaginary with H_∞ norm less than a given scalar $\gamma > 0$. The closed-loop system of plant (14) with controller (16) is given by

$$\begin{aligned} \dot{x}(t) &= (A - B_2 F C_2)x(t) + B_1 w(t), \\ z(t) &= C_1 x(t). \end{aligned} \tag{17}$$

The closed-loop system $(A - B_2 F C_2, B_1, C_1, 0)$ is also state-space symmetric.

The following lemma provides a necessary and sufficient stabilizability condition for state-space symmetric systems by output feedback control.

Lemma 6 (*Tan & Grigoriadis, 2001*). *Consider the state-space symmetric system represented by* (14). *Then, there exists a symmetric static output feedback control law* (16) *stabilizes the closed-loop system if and only if the following condition holds:*

$$B_2^\perp A B_2^{\perp T} < 0. \tag{18}$$

The following theorem provides explicit expressions for the infimum of the closed-loop H_∞ norm under a mixed H_∞ and negative imaginary performance, and the corresponding controller gain range. The idea of Theorem 3 and its proof are motivated by Theorem 8 in Meisami-Azad et al. (2009) and the comments in Bara (2012).

Theorem 3. *Consider the state-space symmetric system represented by* (14) *and the control law represented by* (16). *Suppose that system* (14) *is stabilizable, that is, condition* (18) *holds. Then, the infimum of the closed-loop H_∞ norm γ_{inf} under a mixed H_∞ and negative imaginary performance by symmetric static output feedback control, can be computed from*

$$\gamma_{\text{inf}} \triangleq \inf \gamma = \lambda_{\max}[B_1^T B_2^{\perp T}(-B_2^\perp A B_2^{\perp T})^{-1} B_2^\perp B_1]. \tag{19}$$

For any $\gamma > \gamma_{\text{inf}}$, a symmetric static output feedback gain, which renders the closed-loop system negative imaginary with H_∞ norm less than γ, can be selected as

$$F > B_2^\dagger[\Delta - \Delta B_2^{\perp T}(B_2^\perp \Delta B_2^{\perp T})^{-1} B_2^\perp \Delta]B_2^{\dagger T}, \tag{20}$$

where $\Delta = A + \frac{1}{\gamma} B_1 B_1^T$.

Proof. We know that the closed-loop system (17) is state-space symmetric. According to Theorem 2 and the bounded real lemma, the closed-loop system (17) is negative imaginary with H_∞ norm less than γ if and only if the closed-loop system (17) is stable with H_∞ norm less than γ, which is equivalent to the following matrix inequality holds

$$\begin{pmatrix} (A - B_2 F B_2^T)P + P(A - B_2 F B_2^T) & PB_1 & B_1 \\ B_1^T P & -\gamma I & 0 \\ B_1^T & 0 & -\gamma I \end{pmatrix} < 0. \tag{21}$$

Using a similar argument as in the proof of Theorem 1 and the proof of Lemma 2 in Tan and Grigoriadis (2001), it can be found that $P = I$ is a solution of (21). Hence,

$$\begin{pmatrix} 2A - 2B_2 F B_2^T & B_1 & B_1 \\ B_1^T & -\gamma I & 0 \\ B_1^T & 0 & -\gamma I \end{pmatrix} < 0. \tag{22}$$

Using Schur complement equivalence, we obtain

$$\begin{pmatrix} -\gamma I & 0 \\ 0 & -\gamma I \end{pmatrix} < 0, \tag{23}$$

and

$$A + \frac{1}{\gamma} B_1 B_1^T < B_2 F B_2^T. \tag{24}$$

It is obvious that (23) holds. According to the definition of orthogonal complement, we know that $B_2^{\perp T}$ is of full column rank. By pre- and post-multiplying (24) with $B_2^\perp$ and $B_2^{\perp T}$, it follows that

$$B_2^\perp(A + \frac{1}{\gamma} B_1 B_1^T)B_2^{\perp T} < 0, \tag{25}$$

and hence

$$\frac{1}{\gamma} B_2^\perp B_1 B_1^T B_2^{\perp T} < -B_2^\perp A B_2^{\perp T}. \tag{26}$$

Note that (25) also holds by applying the Generalized Finsler's Lemma on (24). Then, applying Lemma 3 on (26) results in

$$\lambda_{\max}\left[\frac{1}{\gamma} B_1^T B_2^{\perp T}(-B_2^\perp A B_2^{\perp T})^{-1} B_2^\perp B_1\right] < 1. \tag{27}$$

One has that

$$\gamma > \lambda_{\max}[B_1^T B_2^{\perp T}(-B_2^\perp A B_2^{\perp T})^{-1} B_2^\perp B_1]. \tag{28}$$

That is, $\inf \gamma = \lambda_{\max}[B_1^T B_2^{\perp T}(-B_2^\perp A B_2^{\perp T})^{-1} B_2^\perp B_1]$, which provides the infimum of the H_∞ norm γ_{inf} as described in (19). Then, applying the Generalized Finsler's Lemma on (24) again, we obtain that F satisfies the condition in (20). ∎

Remark 5. Let $\gamma = \alpha \gamma_{\text{inf}}$ with $\alpha > 1$, the selected optimal control gain F as described in (20) ensures that the closed-loop system is negative imaginary; meanwhile, the H_∞ norm of the closed-loop system is less than $\alpha \gamma_{\text{inf}}$. Similarly, static output feedback control for mixed H_∞ and positive real control synthesis problem can be considered by using the similar method in Theorem 3.

Remark 6. Compared with Theorem 8 in Meisami-Azad et al. (2009), the explicit formulation of the H_∞ norm and the associated optimal control gain in Meisami-Azad et al. (2009, Theorem 8) are related to the trade-off parameter $\theta \in (0, 1)$. However, our explicit expressions of Theorem 3 in (19) and (20) are independent of the trade-off parameter θ and consistent with the comments in Bara (2012). As pointed out in Bara (2012), the H_∞ norm of a stable state-space symmetric system is a constant value for a given system, which only depends on the system state-space matrices, it should not depend on the trade-off parameter θ. Note that condition (18) is needed to confirm that the system is stabilizable.

5. Numerical examples

Three examples are provided in this section to illustrate the main results of the paper. The first example demonstrates the application of state-space symmetric negative imaginary theorem. The second example validates the explicit expressions of closed-loop optimal H_∞ norm and the associated control gain to Multi-Input Multi-Output (MIMO) systems. One SISO example that shows the relationship between optimal H_∞ norm and optimal control gain is studied in the third example.

Example 1. Consider an RL circuit network as depicted in Fig. 1. This RL circuit network is taken from Meisami-Azad et al. (2009). We take the currents through the three inductors $L_i, i = 1, 2, 3$, as the state variables $x_i(t), i = 1, 2, 3$, take the signal $\frac{V_d}{L_1}$ as the system input $w(t)$, and take the current i as the output $z(t)$. Then, this results in the following open-loop system:

$$\dot{x}(t) = \begin{pmatrix} \dfrac{-R_2}{L_1} & \dfrac{R_2}{L_1} & 0 \\ \dfrac{R_2}{L_2} & \dfrac{-(R_2 + R_3)}{L_2} & \dfrac{R_3}{L_2} \\ 0 & \dfrac{R_3}{L_3} & \dfrac{-(R_3 + R_4)}{L_3} \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} w(t),$$

$$z(t) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) + \frac{L_1}{R_1} w(t). \tag{29}$$

If $L_1 = L_2 = L_3$, then system (29) is state-space symmetric. Let $L_1 = L_2 = L_3 = 1\,H, R_1 = 2\,\Omega, R_2 = 3\,\Omega, R_3 = 4\,\Omega$, and $R_4 = 5\,\Omega$,
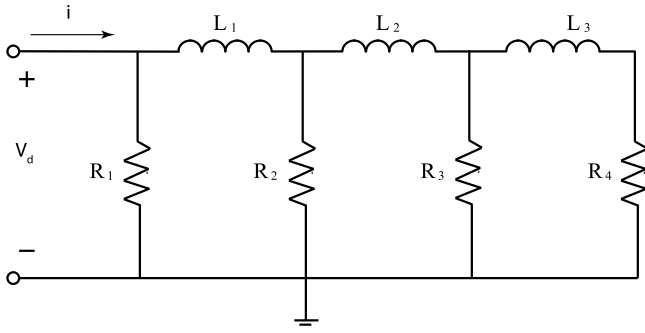
Fig. 1. RL circuit network.

the system matrices A, B, C and D of system (29) are given by

$$A = \begin{pmatrix} -3 & 3 & 0 \\ 3 & -7 & 4 \\ 0 & 4 & -9 \end{pmatrix}, \quad B = C^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad D = \frac{1}{2}. \tag{30}$$

It can be found that $A = A^T < 0$, and hence this system is negative imaginary according to Theorem 2. Moreover, YALMIP (Lofberg, 2004) and SeDuMi were used to find a solution

$$Y = \begin{pmatrix} 0.7833 & 0.4500 & 0.2000 \\ 0.4500 & 0.5324 & 0.2298 \\ 0.2000 & 0.2298 & 0.2721 \end{pmatrix} > 0,$$

which satisfies the inequality and equality conditions of (4), and hence illustrates that system (29) with the data in (30) is negative imaginary. Meanwhile, system (29) with the data in (30) is also asymptotically stable and positive real according to Corollary 1.

Example 2 (*MIMO System*). Consider an MIMO state-space symmetric system (14) with data given by

$$A = \begin{pmatrix} -3 & 2 & 0 & 1 \\ 2 & -2 & -1 & 0 \\ 0 & -1 & -3 & 1 \\ 1 & 0 & 1 & -5 \end{pmatrix}, \quad B_1 = C_1^T = \begin{pmatrix} 3 & 2.5 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix},$$

$$B_2 = C_2^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The open-loop system $(A_1, B_1, C_1, 0)$ is asymptotically stable. Using the explicit formula (2), it can be computed that the H_∞ norm of the open-loop system is $\gamma^* = 35.1145$. Consider the symmetric static output feedback H_∞ control design problem. Using (19) in Theorem 3, the infimum of the closed-loop system H_∞ norm is calculated as $\gamma_{\text{inf}} = 0.4286$. For $\gamma = 1.5\gamma_{\text{inf}} > \gamma_{\text{inf}}$, a symmetric static output feedback controller, which leads to the closed-loop system satisfies negative imaginary properties with H_∞ norm less than γ, can be selected explicitly by using Theorem 3 as follows:

$$F > \begin{pmatrix} 37.1367 & -7.5451 \\ -7.5451 & 7.6253 \end{pmatrix}.$$

Example 3 (*SISO System*). Consider a SISO state-space symmetric system (14) with data given by

$$A = \begin{pmatrix} -5 & 0 & 2 \\ 0 & -1 & -1 \\ 2 & -1 & -2 \end{pmatrix}, \quad B_1 = C_1^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

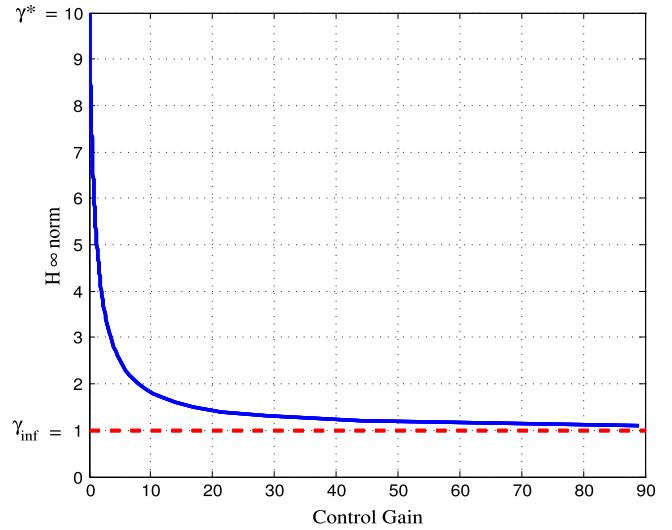$$B_2 = C_2^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Fig. 2. Variation of the closed-loop H_∞ norm with the control gain F.

It can be computed that the H_∞ norm of the open-loop system $(A, B_1, C_1, 0)$ is $\gamma^* = 10$. Using (19) in Theorem 3, we explicitly compute the infimum of closed-loop H_∞ norm as $\gamma_{\text{inf}} = 1$. For $\gamma = 1.6 > \gamma_{\text{inf}}$, a symmetric static output feedback controller, which leads to the closed-loop system satisfies negative imaginary properties with H_∞ norm less than γ, can be chosen by $F > 14$. Fig. 2 shows that the variation of the closed-loop H_∞ norm as a function of the control gain F which confirms the negative imaginary and H_∞ norm performances. When $-1 < F < 0$, the closed-loop H_∞ norm is larger than 10; when $F < -1$, the closed-loop system is unstable.

6. Conclusions

This paper has studied the static output feedback negative imaginary synthesis problem for state-space symmetric systems. A necessary and sufficient condition involving only system state matrix and feedforward matrix has been developed to characterize the state-space symmetric positive real and negative imaginary systems, respectively. An explicit expression for the infimum H_∞ norm of the closed-loop system that satisfies negative imaginary properties, and the associated optimal control gain have been developed. The results presented in the paper have provided a convenient tool to test the state-space symmetric negative imaginary properties, and provided an explicit solution to compute the optimal H_∞ norm and control gain. Especially for large-scale systems, the explicit solution has obvious computational advantages.

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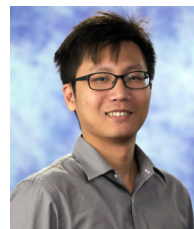
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