Reachable Set Estimation and Synthesis for Periodic Positive Systems

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***Abstract*—This paper investigates the problems of reachable set estimation and synthesis for periodic positive systems with two different exogenous disturbances. The lifting method and the pseudoperiodic Lyapunov function method are adopted for the estimation problem. The reachable set bounding conditions are proposed by employing Lyapunov-based inequalities and the S-procedure technique. Two optimization methods are used to minimize the bounding hyper-pyramids of the reachable set. In addition, the state-feedback controller design conditions that make the reachable set of closed-loop systems lie within a given hyper-pyramid are derived. Finally, numerical examples are presented to illustrate the validity of the obtained conditions.**

***Index Terms*—Hyper-pyramid, periodic positive systems, reachable set estimation, S-procedure, state-feedback control.**

I. Introduction

PERIODIC systems, whose parameters have periodic features, have attracted much attention in the past few decades. Numerous practical systems in the real world have periodic properties and, thus, can be modeled as periodic systems. For example, in economics, the business cycles and seasonal effects can be captured by periodic models. A pendulum, which has cyclic behavior, can be modeled as a periodic system. In terms of control, periodic control is a proven effective way for improving the system performance for time-invariant plants [1], [2]. Because of the theoretical and practical importance of periodic systems, problems concerning periodic systems have been studied in [3] and [4]. Recently, periodic systems with positive characteristics have attracted increasing attention from researchers. Systems are called positive if for any non-negative initial conditions and input signals, their states and output signals stay in the non-negative orthant. Bougatef *et al.* [5] and Rami and Napp [6] investigated the stability and stabilization problems of discrete-time periodic positive systems. Periodic positive systems with time delays were considered in [7] for the stability analysis problem. For readers who are interested in the research results of periodic systems, see also the monograph [8].

As a fundamental concept of control theory, reachability has received growing attention in recent years. The reachable set of a system is defined as the set of all system states that are reachable from the origin under given system inputs. In most cases, the exact characterization of the reachable set for a system is impossible. A common strategy for studying the reachable set is to find a region that is as small as possible to bound the reachable set. The estimation criteria for seeking the bounding ellipsoids for the reachability of linear time-invariant systems (LTIs) with different classes of inputs (such as unit-peak inputs and componentwise unit-energy inputs) were presented in the monograph [9]. In [10], the problem of recursively estimating the state uncertainty set of discrete-time systems was investigated. Over the past decade, with the development of multiple Lyapunov functional approaches, Lyapunov functional methods and some new inequalities, researchers took delays and uncertainties problems into account when investigating the reachability of LTI systems. In [11], sufficient criteria for seeking the bounding ellipsoids for the reachability of LTI systems with time-varying delays and polytopic uncertainties were derived by employing the Lyapunov–Razumikhin method. However, the obtained conditions in [11] are difficult to check as they contain many nonlinear terms. In order to reduce the computational complexity, the modified Lyapunov–Krasovskii (L-K) functional method was used to investigate the problem of the reachable set estimation in [12], which can be checked conveniently. Then, the reachable set estimation of discrete-time systems with integral bounds was addressed in [13]. In addition, a novel maximal L-K functional approach was proposed in [14] to reduce the conservatism of the reachable set bounding criteria. Besides the above work, Nam and Pathirana [15] and Kwon *et al.* [16] considered the reachable set estimation problem for LTI

Manuscript received July 6, 2018; revised December 6, 2018 and February 7, 2019; accepted March 20, 2019. This work was supported in part by the General Research Fund under Grant 17205815 and Grant 17202317, in part by the Natural Science Foundation of Jiangsu Province under Grant BK20160810, in part by the National Natural Science Foundation of China under Grant 61603180, in part by the Tencent Rhinoceros Birds-Scientific Research Foundation for Young Teachers of Shenzhen University, and in part by the National Taipei University of Technology-Shenzhen University Joint Research Program. This paper was recommended by Associate Editor W. Yu. *(Corresponding author: Yukang Cui.)*

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Digital Object Identifier 10.1109/TCYB.2019.2908676

systems with different types of time delays. Many important practical systems are actually nonlinear and the so-called Takagi–Sugeno (T–S) fuzzy model has been extensively used to deal with the problem of analysis and synthesis for complex nonlinear systems under disturbances [17], [18]. Motivated by the safety monitoring of nonlinear systems, the estimation of the reachable set of time-delay T–S fuzzy systems with bounded input disturbances and nonzero initial conditions was studied in [19]. Some researchers investigated the reachable set estimation problem for other types of systems, such as time-delay neural networks and polytopic uncertainties [20]; linear time-varying systems with delays [21]; bilinear systems [22]; switched systems [23], [24]; large-scale nonlinear systems [25]; gene expression system [26]; singular system [27]; and positive systems [28]. Moreover, the reachable set estimation criteria of discrete-time systems were proposed in [29] and [30] and zonotopic guaranteed state estimation for uncertain systems was studied in [31]. In light of the reachability analysis results, researchers investigated the reachable set synthesis problem [23], [32]. These reachable set estimation techniques, however, do not capture the periodic and positivity characteristics of the periodic positive system, which inevitably leads to conservative results. The above reasons motivate this paper.

Notation: The notations employed are fairly standard. $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}^n$ denote the set of positive integers, non-negative integers, real numbers, and n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices. $\bar{\mathbb{R}}_+$ denotes the set of non-negative real numbers and $\mathbb{R}_+$ denotes the set of positive real numbers. Vector x and matrix A with all elements positive (non-negative) are denoted as $x >> 0$ ($x \geq\geq 0$) and $A >> 0$ ($A \geq\geq 0$). $\mathbf{1}$ represents column vectors with each entry being 1. The superscript T represents the matrix transpose. The norms $\|\omega_k\|_1$, $\|\omega\|_{1,1}$, and $\|\omega\|_{\infty,1}$ are defined as $\sum_{i=1}^{n_\omega}|\omega_{ki}|$, $\sum_{k=0}^{\infty}\|\omega_k\|_1$, and $\sup_{k\geq 0}\|\omega_k\|_1$, respectively.

II. Problem Formulation and Preliminaries

The discrete-time periodic positive systems considered can be described by the following difference equation:

$$x_{k+1} = A_k x_k + B_{\omega,k}\omega_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector and $\omega_k \in \mathbb{R}^{n_\omega}$ is the disturbance vector. $A_k = A_{k+lN} \geq\geq 0$ and $B_{\omega,k} = B_{\omega,k+lN} \geq\geq 0$, $l \in \mathbb{N}_0$, are N-periodic real matrices with appropriate dimensions.

Two possible classes of exogenous disturbances ω will be studied for the reachability analysis and synthesis problems

$$\omega \in \bar{\Omega}^+_{1,1} \triangleq \left\{\omega \middle| \|\omega\|_{1,1} \leq 1,\ \omega_k \geq\geq 0\right\} \tag{2}$$

$$\omega \in \bar{\Omega}^+_{\infty,1} \triangleq \left\{\omega \middle| \|\omega\|_{\infty,1} \leq 1,\ \omega_k \geq\geq 0\right\}. \tag{3}$$

The reachable sets of system (1) are denoted as

$$\Re_x\left(\bar{\Omega}^+_{1,1}\right) \triangleq \Big\{x_k \big| x_0 = 0,\ x_k,\ \omega_k \text{ satisfy (1) and } \omega \in \bar{\Omega}^+_{1,1},\ k \geq 0\Big\} \tag{4}$$

$$\Re_x\left(\bar{\Omega}^+_{\infty,1}\right) \triangleq \Big\{x_k \big| x_0 = 0,\ x_k,\ \omega_k \text{ satisfy (1) and } \omega \in \bar{\Omega}^+_{\infty,1},\ k \geq 0\Big\}. \tag{5}$$

Then, we present the following sets for system (1):

$$\begin{aligned}\Re_0\left(\bar{\Omega}^+_{1,1}\right) &\triangleq \left\{x_{lN} \middle| x_{lN} \in \Re_x(\bar{\Omega}^+_{1,1}),\quad l \in \mathbb{N}_0\right\}\\ \Re_1(\bar{\Omega}^+_{1,1}) &\triangleq \left\{x_{1+lN} \middle| x_{1+lN} \in \Re_x(\bar{\Omega}^+_{1,1}),\quad l \in \mathbb{N}_0\right\}\\ &\ \ \vdots\\ \Re_{N-1}(\bar{\Omega}^+_{1,1}) &\triangleq \left\{x_{N-1+lN} \middle| x_{N-1+lN} \in \Re_x(\bar{\Omega}^+_{1,1}),\quad l \in \mathbb{N}_0\right\}.\end{aligned} \tag{6}$$

By such definitions, the reachable set $\Re_x(\bar{\Omega}^+_{1,1})$ can be expressed as $\Re_x(\bar{\Omega}^+_{1,1}) = \bigcup_{i=0,1,\ldots,N-1}\Re_i(\bar{\Omega}^+_{1,1})$. In addition, if the defined sets are for the $\bar{\Omega}^+_{\infty,1}$ case, we have $\Re_x(\bar{\Omega}^+_{\infty,1}) = \bigcup_{i=0,1,\ldots,N-1}\Re_i(\bar{\Omega}^+_{\infty,1})$.

Hyper-pyramids denoted by

$$\mathbf{C}(p) = \left\{\xi \middle| p^{\mathrm{T}}\xi \leq 1,\ \xi \in \overline{\mathbb{R}}^{n_x}_+\right\},\ p \in \mathbb{R}^{n_x}_+ \tag{7}$$

will be employed for the reachable set bounding of system (1).

The following definitions and lemmas, which will be used in the derivation of the reachable set estimation conditions, are first proposed. The definition of positivity for system (1) is given similar to [6].

Definition 1: System (1) is called positive if for any non-negative initial condition $x_{k_0} \geq\geq 0$ at any initial time $k_0 \geq 0$ and all disturbances $\omega_k \geq\geq 0$, trajectory $x_k \geq\geq 0$ for all $k > k_0$.

Lemma 1: The discrete-time periodic system (1) is positive iff $A_k \geq\geq 0,\ B_{\omega,k} \geq\geq 0,\ k \in \mathbb{N}_0$.

Assumption 1: System (1) is positive. That is, $A_k \geq\geq 0,\ B_{\omega,k} \geq\geq 0,\ k \in \mathbb{N}_0$.

Definition 2 [33]: Suppose $\mathbb{V}$ is a linear vector space, $s_l : \mathbb{V} \to \mathbb{R}$. The inequality $s_l(y) \geq 0$ is called regular if $y^* \in \mathbb{V}$ exists such that $s_l(y^*) > 0,\ l = 1, 2, \ldots, M,\ M \in \mathbb{N}$.

Lemma 2 [33]: Suppose $s_l : \mathbb{R}^m \to \mathbb{R},\ s_l(y) = g_l^{\mathrm{T}}y + h_l,\ l = 0, 1, \ldots, M$, be linear functionals defined in the linear space $\mathbb{R}^m$, where $g_l \in \mathbb{R}^m,\ h_l \in \mathbb{R}$, and $M \in \mathbb{N}$. If $s_l(y)$ is regular for $l = 1, 2, \ldots, M$, then the following statements are identical.

1) $s_0(y) \geq 0$, for all $y \in \mathbb{R}^m$ such that $s_l(y) \geq 0,\ l = 1, 2, \ldots, M$.
2) Scalars $\tau_l \geq 0,\ l = 1, 2, \ldots, M$ exist such that $s_0(y) \geq \sum_{l=1}^{M}\tau_l s_l(y),\ \forall y \in \mathbb{R}^m$.

In order to derive the reachable set estimation criteria, two Lyapunov-based inequalities are introduced in the following text which can be obtained by following a manner similar to the proofs of [34, Lemmas 3.1 and 3.2].

Lemma 3: Suppose $V(x, k)$ is a definite positive function [i.e., for any $x \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$, $V(x, k) \geq 0$ with $V(0, 0) = 0$]. If

$$V(x_{k+1}, k + 1) - V(x_k, k) \leq \mathbf{1}^{\mathrm{T}}\omega_k, \forall\ k \in \mathbb{N}_0 \tag{8}$$

then $0 \leq V(x_k, k) \leq 1$.

Lemma 4: Suppose $V(x, k)$ is a definite positive function [i.e., for any $x \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$, $V(x, k) \geq 0$ with $V(0, 0) = 0$]

satisfying $V(x,k)=V(x,k+lN),\ l\in\mathbb{N}_0$. If scalar variables $0\le\alpha_k\le 1$ exist such that

$$V(x_{k+1},k+1)-\alpha_k V(x_k,k)-(1-\alpha_k)\mathbf{1}^{\mathrm{T}}\omega_k\le 0,\forall k\in\mathbb{N}_0 \tag{9}$$

then $0\le V(x_k,k)\le 1$ for all x_0 satisfying $V(x_0,0)\le 1$.

III. Main Results

In this section, the reachable set bounding criteria are derived for different exogenous disturbances. The lifting method and the pseudoperiodic co-positive Lyapunov functional method are employed to derive the reachable set bounding conditions. In order to minimize the bounding hyper-pyramids, two optimization approaches are also adopted. In addition, the state-feedback controller design conditions are obtained in light of the results of reachable set estimation.

A. Reachable Set Estimation Conditions

1) Lifting Approach: Through lifting over a period of N, we can obtain the following LTIs $\Sigma_i,\ i=0,1,\ldots,N-1$:

$$\Sigma_i : y_{k+1,i}=\mathcal{A}_i y_{k,i}+\mathcal{B}_{w,i}\omega_{k,i} \tag{10}$$

where

$$\begin{aligned} y_{k,i}&=x_{i+kN}\\ \omega_{k,i}&=\begin{bmatrix}\omega_{i+kN}^{\mathrm{T}} & \omega_{i+1+kN}^{\mathrm{T}} & \cdots & \omega_{i+N-1+kN}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\\ \mathcal{A}_i&=A_{i+N-1}A_{i+N-2}\cdots A_N A_{N-1}\cdots A_i\\ \mathcal{B}_{w,i}&=\big[A_{i+N-1}A_{i+N-2}\cdots A_N A_{N-1}\cdots A_{i+1}B_{\omega,i}\\ &\qquad A_{i+N-1}A_{i+N-2}\cdots A_N A_{N-1}\cdots A_{i+2}B_{\omega,i+1}\\ &\qquad\cdots\quad B_{\omega,i+N-1}\big]\\ y_{0,i}&=x_i. \end{aligned} \tag{11}$$

The reachable set can be estimated through the obtained time-invariant systems. We obtain the following reachability estimation conditions for system (1).

Theorem 1: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{1,1}^{+}$ in (2). If there exist vectors $p_i\in\mathbb{R}_{+}^{n_x},\ i=0,1,\ldots,N-1$, such that

$$\mathcal{A}_i^{\mathrm{T}}p_i-p_i\le\le 0,\ \ \mathcal{B}_{w,i}^{\mathrm{T}}p_i\le\le\mathbf{1},\ \ i=0,1,\ldots,N-1 \tag{12}$$

$$\Gamma_i^{\mathrm{T}}p_i\ \le\ \mathbf{1},\ \ i=1,2,\ldots,N-1 \tag{13}$$

where

$$\Gamma_i=\big[A_{i-1}A_{i-2}\cdots A_1B_{\omega,0}\quad A_{i-1}A_{i-2}\cdots A_2B_{\omega,1}\quad\cdots\quad A_{i-1}B_{\omega,i-2}\quad B_{\omega,i-1}\big] \tag{14}$$

then the reachable set $\Re_x(\bar{\Omega}_{1,1}^{+})$ of the positive periodic system (1) is bounded by the union of a set of hyper-pyramids denoted by

$$\bigcup_{i=0,1,\ldots,N-1}\mathbf{C}(p_i). \tag{15}$$

Proof: The disturbance $\omega\in\bar{\Omega}_{1,1}^{+}$ implies

$$\sum_{k=0}^{\infty}\mathbf{1}^{\mathrm{T}}\omega_k\le 1. \tag{16}$$

We adopt a linear co-positive Lyapunov function for system Σ_i

$$V_i(y_{k,i})=p_i^{\mathrm{T}}y_{k,i} \tag{17}$$

where $p_i=[p_{i1},p_{i2},\ldots,p_{in_x}]^{\mathrm{T}}\in\mathbb{R}_{+}^{n_x}$. The initial condition $y_{0,i}$ can be obtained as follows for $i\in 1,2,\ldots,N-1$:

$$\begin{aligned} y_{0,i}=x_i&=A_{i-1}x_{i-1}+B_{\omega,i-1}\omega_{i-1}\\ &=A_{i-1}\big(A_{i-2}x_{i-2}+B_{\omega,i-2}\omega_{i-2}\big)+B_{\omega,i-1}\omega_{i-1}\\ &=A_{i-1}A_{i-2}x_{i-2}+A_{i-1}B_{\omega,i-2}\omega_{i-2}+B_{\omega,i-1}\omega_{i-1}\\ &=\cdots\\ &=A_{i-1}A_{i-2}\cdots A_0x_0+A_{i-1}A_{i-2}\cdots A_1B_{\omega,0}\omega_0\\ &\quad+A_{i-1}A_{i-2}\cdots A_2B_{\omega,1}\omega_1+\cdots+B_{\omega,i-1}\omega_{i-1}\\ &=A_{i-1}A_{i-2}\cdots A_1B_{\omega,0}\omega_0+A_{i-1}A_{i-2}\cdots A_2B_{\omega,1}\omega_1\\ &\quad+\cdots+B_{\omega,i-1}\omega_{i-1}\\ &=\Gamma_i\begin{bmatrix}\omega_0^{\mathrm{T}} & \omega_1^{\mathrm{T}} & \cdots & \omega_{i-1}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}. \end{aligned} \tag{18}$$

From condition (13), we have

$$\begin{aligned} V_i(y_{0,i})=p_i^{\mathrm{T}}y_{0,i}&=p_i^{\mathrm{T}}\Gamma_i\begin{bmatrix}\omega_0^{\mathrm{T}} & \omega_1^{\mathrm{T}} & \cdots & \omega_{i-1}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\\ &\le\mathbf{1}^{\mathrm{T}}\begin{bmatrix}\omega_0^{\mathrm{T}} & \omega_1^{\mathrm{T}} & \cdots & \omega_{i-1}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\\ &=\sum_{m=0}^{i-1}\mathbf{1}^{\mathrm{T}}\omega_m. \end{aligned} \tag{19}$$

On the other hand, condition (12) ensures that

$$V_i(y_{k+1,i})-V_i(y_{k,i})\le\mathbf{1}^{\mathrm{T}}\omega_{k,i} \tag{20}$$

where $\omega_{k,i}$ is defined in (11). The proof of (20) is as follows:

$$\begin{aligned} V_i(y_{k+1,i})-V_i(y_{k,i})&=p_i^{\mathrm{T}}y_{k+1,i}-p_i^{\mathrm{T}}y_{k,i}\\ &=p_i^{\mathrm{T}}\big(\mathcal{A}_iy_{k,i}+\mathcal{B}_{w,i}\omega_{k,i}\big)-p_i^{\mathrm{T}}y_{k,i}\\ &=\big(p_i^{\mathrm{T}}\mathcal{A}_i-p_i^{\mathrm{T}}\big)y_{k,i}+p_i^{\mathrm{T}}\mathcal{B}_{w,i}\omega_{k,i}\\ &\le\mathbf{1}^{\mathrm{T}}\omega_{k,i}. \end{aligned} \tag{21}$$

By adding (20) from $k=0$ to $k=l-1$, we have

$$V_i(y_{l,i})-V_i(y_{0,i})\le\sum_{j=0}^{l-1}\mathbf{1}^{\mathrm{T}}\omega_{j,i}=\sum_{m=i}^{i+lN-1}\mathbf{1}^{\mathrm{T}}\omega_m \tag{22}$$

for $l\in\mathbb{N}$. As $V_i(y_{0,i})\le\sum_{m=0}^{i-1}\mathbf{1}^{\mathrm{T}}\omega_m$, (22) implies

$$\begin{aligned} V_i(y_{l,i})&\le V_i(y_{0,i})+\sum_{m=i}^{i+lN-1}\mathbf{1}^{\mathrm{T}}\omega_m=\sum_{m=0}^{i+lN-1}\mathbf{1}^{\mathrm{T}}\omega_m\\ &\le\sum_{m=0}^{\infty}\mathbf{1}^{\mathrm{T}}\omega_m\le 1. \end{aligned} \tag{23}$$

This means that the defined set $\Re_i(\bar{\Omega}_{1,1}^{+})$ is restricted by hyper-pyramid $\mathbf{C}(p_i)$. As $\Re_x(\bar{\Omega}_{1,1}^{+})=\bigcup_{i=0,1,\ldots,N-1}\Re_i(\bar{\Omega}_{1,1}^{+})$, we conclude that the reachable set $\Re_x(\bar{\Omega}_{1,1}^{+})$ is restricted by the union of a set of hyper-pyramids given by (15). ■

Theorem 2: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{\infty,1}^{+}$ in (3). If there exist vectors

$p_i \in \mathbb{R}_+^{n_x}$, and scalars $0 \le \alpha_i \le 1$, $i = 0, 1, \dots N-1$, such that

$$\mathcal{A}_i^{\mathrm{T}} p_i - \alpha_i p_i \le\le 0,\ \mathcal{B}_{w,i}^{\mathrm{T}} p_i \le\le \frac{1-\alpha_i}{N}\mathbf{1},\ i = 0, 1, \dots, N-1 \tag{24}$$

$$\Gamma_i^{\mathrm{T}} p_i \le \frac{1}{i}\mathbf{1},\ \ i = 1, 2, \dots, N-1 \tag{25}$$

then the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^+)$ of periodic positive systems (1) is bounded by the union of a set of hyper-pyramids denoted by

$$\bigcup_{i=0,1,\dots,N-1} \mathbf{C}(p_i). \tag{26}$$

Proof: The disturbance $\omega \in \bar{\Omega}_{\infty,1}^+$ implies

$$\mathbf{1}^{\mathrm{T}}\omega_k \le 1. \tag{27}$$

We adopt a linear co-positive Lyapunov function for system Σ_i

$$V_i(y_{k,i}) = p_i^{\mathrm{T}} y_{k,i} \tag{28}$$

where $p_i = [p_{i1}, p_{i2}, \dots, p_{in_x}]^{\mathrm{T}} \in \mathbb{R}_+^{n_x}$. The initial condition $y_{0,i}$ can be obtained as

$$y_{0,i} = \Gamma_i \begin{bmatrix} \omega_0^{\mathrm{T}} & \omega_1^{\mathrm{T}} & \cdots & \omega_{i-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{29}$$

From condition (25), we have

$$\begin{aligned} V_i(y_{0,i}) &= p_i^{\mathrm{T}} y_{0,i} = p_i^{\mathrm{T}}\Gamma_i \begin{bmatrix} \omega_0^{\mathrm{T}} & \omega_1^{\mathrm{T}} & \cdots & \omega_{i-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\ &\le \frac{1}{i}\mathbf{1}^{\mathrm{T}} \begin{bmatrix} \omega_0^{\mathrm{T}} & \omega_1^{\mathrm{T}} & \cdots & \omega_{i-1}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \le 1 \end{aligned} \tag{30}$$

holds for $i = 1, 2, \dots, N-1$. In addition, for $i = 0$, we have $V_i(y_{0,i}) = V_0(y_{0,0}) = V_0(x_0) = V_0(0) = 0$. On the other hand, condition (24) ensures that

$$V_i(y_{k+1,i}) - \alpha_i V_i(y_{k,i}) - \frac{1-\alpha_i}{N}\mathbf{1}^{\mathrm{T}}\omega_{k,i} \le 0. \tag{31}$$

The proof of (31) is as follows:

$$\begin{aligned} &V_i(y_{k+1,i}) - \alpha_i V_i(y_{k,i}) - \frac{1-\alpha_i}{N}\mathbf{1}^{\mathrm{T}}\omega_{k,i} \\ &= p_i^{\mathrm{T}} y_{k+1,i} - \alpha_i p_i^{\mathrm{T}} y_{k,i} - \frac{1-\alpha_i}{N}\mathbf{1}^{\mathrm{T}}\omega_{k,i} \\ &= p_i^{\mathrm{T}}(\mathcal{A}_i y_{k,i} + \mathcal{B}_{w,i}\omega_{k,i}) - \alpha_i p_i^{\mathrm{T}} y_{k,i} - \frac{1-\alpha_i}{N}\mathbf{1}^{\mathrm{T}}\omega_{k,i} \\ &= (p_i^{\mathrm{T}}\mathcal{A}_i - \alpha_i p_i^{\mathrm{T}}) y_{k,i} + \left(p_i^{\mathrm{T}}\mathcal{B}_{w,i} - \frac{1-\alpha_i}{N}\mathbf{1}^{\mathrm{T}}\right)\omega_{k,i} \\ &\le 0. \end{aligned} \tag{32}$$

From (31), we have

$$V_i(y_{k+1,i}) - \alpha_i V_i(y_{k,i}) \le \frac{1-\alpha_i}{N}\mathbf{1}^{\mathrm{T}}\omega_{k,i} \le 1-\alpha_i \tag{33}$$

which means

$$V_i(y_{k+1,i}) - 1 \le \alpha_i(V_i(y_{k,i}) - 1). \tag{34}$$

As $V_i(y_{0,i}) - 1 \le 0$, $i = 0, 1, \dots, N-1$, the above inequality ensures

$$V_i(y_{k,i}) \le 1$$

holds for $k \in \mathbb{N}_0$. This means that the defined set $\Re_i(\bar{\Omega}_{\infty,1}^+)$ is restricted by the hyper-pyramid $\mathbf{C}(p_i)$. As $\Re_x(\bar{\Omega}_{\infty,1}^+) = \bigcup_{i=0,1,\dots,N-1}\Re_i(\bar{\Omega}_{\infty,1}^+)$, we conclude that the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^+)$ is restricted by the union of a set of hyper-pyramids denoted by (26). ■

2) Pseudoperiodic Lyapunov Functional Approach: The reachable set estimation criteria can also be derived through the pseudoperiodic co-positive Lyapunov functional approach. By using the pseudoperiodic co-positive Lyapunov functional method, the following theorems are proposed.

Theorem 3: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{1,1}^+$ in (2). If there exist vectors $p_i \in \mathbb{R}_+^{n_x}$, $i = 0, 1, \dots, N$, such that

$$A_i^{\mathrm{T}} p_{i+1} - p_i \le\le 0, \quad B_{\omega,i}^{\mathrm{T}} p_{i+1} \le\le \mathbf{1},\ \ i = 0, 1, \dots, N-1 \tag{35}$$

$$p_N = p_0 \tag{36}$$

then the reachable set $\Re_x(\bar{\Omega}_{1,1}^+)$ of periodic positive system (1) is bounded by the union of a set of hyper-pyramids denoted by

$$\bigcup_{i=0,1,\dots,N-1} \mathbf{C}(p_i). \tag{37}$$

Proof: Construct a pseudoperiodic co-positive Lyapunov functional as

$$V(x_k, k) = p_k^{\mathrm{T}} x_k \tag{38}$$

where $p_k = p_{k+lN} \in \mathbb{R}_+^{n_x}$, $l \in \mathbb{N}_0$. In order to employ Lemma 3, we define

$$H_k = V(x_{k+1}, k+1) - V(x_k, k). \tag{39}$$

Then, along the trajectory of system (1), we have

$$\begin{aligned} H_k &= p_{k+1}^{\mathrm{T}} x_{k+1} - p_k^{\mathrm{T}} x_k \\ &= p_{k+1}^{\mathrm{T}}(A_k x_k + B_{\omega,k}\omega_k) - p_k^{\mathrm{T}} x_k \\ &= (p_{k+1}^{\mathrm{T}} A_k - p_k^{\mathrm{T}}) x_k + p_{k+1}^{\mathrm{T}} B_{\omega,k}\omega_k. \end{aligned} \tag{40}$$

Due to the periodicity of p_k, conditions (35) and (36) lead to

$$A_k^{\mathrm{T}} p_{k+1} - p_k \le\le 0,\ B_{\omega,k}^{\mathrm{T}} p_{k+1} \le\le \mathbf{1} \tag{41}$$

accordingly, $H_k \le \mathbf{1}^{\mathrm{T}}\omega_k$. By applying Lemma 3, it follows that $V(x_k, k) = p_k^{\mathrm{T}} x_k \le 1$, which implies that the reachable set $\Re_x(\bar{\Omega}_{1,1}^+)$ is restricted by the union of a set of hyper-pyramids (37). ■

Theorem 4: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{\infty,1}^+$ in (3). If there exist vectors $p_i \in \mathbb{R}_+^{n_x}$, $i = 0, 1, \dots, N$, and scalars $0 \le \alpha_i \le 1$, $i = 0, 1, \dots, N-1$, such that

$$\begin{aligned} A_i^{\mathrm{T}} p_{i+1} - \alpha_i p_i \le\le 0, B_{\omega,i}^{\mathrm{T}} p_{i+1} \le\le (1-\alpha_i)\mathbf{1} \\ i = 0, 1, \dots, N-1 \end{aligned} \tag{42}$$

$$p_N = p_0 \tag{43}$$

then the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^+)$ of the periodic positive system (1) is bounded by the union of a set of hyper-pyramids denoted by

$$\bigcup_{i=0,1,\dots,N-1} \mathbf{C}(p_i). \tag{44}$$

Proof: Construct a pseudoperiodic co-positive Lyapunov functional as

$$V(x_k, k) = p_k^{\mathrm{T}} x_k \tag{45}$$

where $p_k = p_{k+lN} \in \mathbb{R}_+^{n_x}$, $l \in \mathbb{N}_0$. In order to employ Lemma 4, we define

$$I_k = V(x_{k+1}, k+1) - \alpha_k V(x_k, k) - (1 - \alpha_k)\mathbf{1}^{\mathrm{T}}\omega_k \tag{46}$$

where $\alpha_k = \alpha_{k+lN}$, $l \in \mathbb{N}_0$. Then, along the trajectory of system (1), we have

$$\begin{aligned} I_k &= p_{k+1}^{\mathrm{T}} x_{k+1} - \alpha_k p_k^{\mathrm{T}} x_k - (1-\alpha_k)\mathbf{1}^{\mathrm{T}}\omega_k \\ &= p_{k+1}^{\mathrm{T}}\big(A_k x_k + B_{\omega,k}\omega_k\big) - \alpha_k p_k^{\mathrm{T}} x_k - (1-\alpha_k)\mathbf{1}^{\mathrm{T}}\omega_k \\ &= \big(p_{k+1}^{\mathrm{T}} A_k - \alpha_k p_k^{\mathrm{T}}\big)x_k + \big(p_{k+1}^{\mathrm{T}} B_{\omega,k} - (1-\alpha_k)\mathbf{1}^{\mathrm{T}}\big)\omega_k. \end{aligned} \tag{47}$$

Due to the periodicity of α_k and p_k, conditions (42) and (43) lead to

$$A_k^{\mathrm{T}} p_{k+1} - \alpha_k p_k \leq\leq 0, \quad B_{\omega,k}^{\mathrm{T}} p_{k+1} \leq\leq (1-\alpha_k)\mathbf{1} \tag{48}$$

accordingly, $I_k \leq 0$. By using Lemma 4, we have $V(x_k, k) = p_k^{\mathrm{T}} x_k \leq 1$, which implies that the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^+)$ is restricted by the union of a set of hyper-pyramids (44). ■

Remark 1: In the proofs of Theorems 3 and 4, we adopted a class of Lyapunov functions that have matrices (P_i, $i = 0, 1, \ldots, N-1$) with periodicity corresponding to that of the N-periodic systems. Such Lyapunov functions are referred to as pseudoperiodic Lyapunov functions, which lead to less conservative results.

3) Optimization of Bounding Hyper-Pyramids: Note that any p_i obtained in Theorems 1–4 can be used to estimate the bounding hyper-pyramids for the reachability of systems. Usually, the bounding hyper-pyramids are required to be as small as possible. This goal can be obtained by seeking the solutions of the two following optimization problems.

Minimal Volume Problem:

$$\min\left(-\sum_{l=1}^{n_x} \ln p_{il}\right) \quad \text{subject to the conditions in Theorems 1–4} \tag{49}$$

where p_{il} is the lth element of vector p_i, $i = 0, 1, \ldots, N-1$, $l = 1, 2, \ldots, n_x$.

Sequentially Minimal Axis Problem:

$$\min\left(\frac{1}{p_{il}}\right), \text{ subject to the conditions in Theorems 1–4} \tag{50}$$

where p_{il} is the lth element of vector p_i, $i = 0, 1, \ldots, N-1$, $l = 1, 2, \ldots, n_x$.

Denote $p_i^{0,\mathrm{opt}}$ as the optimal p_i obtained by solving MVP, and $p_i^{l,\mathrm{opt}}$ as the optimal p_i obtained by solving SMAP along x_l-coordinate, then $\Re_i(\bar{\Omega}_{1,1}^+)$ or $\Re_i(\bar{\Omega}_{\infty,1}^+)$ can be restricted by the intersection of hyper-pyramids $\bigcap_{l=0,1,\ldots,n_x} \mathbf{C}(p_i^{l,\mathrm{opt}})$, and the overall reachable sets $\Re_x(\bar{\Omega}_{1,1}^+)$ or $\Re_x(\bar{\Omega}_{\infty,1}^+)$ can be bounded by $\bigcup_{i=0,1,\ldots,N-1}(\bigcap_{l=0,1,\ldots,n_x} \mathbf{C}(p_i^{l,\mathrm{opt}}))$.

Remark 2: For disturbances $\omega \in \bar{\Omega}_{\infty,1}^+$, the computational cost of Theorem 4 is higher than that of Theorem 2 because it is harder to seek the coupled (α_k, p_k) pairs in the conditions of Theorem 4. However, the lifting method is inapplicable to the reachable set estimation problem of more general periodic positive systems and the reachable set synthesis problem. In addition, the genetic algorithm (GA) [23], [35], [36] can be employed to optimize value of the variables α_k in Theorem 4.

As special cases of the pseudoperiodic Lyapunov functions, if all of the p_i are set to be the identical, then Theorems 3 and 4 reduce to the common Lyapunov functional approaches. We propose several corollaries for comparison in the following text.

Corollary 1: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{1,1}^+$ in (2). If vector $p \in \mathbb{R}_+^{n_x}$ exists such that

$$A_i^{\mathrm{T}} p - p \leq\leq 0, \quad B_{\omega,i}^{\mathrm{T}} p \leq\leq \mathbf{1}, \quad i = 0, 1, \ldots, N-1 \tag{51}$$

then the reachable set $\Re_x(\bar{\Omega}_{1,1}^+)$ of the periodic positive system (1) is bounded by the given hyper-pyramid $\mathbf{C}(p)$.

Corollary 2: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{\infty,1}^+$ in (3). If there exist vector $p \in \mathbb{R}_+^{n_x}$ and scalars $0 \leq \alpha_i \leq 1$, $i = 0, 1, \ldots, N-1$, such that

$$A_i^{\mathrm{T}} p - \alpha_i p \leq\leq 0, \quad B_{\omega,i}^{\mathrm{T}} p \leq\leq (1-\alpha_i)\mathbf{1}, \quad i = 0, 1, \ldots, N-1 \tag{52}$$

then the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^+)$ of the periodic positive system (1) is bounded by the given hyper-pyramid $\mathbf{C}(p)$.

Corollary 3: Consider system (1) with zero initial conditions and the class of disturbances $\bar{\Omega}_{\infty,1}^+$ in (3). If vector $p \in \mathbb{R}_+^{n_x}$ and scalar $0 \leq \alpha \leq 1$ exist such that

$$A_i^{\mathrm{T}} p - \alpha p \leq\leq 0, \quad B_{\omega,i}^{\mathrm{T}} p \leq\leq (1-\alpha)\mathbf{1}, \quad i = 0, 1, \ldots, N-1 \tag{53}$$

then the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^+)$ of the periodic positive system (1) is bounded by the given hyper-pyramid $\mathbf{C}(p)$.

Remark 3: Decision variables in Corollaries 1–3 are fewer than those in Theorems 3 and 4. However, as common variables P and α are expected to satisfy all subsystems, Corollaries 1–3 are more conservative than Theorems 3 and 4.

B. State-Feedback Controller Design

This section investigates the problem of state-feedback controller design for periodic positive systems with two different exogenous disturbances. Consider system (1) with control inputs

$$x_{k+1} = A_k x_k + B_{u,k} u_k + B_{\omega,k}\omega_k \tag{54}$$

where $u_k \in \mathbb{R}^{n_u}$ is the control input vector and $B_{u,k} = B_{u,k+lN}$, $l \in \mathbb{N}_0$, are N-periodic constant real matrices. For system (54), a periodic state-feedback controller is designed as

$$u_k = K_k x_k \tag{55}$$

where $K_k = K_{k+lN}$, $l \in \mathbb{N}_0$, are N-periodic controller gain matrices to be determined. With this controller, the closed-loop system is given as

$$x_{k+1} = \big(A_k + B_{u,k} K_k\big)x_k + B_{\omega,k}\omega_k. \tag{56}$$

In this section, the following reachable set synthesis problem will be investigated.

1) *CDP:* Given a hyper-pyramid $\mathbf{C}(\eta)$, where $\eta \in \mathbb{R}_{+}^{n_x}$, design a state-feedback controller (55) such that the reachable set of the closed-loop system (56) is bounded in the given hyper-pyramid $\mathbf{C}(\eta)$.

Remark 4: It is worth mentioning that matrices A_k and $B_{u,k}$ are not required to be non-negative. The positivity of the closed-loop system is ensured as long as $A_k + B_{u,k}K_k$ and $B_{\omega,k}$ are non-negative.

Then, the pseudoperiodic co-positive Lyapunov function approach is used to study the state-feedback controller design problem. We have the following theorems for the proposed state-feedback controller design problem.

Theorem 5: Consider system (54) under zero initial conditions and the class of disturbances $\bar{\Omega}_{1,1}^{+}$ in (2). Given a vector $\eta \in \mathbb{R}_{+}^{n_x}$, if there exist vectors $p_i \in \mathbb{R}_{+}^{n_x}$, $i = 0, 1, \ldots, N$, and $z_i \in \mathbb{R}^{n_x}$, $i = 0, 1, \ldots, N-1$, such that

$$A_i^{\mathrm{T}} p_{i+1} + z_i - p_i \leq\leq 0 \tag{57}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i+1} \leq\leq \mathbf{1} \tag{58}$$

$$\left(\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}\right) A_i + B_{u,i} \widetilde{p}_i z_i^{\mathrm{T}} \geq\geq 0 \tag{59}$$

$$\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1} > 0 \tag{60}$$

$$B_{\omega,i} \geq\geq 0 \tag{61}$$

$$p_i \geq\geq \eta \tag{62}$$

$$p_N = p_0 \tag{63}$$

where $\widetilde{p}_i \in \mathbb{R}^{n_u}$, $i = 0, 1, \ldots, N-1$, are given nonzero vectors, then there exists a state-feedback controller in the form of (55) such that the reachable set $\Re_x(\bar{\Omega}_{1,1}^{+})$ of the closed-loop system is bounded in the given hyper-pyramid $\mathbf{C}(\eta)$, and a desired controller can be proposed with the controller gains denoted by

$$K_i = \frac{\widetilde{p}_i z_i^{\mathrm{T}}}{\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}}. \tag{64}$$

Proof: Note that $\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}$ is a scalar, thus (64) implies

$$K_i^{\mathrm{T}} = \frac{z_i \widetilde{p}_i^{\mathrm{T}}}{\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}}. \tag{65}$$

Right-multiplying $B_{u,i}^{\mathrm{T}} p_{i+1}$ on each side of (65), one can obtain

$$K_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1} = z_i. \tag{66}$$

Applying the change of variable z_i to (57), one has

$$\left(A_i + B_{u,i} K_i\right)^{\mathrm{T}} p_{i+1} - p_i \leq\leq 0. \tag{67}$$

In addition, (59) and (60) imply

$$A_i + B_{u,i} \frac{\widetilde{p}_i z_i^{\mathrm{T}}}{\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}} \geq\geq 0 \tag{68}$$

which means

$$A_i + B_{u,i} K_i \geq\geq 0. \tag{69}$$

This equation and (61) ensure the positivity of the closed-loop system. According to Theorem 3, conditions (58) and (67) ensure that the reachable set of the closed-loop system is restricted by the union of a set of hyper-pyramids $\bigcup_{i=0,1,\ldots,N-1} \mathbf{C}(p_i)$. Furthermore, $p_i \geq\geq \eta$ means that hyper-pyramids $\mathbf{C}(p_i)$ are contained in the given hyper-pyramid $\mathbf{C}(\eta)$. Thus, the reachable set $\Re_x(\bar{\Omega}_{1,1}^{+})$ of the closed-loop system is restricted by the given hyper-pyramid $\mathbf{C}(\eta)$. ■

Theorem 6: Consider system (54) under zero initial conditions and the class of disturbances $\bar{\Omega}_{\infty,1}^{+}$ in (3). Given a vector $\eta \in \mathbb{R}_{+}^{n_x}$, if there exist vectors $p_i \in \mathbb{R}_{+}^{n_x}$, $i = 0, 1, \ldots, N$, $z_i \in \mathbb{R}^{n_x}$, $i = 0, 1, \ldots, N-1$, and scalars $0 \leq \alpha_i \leq 1$, $i = 0, 1, \ldots, N-1$, such that

$$A_i^{\mathrm{T}} p_{i+1} + z_i - \alpha_i p_i \leq\leq 0 \tag{70}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i+1} \leq\leq (1 - \alpha_i)\mathbf{1} \tag{71}$$

$$\left(\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}\right) A_i + B_{u,i} \widetilde{p}_i z_i^{\mathrm{T}} \geq\geq 0 \tag{72}$$

$$\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1} > 0 \tag{73}$$

$$B_{\omega,i} \geq\geq 0 \tag{74}$$

$$p_i \geq\geq \eta \tag{75}$$

$$p_N = p_0 \tag{76}$$

where $\widetilde{p}_i \in \mathbb{R}^{n_u}$, $i = 0, 1, \ldots, N-1$, are given nonzero vectors, then there exists a state-feedback controller in the form of (55) such that the reachable set $\Re_x(\bar{\Omega}_{\infty,1}^{+})$ of the closed-loop system is bounded in the given hyper-pyramid $\mathbf{C}(\eta)$, and a desired controller can be proposed with the controller gains denoted by

$$K_i = \frac{\widetilde{p}_i z_i^{\mathrm{T}}}{\widetilde{p}_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}} p_{i+1}}. \tag{77}$$

We omit the proof of Theorem 6 here as it is similar to the proof of Theorem 5. The controller design method used in this paper is inspired by the work of [37], and the choice of vectors $\widetilde{p}_i$ is free.

Remark 5: The synthesis problem of positive systems is more difficult compared with that of general systems. A common method used in the literature for the synthesis problem is a two-step iterative method. In order to solve the synthesis problem directly, vectors z_i and $\widetilde{p}_i$ are introduced. The synthesis problem of the single control input case can be solved efficiently by the proposed corollaries. However, the proposed method may lead to a certain level of conservatism for the synthesis problem of the multiple control input case as the rank of controller gain matrices is restricted to be 1.

IV. Illustrative Example

We use four numerical examples to illustrate the effectiveness of the results obtained in the above section. Examples 1 and 2 are provided to show the determination of the bounding hyper-pyramids for the reachability of the positive periodic system with different disturbances, $\omega \in \bar{\Omega}_{1,1}^{+}$ and $\omega \in \bar{\Omega}_{\infty,1}^{+}$, respectively. Then, in Examples 3 and 4, we propose different state-feedback controllers to ensure that the state of the closed-loop system can be bounded by the given hyper-pyramid under either exogenous disturbances $\omega \in \bar{\Omega}_{1,1}^{+}$ or $\omega \in \bar{\Omega}_{\infty,1}^{+}$.

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A. Example 1

Consider system (1) with two modes

$$A_0 = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix},\ B_{\omega,0} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.8 \end{bmatrix},\ B_{\omega,1} = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}. \tag{78}$$

As $A_i \geq\geq 0$, $B_{\omega,i} \geq\geq 0$, for $i = 0, 1$, this system is positive. Then, the bounding hyper-pyramids can be determined for the reachability of the system with exogenous disturbances $\omega \in \bar{\Omega}^+_{1,1}$ by the proposed methods.

By solving the MVP and SMAP in regard to Theorem 1, the following optimal solutions are obtained.

When minimizing the capacity of $\mathbf{C}(p_0)$

$$p_0^{0,\text{opt}} = \begin{bmatrix} 1.9871 \\ 1.6901 \end{bmatrix}. \tag{79}$$

The optimal capacity of $\mathbf{C}(p_0)$ is 0.1489.

When minimizing the capacity of $\mathbf{C}(p_1)$

$$p_1^{0,\text{opt}} = \begin{bmatrix} 1.9223 \\ 1.5386 \end{bmatrix}. \tag{80}$$

The optimal capacity of $\mathbf{C}(p_1)$ is 0.1691. When minimizing $(1/p_{01})$

$$p_0^{1,\text{opt}} = \begin{bmatrix} 1.9879 \\ 1.6898 \end{bmatrix}. \tag{81}$$

When minimizing $(1/p_{02})$

$$p_0^{2,\text{opt}} = \begin{bmatrix} 1.7272 \\ 1.8134 \end{bmatrix}. \tag{82}$$

When minimizing $(1/p_{11})$

$$p_1^{1,\text{opt}} = \begin{bmatrix} 1.9230 \\ 1.5384 \end{bmatrix}. \tag{83}$$

When minimizing $(1/p_{12})$

$$p_1^{2,\text{opt}} = \begin{bmatrix} 1.6667 \\ 1.6666 \end{bmatrix}. \tag{84}$$

By solving the MVP and SMAP in regard to Theorem 3, the following optimal solutions are obtained.

When minimizing the capacity of $\mathbf{C}(p_0)$

$$p_0^{0,\text{opt}} = \begin{bmatrix} 1.9867 \\ 1.6900 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.8980 \\ 1.5508 \end{bmatrix}. \tag{85}$$

The optimal capacity of $\mathbf{C}(p_0)$ is 0.1489.

When minimizing the capacity of $\mathbf{C}(p_1)$

$$p_0 = \begin{bmatrix} 1.8492 \\ 1.6925 \end{bmatrix},\ p_1^{0,\text{opt}} = \begin{bmatrix} 1.9209 \\ 1.5394 \end{bmatrix}. \tag{86}$$

The optimal capacity of $\mathbf{C}(p_1)$ is 0.1691. When minimizing $(1/p_{01})$

$$p_0^{1,\text{opt}} = \begin{bmatrix} 1.9880 \\ 1.6899 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.8986 \\ 1.5507 \end{bmatrix}. \tag{87}$$

When minimizing $(1/p_{02})$

$$p_0^{2,\text{opt}} = \begin{bmatrix} 1.7272 \\ 1.8133 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.7531 \\ 1.6234 \end{bmatrix}. \tag{88}$$

When minimizing $(1/p_{11})$

$$p_0 = \begin{bmatrix} 1.8467 \\ 1.6923 \end{bmatrix},\ p_1^{1,\text{opt}} = \begin{bmatrix} 1.9229 \\ 1.5385 \end{bmatrix}. \tag{89}$$

When minimizing $(1/p_{12})$

$$p_0 = \begin{bmatrix} 1.6670 \\ 1.6670 \end{bmatrix},\ p_1^{2,\text{opt}} = \begin{bmatrix} 1.6671 \\ 1.6664 \end{bmatrix}. \tag{90}$$

Moreover, the bounding hyper-pyramids can be calculated by Corollary 1. However, when checking the inequalities in (51), that is

$$A_0^{\text{T}} p - p \leq\leq 0,\ B_{\omega,0}^{\text{T}} p \leq\leq \mathbf{1}$$
$$A_1^{\text{T}} p - p \leq\leq 0,\ B_{\omega,1}^{\text{T}} p \leq\leq \mathbf{1}$$

it is shown that they are infeasible for system (78), which means that Corollary 1 is more conservative than Theorem 3.

The bounding hyper-pyramids obtained $\bigcap_{l=0,1,2} \mathbf{C}(p_0^{l,\text{opt}})$ for $\Re_0(\bar{\Omega}^+_{1,1})$ are shown in Figs. 1(a) and 2(a), and the bounding hyper-pyramids $\bigcap_{l=0,1,2} \mathbf{C}(p_1^{l,\text{opt}})$ for $\Re_1(\bar{\Omega}^+_{1,1})$ are shown in Figs. 1(b) and 2(b). Furthermore, the boundaries $\bigcup_{i=0,1}(\bigcap_{l=0,1,2} \mathbf{C}(p_i^{l,\text{opt}}))$ for the reachable set $\Re_x(\bar{\Omega}^+_{1,1})$ are shown in Figs. 1(c) and 2(c). The system state under exogenous disturbances $\omega_k = (1-c) \times c^k$, $c = 0, 0.1, \ldots, 0.9$ ($\omega \in \bar{\Omega}^+_{1,1}$), is also presented in Figs. 1 and 2.

B. Example 2

Consider system (1) with two modes

$$A_0 = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.0 \end{bmatrix},\ B_{\omega,0} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix},\ B_{\omega,1} = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}. \tag{91}$$

As $A_i \geq\geq 0$, $B_{\omega,i} \geq\geq 0$, for $i = 0, 1$, the above system is positive. Then, the bounding hyper-pyramids can be determined for the reachable set of the system with exogenous disturbances $\omega \in \bar{\Omega}^+_{\infty,1}$ by the proposed methods.

By optimizing α_i with a fixed step size of 0.0001 and solving the MVP and SMAP in regard to Theorem 2, the following solutions are obtained.

When minimizing the capacity of $\mathbf{C}(p_0)$

$$\alpha_0 = 0.1378,\ p_0^{0,\text{opt}} = \begin{bmatrix} 1.6144 \\ 0.8988 \end{bmatrix}. \tag{92}$$

The optimal capacity of $\mathbf{C}(p_0)$ is 0.3446. When minimizing the capacity of $\mathbf{C}(p_1)$

$$\alpha_1 = 0.1106,\ p_1^{0,\text{opt}} = \begin{bmatrix} 1.2652 \\ 1.9165 \end{bmatrix}. \tag{93}$$

The optimal capacity of $\mathbf{C}(p_1)$ is 0.2062.

When minimizing $(1/p_{01})$

$$\alpha_0 = 0.3044,\ p_0^{1,\text{opt}} = \begin{bmatrix} 2.1003 \\ 0.4592 \end{bmatrix}. \tag{94}$$

When minimizing $(1/p_{02})$

$$\alpha_0 = 0.1958,\ p_0^{2,\text{opt}} = \begin{bmatrix} 1.5653 \\ 0.8904 \end{bmatrix}. \tag{95}$$

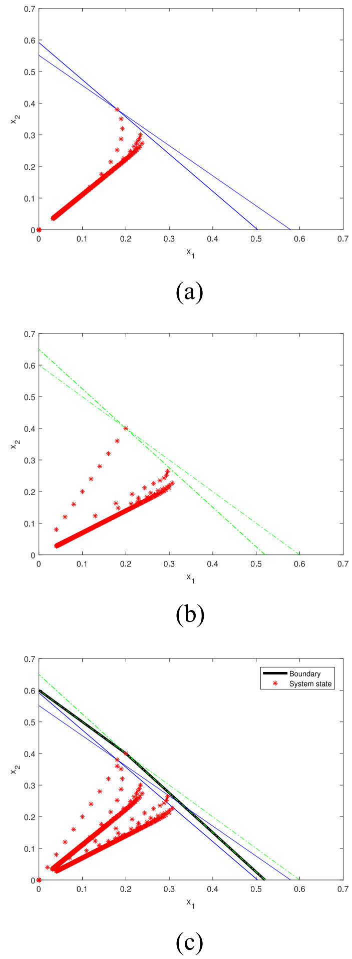

Fig. 1. Bounding hyper-pyramids obtained by Theorem 1. (a) $\Re_0(\bar{\Omega}^+_{1,1})$. (b) $\Re_1(\bar{\Omega}^+_{1,1})$. (c) $\Re_x(\bar{\Omega}^+_{1,1})$.

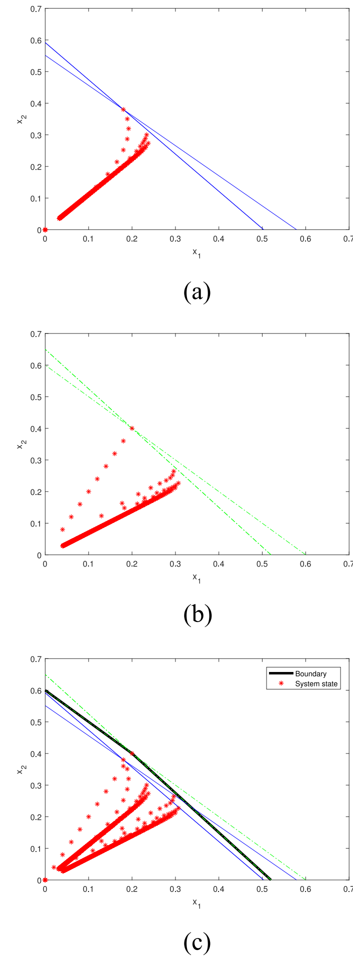

Fig. 2. Bounding hyper-pyramids obtained by Theorem 3. (a) $\Re_0(\bar{\Omega}^+_{1,1})$. (b) $\Re_1(\bar{\Omega}^+_{1,1})$. (c) $\Re_x(\bar{\Omega}^+_{1,1})$.

When minimizing $(1/p_{11})$

$$\alpha_1 = 0.1674,\ p_1^{1,\mathrm{opt}} = \begin{bmatrix} 0.6444 \\ 1.1728 \end{bmatrix}. \tag{96}$$

When minimizing $(1/p_{12})$

$$\alpha_1 = 0.2540,\ p_1^{2,\mathrm{opt}} = \begin{bmatrix} 0.5254 \\ 2.6793 \end{bmatrix}. \tag{97}$$

For Theorem 4, GA is employed to seek an optimal solution. The chromosome size is chosen to be 16, the generation number is chosen to be 200, rate of crossover is chosen to be 0.6, mutation rate is chosen to be 0.01, and elitist selection is not employed. By employing GA and solving the MVP and SMAP in regard to Theorem 4, the following optimal solutions are obtained.

When minimizing the capacity of $\mathbf{C}(p_0)$

$$\begin{aligned} &\alpha_0 = 0.3905,\ \alpha_1 = 0.3150 \\ &p_0^{0,\mathrm{opt}} = \begin{bmatrix} 2.3320 \\ 1.5058 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.9591 \\ 2.1757 \end{bmatrix}. \end{aligned} \tag{98}$$

The optimal capacity of $\mathbf{C}(p_0)$ is 0.1424.

When minimizing the capacity of $\mathbf{C}(p_1)$

$$\begin{aligned} &\alpha_0 = 0.3535,\ \alpha_1 = 0.3459 \\ &p_0 = \begin{bmatrix} 2.2162 \\ 1.4415 \end{bmatrix},\ p_1^{0,\mathrm{opt}} = \begin{bmatrix} 1.6984 \\ 3.0675 \end{bmatrix}. \end{aligned} \tag{99}$$

The optimal capacity of $\mathbf{C}(p_1)$ is 0.0960. When minimizing $(1/p_{01})$

$$\begin{aligned} &\alpha_0 = 0.5986,\ \alpha_1 = 0.4740 \\ &p_0^{1,\mathrm{opt}} = \begin{bmatrix} 3.0767 \\ 0.7277 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.4518 \\ 1.1100 \end{bmatrix}. \end{aligned} \tag{100}$$

When minimizing $(1/p_{02})$

$$\begin{aligned} &\alpha_0 = 0.5592,\ \alpha_1 = 0.3264 \\ &p_0^{2,\mathrm{opt}} = \begin{bmatrix} 0.9478 \\ 1.9293 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.1720 \\ 2.0638 \end{bmatrix}. \end{aligned} \tag{101}$$

When minimizing $(1/p_{11})$

$$\begin{aligned} &\alpha_0 = 0.4312,\ \alpha_1 = 0.4319 \\ &p_0 = \begin{bmatrix} 1.1173 \\ 1.5211 \end{bmatrix},\ p_1^{1,\mathrm{opt}} = \begin{bmatrix} 2.1863 \\ 1.3153 \end{bmatrix}. \end{aligned} \tag{102}$$

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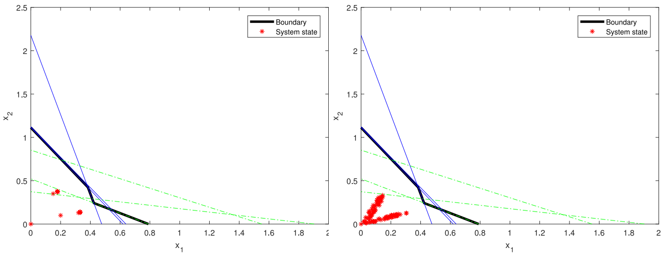
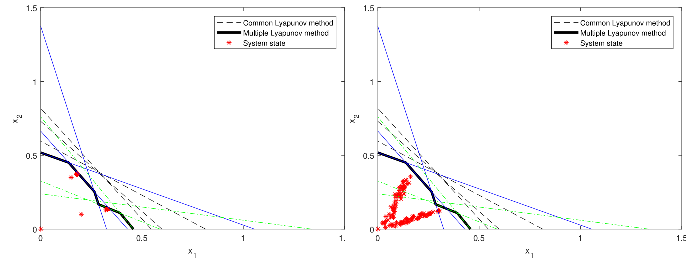


(a) (b)

Fig. 3. Reachable set boundary obtained by Theorem 2. (a) $\omega_k \equiv 1$. (b) $\omega_k = \text{rand}(k)$.



(a) (b)

Fig. 4. Reachable set boundary obtained by Theorem 4. (a) $\omega_k \equiv 1$. (b) $\omega_k = \text{rand}(k)$.

When minimizing $(1/p_{12})$

$$\alpha_0 = 0.4321,\ \alpha_1 = 0.6332$$
$$p_0 = \begin{bmatrix} 2.1090 \\ 0.5196 \end{bmatrix},\ p_1^{2,\text{opt}} = \begin{bmatrix} 0.7483 \\ 4.1823 \end{bmatrix}. \tag{103}$$

Moreover, the bounding hyper-pyramids can be decided by Corollaries 2 and 3. Taking Corollary 3, for instance, by optimizing α with a fixed step size 0.0001 and solving the MVP and SMAP in regard to Corollary 3, the following solutions are obtained.

When minimizing the capacity of $\mathbf{C}(p)$

$$\alpha = 0.4221,\ p^{0,\text{opt}} = \begin{bmatrix} 1.6716 \\ 1.3691 \end{bmatrix}. \tag{104}$$

The optimal capacity of $\mathbf{C}(p)$ is 0.2185. When minimizing $(1/p_1)$

$$\alpha = 0.4500,\ p^{1,\text{opt}} = \begin{bmatrix} 1.8333 \\ 1.2222 \end{bmatrix}. \tag{105}$$

When minimizing $(1/p_2)$

$$\alpha = 0.3733,\ p^{2,\text{opt}} = \begin{bmatrix} 1.2289 \\ 1.6794 \end{bmatrix}. \tag{106}$$

The obtained bounding hyper-pyramids for $\Re_0(\bar{\Omega}^+_{\infty,1})$ (the solid line) and $\Re_1(\bar{\Omega}^+_{\infty,1})$ (the dotted line) and boundaries $\bigcup_{i=0,1}(\bigcap_{l=0,1,2}\mathbf{C}(p_i^{l,\text{opt}}))$ for the reachable set $\Re_x(\bar{\Omega}^+_{\infty,1})$ are presented in Figs. 3 and 4. The system state under exogenous disturbances $\omega_k \equiv 1$ $(\omega \in \bar{\Omega}^+_{\infty,1})$ is shown in Figs. 3(a) and 4(a). The system state under exogenous disturbances $\omega_k = \text{rand}(k)$ $(\omega \in \bar{\Omega}^+_{\infty,1})$ is presented in Figs. 3(b) and 4(b), where rand(.) is a random number picked from a uniform distribution over [0, 1]. Moreover, the conservatism of Corollary 3 is also presented in Fig. 4.

C. Example 3

Consider system (54) with two modes

$$A_0 = \begin{bmatrix} 1.1 & 0.3 \\ -0.2 & 1.2 \end{bmatrix},\ B_{u,0} = \begin{bmatrix} 1.2 \\ 0.3 \end{bmatrix},\ B_{\omega,0} = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 1.2 & 0.4 \\ 0.6 & 1.1 \end{bmatrix},\ B_{u,1} = \begin{bmatrix} 1.1 \\ -0.7 \end{bmatrix},\ B_{\omega,1} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}. \tag{107}$$

Since the product of A_0 and A_1 is not Schur stable, the state of the open-loop system may diverge to infinity. In the following text, we will obtain a state-feedback controller for the above system with disturbances $\omega \in \bar{\Omega}^+_{1,1}$.

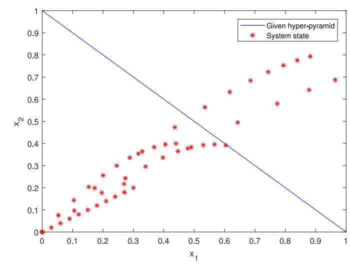


Fig. 5. State of the open-loop system in Example 3.

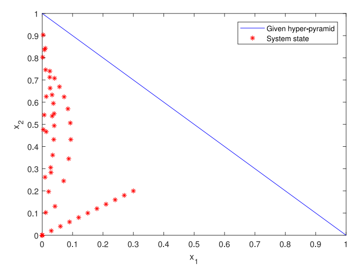


Fig. 6. State of the closed-loop system in Example 3.

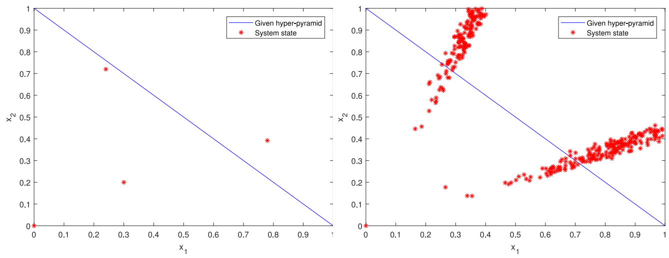
Given $\eta = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$ and disturbances $\omega \in \bar{\Omega}^+_{1,1}$, we can employ Theorem 5 to design a state-feedback controller. With $\widetilde{p}_0 = 1$, the following feasible results are obtained:

$$p_0 = \begin{bmatrix} 7.3366 \\ 0.8827 \end{bmatrix},\ p_1 = \begin{bmatrix} 2.5860 \\ 0.9739 \end{bmatrix}$$
$$z_0 = \begin{bmatrix} 2.0822 \\ -0.9456 \end{bmatrix},\ z_1 = \begin{bmatrix} -7.4720 \\ -2.8156 \end{bmatrix}. \tag{108}$$

The corresponding state-feedback controller is given as

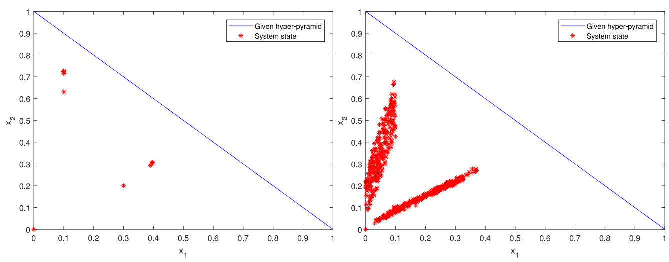
$$K_0 = \begin{bmatrix} 0.6133 & -0.2785 \end{bmatrix},\ K_1 = \begin{bmatrix} -1.0026 & -0.3778 \end{bmatrix}. \tag{109}$$

The trajectory of the open-loop system and the closed-loop system under exogenous disturbances $\omega_k = (1-c) \times c^k,\ c = 0, 0.1, \ldots, 0.9$ $(\omega \in \bar{\Omega}^+_{1,1})$ is shown in Figs. 5 and 6, respectively. From these figures, it is shown that the open-loop system state cannot be bounded by the given hyper-pyramid, while the closed-loop system state can be bounded.



(a) (b)

Fig. 7. State of the open-loop system in Example 4. (a) $\omega_k \equiv 1$. (b) $\omega_k = \text{rand}(k)$.



(a) (b)

Fig. 8. State of the closed-loop system in Example 4. (a) $\omega_k \equiv 1$. (b) $\omega_k = \text{rand}(k)$.

D. Example 4

Consider system (54) with two modes

$$A_0 = \begin{bmatrix} 1.1 & 0.3 \\ 0.2 & 0.2 \end{bmatrix},\ B_{u,0} = \begin{bmatrix} 1.2 \\ 0.3 \end{bmatrix},\ B_{\omega,0} = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}$$
$$A_1 = \begin{bmatrix} 0.2 & 0.4 \\ 0.6 & 0.7 \end{bmatrix},\ B_{u,1} = \begin{bmatrix} 1.1 \\ 0.7 \end{bmatrix},\ B_{\omega,1} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}. \quad (110)$$

The trajectory of the open-loop system under exogenous disturbances $\omega_k \equiv 1$ ($\omega \in \bar{\Omega}^+_{\infty,1}$) and $\omega_k = \text{rand}(k)$ ($\omega \in \bar{\Omega}^+_{\infty,1}$) is presented in Fig. 7. As can be seen from these figures, the trajectory of the open-loop system cannot be restricted by the prescribed hyper-pyramid $\mathbf{C}(\eta)$, where $\eta = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$.

Thus, we have to obtain a state-feedback controller such that the reachable set of closed-loop systems is covered in the given hyper-pyramid.

For disturbances $\omega \in \bar{\Omega}^+_{\infty,1}$, Theorem 6 can be used to obtain a state-feedback controller. With $\widetilde{p}_0 = 1$, we obtain the following feasible results:

$$\alpha_0 = \alpha_1 = 0.3700$$
$$p_0 = \begin{bmatrix} 1.8038 \\ 1.0036 \end{bmatrix},\ p_1 = \begin{bmatrix} 1.2865 \\ 1.2112 \end{bmatrix}$$
$$z_0 = \begin{bmatrix} -1.0772 \\ -0.3575 \end{bmatrix},\ z_1 = \begin{bmatrix} -0.4881 \\ -0.9764 \end{bmatrix}. \quad (111)$$

The corresponding state-feedback controller is designed as

$$K_0 = \begin{bmatrix} -0.5648 & -0.1874 \end{bmatrix}$$
$$K_1 = \begin{bmatrix} -0.1817 & -0.3634 \end{bmatrix}. \quad (112)$$

Using the determined controller, the trajectory of the closed-loop system with exogenous disturbances $\omega_k \equiv 1$ ($\omega \in \bar{\Omega}^+_{\infty,1}$) and $\omega_k = \text{rand}(k)$ ($\omega \in \bar{\Omega}^+_{\infty,1}$) is presented in Fig. 8. As can be seen from these figures, the trajectory of the closed-loop system can be restricted by the obtained hyper-pyramid.

V. Conclusion

In this paper, the problems of reachable set estimation and synthesis for discrete-time periodic positive systems with two different disturbances have been studied. The lifting method and the pseudoperiodic co-positive Lyapunov functional approach have been introduced to deduce the reachable set bounding conditions. Two optimization methods have been adopted to minimize the bounding hyper-pyramids. In addition, in light of the reachable set estimation conditions, we have presented the state-feedback controller design conditions. The techniques and ideas utilized in this paper can be applied to other periodic positive systems, such as periodic time-delay systems with positive characteristics, and periodic positive systems with uncertainties.

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